Indian Institute of Science

Variational Methods in Mechanics and Design

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Hello so far we have talked about calculus of variations and its role is role in mechanics and we also been emphasizing the fact that calculus of variations has a big role to play even in optimal structural design we have considered this problem in the last couple of lectures and developed enough theory for it today we will actually solve the problem and the problem that we are considering is a bar okay.

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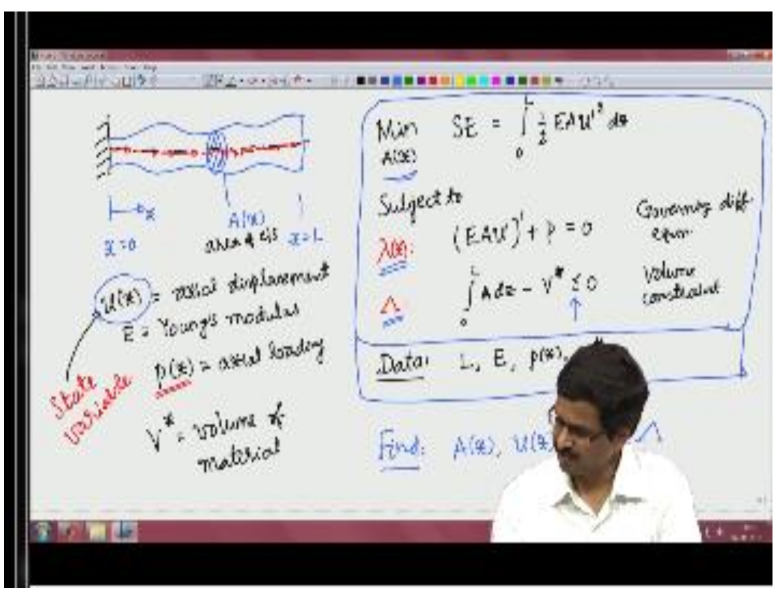

So it is a normally you take a bar to be of uniform cross-section but since we are designing a bar today let us take the axis and show that the bar can have an arbitrary area of cross section this is the problem that we have been taking as a motivating example for including constraints in calculus of variations so everywhere if I take the cross section that is A of x where x is this x = 0 at that point and at this point x=L because the length of the bar is F okay and what we are trying to do is to make this bar as stiff as possible meaning.

We want to minimize the strain energy of the bar which is given as a functional 0 to L half EAU$'$2 x DX let us recall that u of x is the axial displacement when there is a force applied on the bar axial displacement and of course E is the Young's modulus of the material that is the stress-strain relationship and ax is the area of cross section so that is the area of cross section which is our design variable so we minimize with respect to E (y) the strain energy and we have constraints here because this U (x) in this problem is what we call state variable state variable it.

Indicates the state of deformation of the bar here that has to be governed by a differential equation which we had derived earlier which I will write a u prime + P = 0 where p of x is the axial loading, axial loading so it is going to at every point there will be some loading more less depends on what P (x) function that with you is axial loading that is P (x) and the moment we have a function type equation then we have to put a corresponding lambda X which is the Lagrange multiplier okay.

This is the governing equation governing differential equation differential equation which is the function type constraint so corresponding multiplier is also a function and then we have the volume constraint we have been writing this problem in last three lectures this is the resource constraint or volume constraint which is a functional type constraint hence the corresponding multiplier is just an unknown constant which we denote as upper case λ okay, so this is our problem so this is our problem that is we want to minimize the strain energy so that the structure that is the bar will be as stiff as it can be we are minimizing strain energy means that we are not letting it deform a lot that means that it is very stiff under the given loading we should not forget actually to write the data for the problem data.

So we have given the length of the bar we are given the young's of the material of the bar we are given the loading P (x)and we have specified an upper bound and the volume of material so V star here is volume of material okay so this actually completes our problem otherwise we do not know who do not know what is given and what is to be found okay ,we are in the length of the bar X modulus load P (x) and V Star now we need to find what should we find we have to find obviously A (x) because that is our optimization .

Variable in the process we have to deal with our state variable also U (x) that is also unknown we may not be interested in it but as part of the process will end up finding it and we also do not know this Lagrange multiplier function corresponding to the governing differential equation that is λ x and we do not know this upper case lambda okay.

Notice that we have put an inequality constraint as we have discussed in the last lecture and the one before that and much before when we discussed finite variable optimization whenever you have any quality concern there are two possibilities one it can be active the other it can be enacted so we have to find out whether it is active or inactive by working out the problem as we will do today okay, noting this problem the next step is to write the Lagrangian so we will move on and write the Lagrangian .

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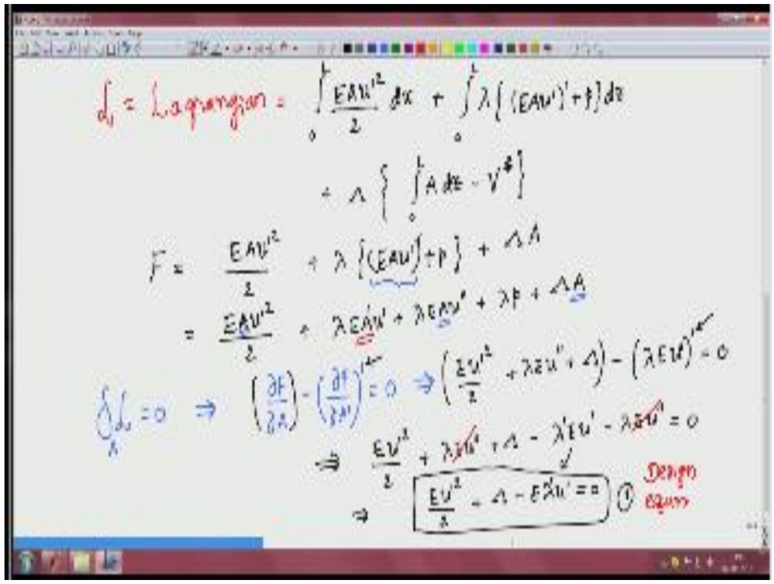

So the Lagrangian for this problem is the objective function 0 to l EAU'^2 by 2 dx +we have the governing differential equation which is a function type constraint which has to be integrated from 0 (L) multiplied by that λ times the governing differential equation itself which is EAU + P dx and we have λ times the constraint which is the functional type constraint which is ADX – V Star that is our Lagrangian once you have a Lagrangian we have to see what our integrand of the Lagrangian so that we can write our Lagrange equation by taking variation with respective variables .

That we have the variables that we have in the Lagrangian E(x) and U(x) A (x) is our design variable U of X is our state variable we also have λ x which is a Lagrange multiplier function that we do not take variation with respect to because that is not something to we have control over the problem