**Indian Institute of Science**

**Variational Methods in Mechanics and Design**

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**NPTEL Online Certification Course**

Hello so we solved a problem which is the bar optimization problem that is stiffest bar for given amount of material and we also discuss this optimality criteria method and look at the MATLAB code. Now we go back to calculus of variations once again because there are a few things that we need to learn that few more things and that is to recall let us write down the problem.

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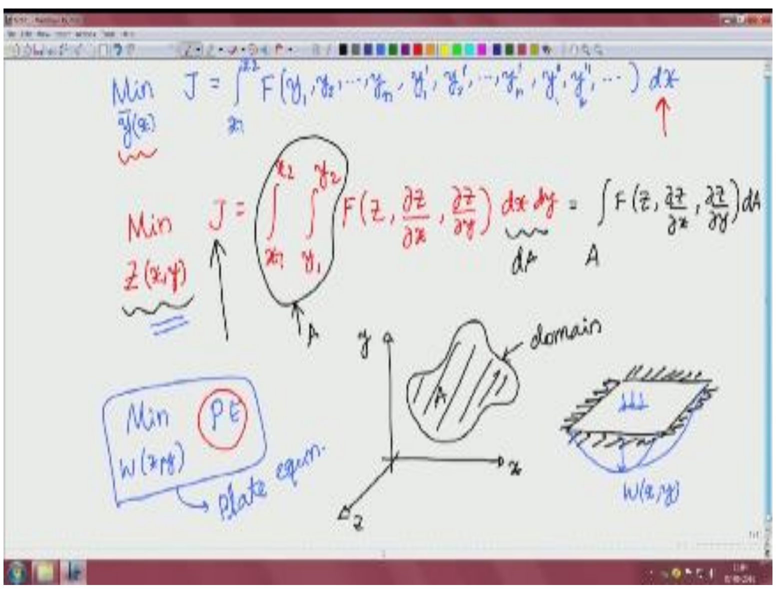

So far what we have discussed in calculus of variations is minimize a functional J with respect to a function y(x) are many let us say we take one function okay or a vector of function let us take $\bar{y}$

y1 y2 y3  n and we have an integral in terms of only one variable which goes which is x, x1 to x2 we had integrand that depends on y1 y2 and any number of its derivative well n functions and then the first derivative second derivative and so forth up to whatever you want right so this is y1 y2 double derivative and so forth.

But our independent variable which is this if was still x that is we are doing one dimensional problems, now what we will do is we will graduate to more than one dimension because most of the problems that we encounter in mechanics are not one-dimensional bars and beams are but plates are not membranes or not and general theory structures of course all three dimensional or 2d structures, right.

So you want to go to more than one dimension in which case all our functions will not be a function of x alone there will be functions of x and y okay that is a difference. In terms of the constraints functional type constraints and function type constraints global and local constraints there will be the same all the change that will occur now will be in terms of the number of independent variables, okay.

So far it is only x that is variable now our functional will have x1 x2 will make now y as our independent variable okay, instead of making y as a function of x will make y as independent variable and make z as a function of x and y it is a 2d problem now okay, and our integrand then will be a function of z and its first derivative.

When I say first derivative now there are two first derivatives are partial δz/δx and then δz/δy and then we write dx dy or domain in that case was just a straight line region with one variable. Now it is a two-dimensional domain so what we have here okay, so this particular thing I can also write this as over some area and then I write my integrand fz δz/δx, δz/δy by dA  so this is dA and this region is A.

So if i take x and y let us say this is x and this is y some arbitrary region okay that is over A over that we have a function that is defined if i say z axis going perpendicular to this xy plane and that zxy surface whose projection is this area that the domain okay this is our domain for the function zxy, so we need to find a function of two variables such that the functional is minimized, okay that is the problem we consider and later two variables can become three variables also. Mechanics from the mechanics perspective we can also think about physical problems that look like this, if you imagine let us say we have a plate okay, let us say it is plate is clamped here clamped here maybe an all edges let us say plate is clapped, okay. Now at every point x and y here there will be deformation so there will be first of all there will be some loading on the plate and the plate is going to deform, right so plate may deform something like this depending on what kind of loading at your port.

Now this deformation is w which is a function of x and y for a beam it is only x now for w which is a function of x and y that is a 2d problem, okay so here also the plate will be minimizing its potential energy so we can write minimize with respect to w which is a function of x and y this is w we minimize potential energy if I do that then I will be getting this is w if I solve this I should get the plate equation you are most likely familiar with the beam equation and how it is derived and so forth. Now we can also do plate equation if we know how to write the potential energy and minimize it with respect to wxy is a function of two independent variables x and y that is a problem we will consider now, okay.

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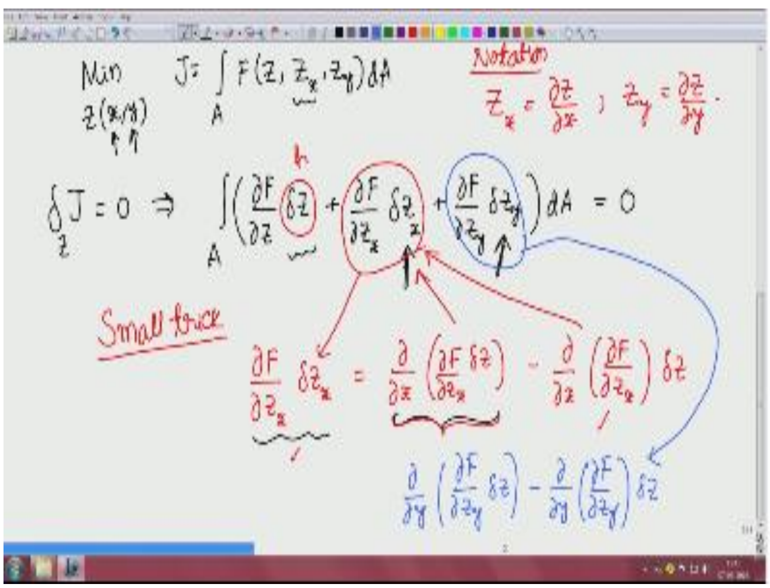

So let us start by writing on the problem from first principles like we had done when we had only Y(x) now we have z x+y J integral over some area I will not write x1 to x2, y1 to y2 that is good for rectangular domains but if i write area A like this it can be any arbitrary domain such as the one we have shown here okay, so I minimize this and I can write my integrand first in terms of Z and I will use this notation because we will have lots of things to write the notation that we are using from now onwards when I write Z sub X means $\delta z/\delta x$ likewise Z sub y equal to $\delta z/\delta y$ and here were to complete the dA.

These are problem we want to solve now when I take variation of j with respect to z that is our variable is that zxy okay, and we have to equate to 0 what we get if you remember the definition of the two variation where we have $z+\varepsilon h$ now are the earlier age was only a function of x now h will be a function of x and y that h can also be called $\Delta Z$ because there is a variation of Z okay, then we take derivative normal derivative with respect to $\varepsilon$ and then subset of $\varepsilon$ equal to 0 if you do all that what we get will be this integral $\delta f/\delta Z$ times $\Delta Z$.

Okay the $\Delta Z$ earlier we were writing it as h okay, now I am using $\Delta z$ to clearly indicate that it is a variation of Z so we have a surface be part of the surface a little bit that perturbation is h that is how we find something is local minimum or not right, so we put up the function and then see whether objective function increases or not if it increases then what we have is actually a local minimum that is our trick.

So we need to consider variation and then since it also depends on zx meaning $\delta z/\delta X$ we have to do $\delta f/\delta Z$ sub x meaning $\delta f/\delta Z$ by $\delta X$ okay, that means wherever f has that X terms that is what we differentiate into perturbation of  zx okay, that is do $\Delta Z/\delta X$ and we will also have $\delta f/\delta ZY$ $\Delta$ ZY and then we have dA as usual, this should be equal to 0 that is what we have okay. Now if you remember when we had the first derivative Y' or dy / DX in order to derive they are diagrams equation we have done integration / parts in the case of two dimensions we'll see we have two variables x and y we have to use something that is equivalent to integration / parts in two dimensions that is a greens theorem okay before we do that our objective if you recall is to convert these $\delta$ Z Z sub X $\delta$ Z sub y x $\delta z$.

We can then w know that δ Z is arbitrary and then we can say that using fundamental lemma of calculus of variations what multiplies that is equal to 0 we get the necessary condition in the form of a differential equation so what we do here in order to convert these things into δ Z we use a a small trick okay it is just very small trick in fact trick is probably not the right word but if you do not know how to proceed somebody tells you then they have given you a trick.

So what we do is we write ∂ f / ∂ Z X δ Z X as ∂ / ∂ X of F and δ z is that actually what we will do is we will write do f / ∂ Z X into δ Z now if I were to use product rule and expand it I will get this one but will also get what I will subtract now because should be on the other side this will be ∂ / ∂ X of ∂ f / ∂ Z X x δ Z okay so notice that if this particular thing is expanded I get this term plus this term and I have gotten it to the right hand side.

In other words I will be replacing this thing which is what we have with this and that okay we throw this away instead of that what is equal these two things we substitute the reason we are doing is whatever δ Z sub X that we had now that has been taken out okay so that is what we are trying to do so we are taking ∂ / ∂ X of this then that gives us this all right that is our aim is like replacing δ Z sub X with something that involves only δ Z similarly for this other term right so that one we will replace with ∂ / ∂ y of ∂ f / ∂ Z Y δ Z minus ∂ / ∂ Y of ∂ f / ∂ Z Y δ Z okay.

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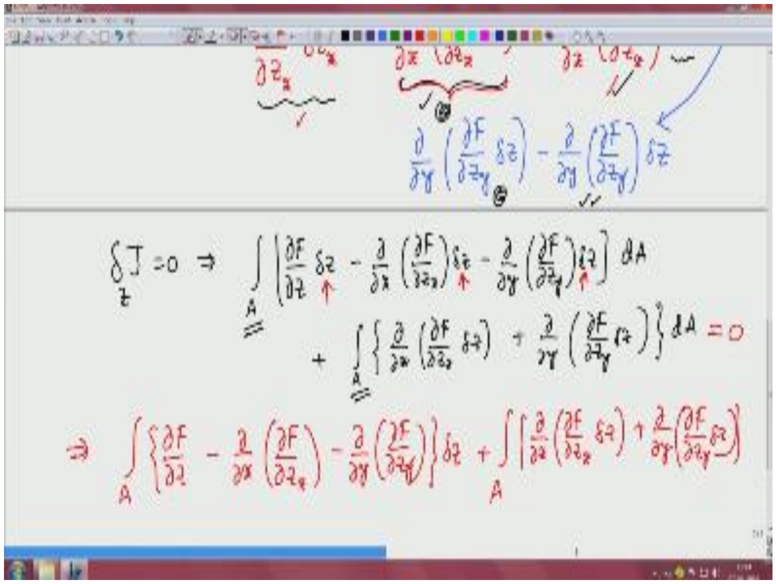

With this we can rewrite our necessary condition in this manner so the δ said variants respect to the z function that we have equal to zero we will now have integral a over the domain a ∂ f / ∂ Z that is this term that remains the same so we have ∂ f / ∂ Z x δ Z and then I will take what I am replacing this and this I'll take the one that has δ Z and write this separately so this will be minus ∂ /∂ X x∂ f / ∂ Z X δ Z and then this also minus ∂ / ∂ Y x 2f / ∂ Z Y time δ that so all these have δ Z let us put it together.

And the other part that is we have this quantity and this quantity I will write separately okay so what have there is ∂ / ∂ X of ∂ f /∂ Z X δ Z + ∂ / ∂ Y of 0 f/ ∂ Z Y δ Z so this will be da here likewise we will have ta there okay both are still integrate over the two-dimensional domain orbital domain but the first term if we see either all δ Z okay so I will write that implies of course this whole thing is equal to 0 that is our necessary condition so I got is a into ∂ f /∂ z - ∂ / ∂ X of ∂ F /∂ z at X - ∂ / ∂ Y of ∂ f /∂ Z Y whole thing multiplied / Z okay that is first part second part is ∂ /∂ x of ∂ f / ∂ Z X δ Z + ∂ / ∂ Y of two f / ∂ Z Y δ Z times δ z ta oh so this δ Z is not there okay this is simply da is equal to zero.

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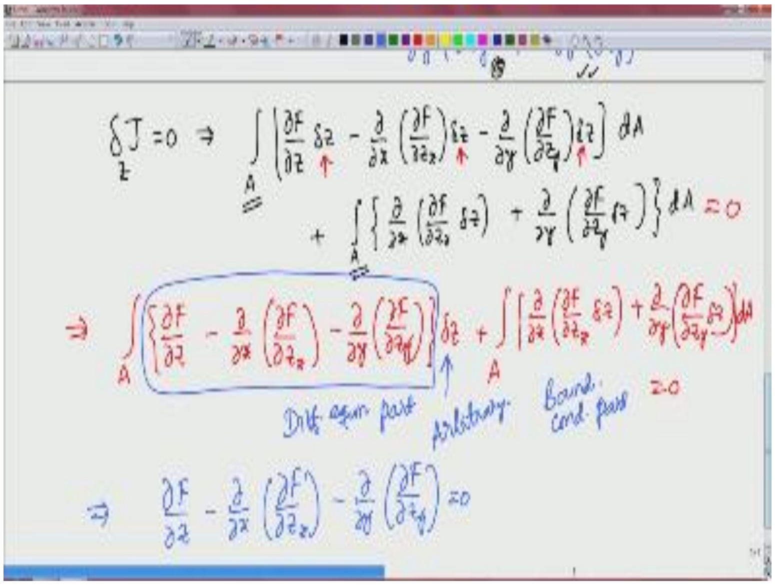

So we have two parts and this is arbitrary this is arbitrary which is clearly outside so what multiplies a current of fundamental lemma must be equal to zero we have two parts this differential equation part and the boundary condition part let us remember this is the differential equation part of the necessary condition whereas this is the boundary condition part both parts are equal to zero we make individual ones equal to zero because the reason we had given somehow with some δ Z you make it work.

So that individual parts are not zero but together they are zero but that is not enough because δ set is arbitrary it should work for anything that is we make it individually 0 when this is 0 since δ Z is the variation and hence it is arbitrary we get the differential equation which is ∂ f / ∂ z minus∂ / ∂ x of ∂ f / ∂ z - ∂ / ∂ Y of ∂ f / ∂ Z Y that is equal to 0 okay.

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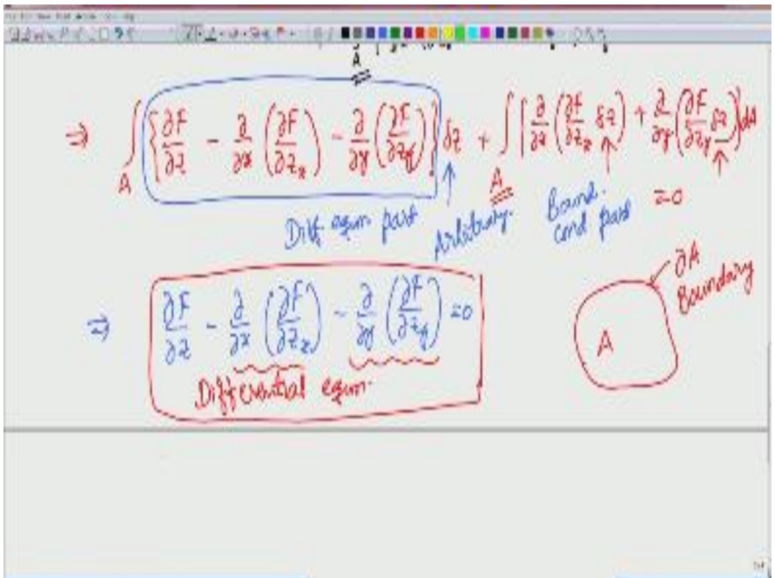


So that is our differential equation okay fact if you recall this actually Euler Lagrange equation there are two independent variables recall it was $\partial f / \partial y$ -d / DX of $\partial f / \partial y'$ that is exactly what we have only thing is with partial derivatives okay now the boundary condition part is little tricky so because there is no $\delta Z$ jetting out here it is inside $\delta z$ it is inside to get it outside we use greens theorem differential equation part Euler Lagrange equation should be clear to you okay.

Whatever was total derivative becomes partial derivative that is all okay there is nothing different here the boundary condition integration by parts boundary conditions for one-dimensional boundary is only x1 and x2 whereas here if I have area like this is my area or which I am integrating okay so here this is inside this is the boundary that is we denote it as $\partial$ a that is a boundary so that is what we want to get out of this is still integrate over the area we want to reduce it in the boundary and that is where we use what is called greens theorem okay.

This green color may not be visible to you but since it is green theorem at least named after person not the color green ok but anyways greens theorem right.

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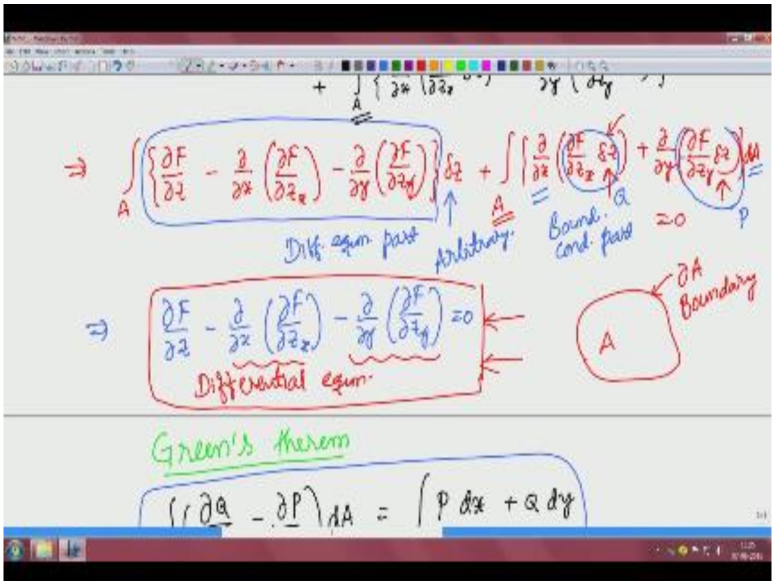


So let us look at what that is what it says is that if I have some area and I have a quantity that looks like this ∂ cube / ∂ X minus ∂ P / ∂ Y over the area this theorem says that is equivalent to doing over the boundary p DX +Q dy okay that is what this is all right. Now let us look at what we have so if we compare this quantity 0 q by ∂ X that is what we have this is Q in this part is Q and then minus ∂ P by ∂ Y is what we have here so minus of that you become key all right so Q P over the area DA now we reduce it to something on the boundary okay so I have to write this, this is the boundary part over the boundary like boundary condition okay which will become pdx p again ∂ f by ∂ Z X Δ Z right.

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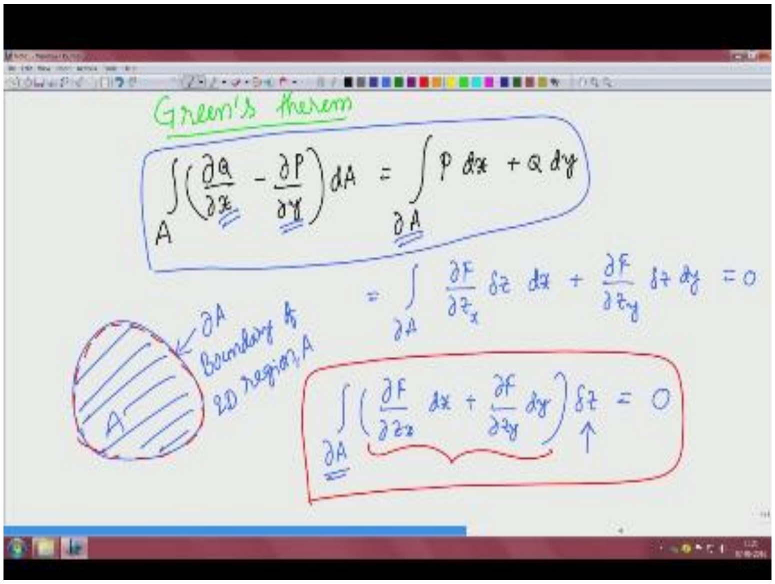


So what you have is ∂ f by  ∂ Z sub X Δ is said that is what we want Δ Z is arbitrary we have and then p into DX okay plus we have Q dy the Q is ∂ f by ∂ Z Y Δ Z ∂ f by ∂ z y Δ z dy that should be equal to 0 because that that is what the condition we had account to green's theorem which is this we got a boundary condition that over the boundary of that region again if this is A this is our ∂ a which is the boundary of region a this is the boundary of 2d region a that we have this is the region we have the boundary and that is where the boundary condition boundary condition says ∂ A.

Where to go around the boundary like this we go around the boundary on that what should be 0 is this that is ∂ f by ∂ Z X DX and ∂ f by ∂ Z Y dy times Δ Z equal to 0 if Δ Z is 0 then there is no problem integrally satisfied that is the what we call essential boundary condition meaning ZD specified if you go back to the plate problem if W X Y is a function if the plate is clamped around the edge there Δ W that is a variation of W just like Δ Z is 0that will be 0 so then you are fine if it is free then this thing should be 0on the boundary ∂ by ∂ Z X DX and TY.

And that is how we get the boundary condition will be the Neumann boundary condition or natural boundary condition okay so we have derived both the differential equation which is this and the boundary condition which is this we have derided for a 2dproblem okay so now with one variable we are able to solve and we saw some applications in this lecture we also see how to solve the 2d problems and 2dproblems are they are both in geometry and mechanics they are also in structural design let us say I want to design a 2d plate okay.

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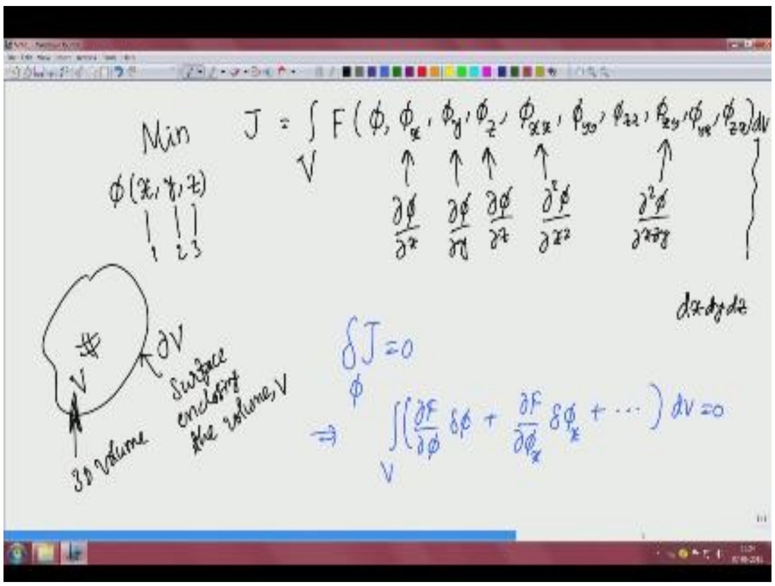

Where I want to determine a function that is a function of x and y okay I can get that right so we can deal with two independent variables now and we can also pose this problem for 3d okay so now let us take a problem minimize J now I will take a function since independent variables maybe XYZ let me take a function something like let us say Ø which is a function of x y and z that is our design variable or integrand now will be over a 3d domain or I will know it as volume V so some v when I put this thing that means that it is 3d volume the inside part of it is 3d volume.

And that is V if I show Dovie that is a boundary surface this is the surface enclosing the volume in closing the volume be okay so here I can write my integrand in terms of T and then V sub X and P sub y p sub Z what does it mean it means that ∂ by ∂ X V is a function of XYZ and this is

∂ V by ∂ Y and this is root 3 by 2 Z if I want which we did not discuss in 2d case which you could as well we can take the second derivatives also Ø x/x x that what we mean is ∂ square Ø by ∂ x square.

And then I can have why, why I can have Z, Z I can have XY right that will be ∂ square Ø by I ∂ X ∂ Y and then I can have Y Z I can have said X and then over the volume right I can have a general function three independent variables when I say DV here what I really mean is DX dy DZ okay DX dy DZ we can have soon one two three independent variables how do we do it is exactly the same way first we take the variation of j with respect to the function v equal to 0recalling or go to variation definition we write it as ∂ f by ∂ Ø into Δ Ø.

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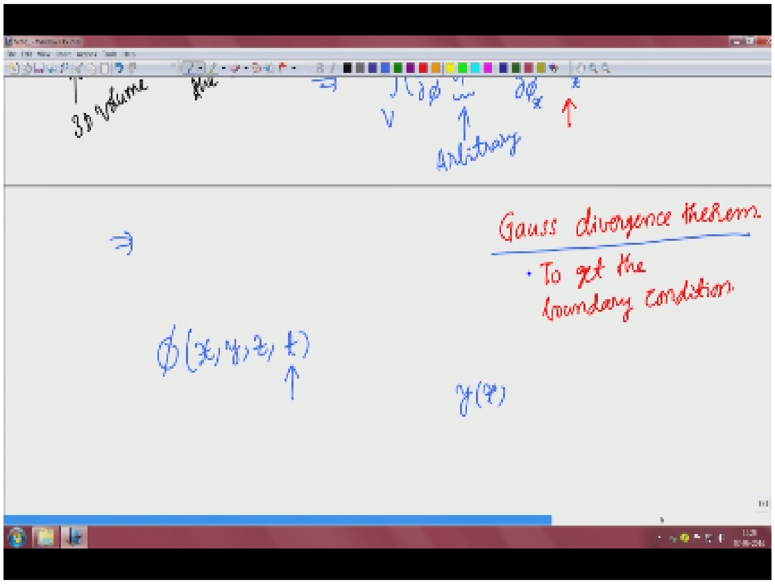


The variation in Ø perturbation and Ø plus f by ∂ V X into Δ V X and then the rest of it how many our Terms you are is there are you will do all that and that will be DV or the volume is equal to zero when we do that so this implies this is already Delphi we do not need anything about the Delphi is arbitrary right that is what we use to get the differential equation and the boundary condition now when you have things like this we the same trick that we use before that is we replace that with something that involves only Delphi are not del P sub X okay.

So when we do all that we get we collect all the terms that will have Delphi and that gives us the differential equation and then Delphi  which will be in the boundary part of it like recalling again to d where we get something like this where the del z was inside we are to use greens theorem in the case of three dimensions we have to use Gauss divergence theorem that is the integration by parts equivalent in three dimensions Gauss divergence theorem okay to get the boundary conditions that is to get the boundary condition.

On the surface of the three-dimensional volume again you see here this is our Dovie surface enclosing the volume V for a 3dproblem like a 3d elastic solid you want to do work with that there is a volume and there is a surface so surface will have the boundary condition wall you will be interior okay  this is to get the boundary condition remember that integration by parts became greens theorem for two dimensions and became Gauss divergence theorem for three dimensions that was too d green Gauss 3d okay an integration by parts for the one variable that is how we get in fact if we were to do this you can see in the accompanying slides of this lecture the detailed derivation of how it comes about.

We have done it by hand step by step today for two dimensional case you can also do it for the three-dimensional case where we will be able to look at the differential equation and the boundary conditions for the problem so we can solve in three dimensions that means that independent variables will have X Y Z in calculus of variations one can go to more independent variables we run out of symbols but you can call it X 1 X 2 X 3X whatever p but such problems are rare in engineering in physics quantum mechanics sometimes people need to use more variables but normally three are good enough for us time will also be variable.

So if I go to dynamic problems I can have Ø  a function of XYZ and T, t is like a fourth variable most often you do not have to treat it like a fourth variable things work out with just three variable but one can do this so if you recall the generalization we started out with only Y of X and then we went with y1 y 2 YN and then we had one derivative and several derivatives now we have not just one Independent,  independent variables so it will be nice if you look to the accompanying slides of these lectures and understand how we go from 2d to 3dwe will do some problems and then you can refer to the slides and understand both okay the next time we will do 2dand 3d worked out examples to understand how differential equation comes up and how boundary conditions could be applied thank you.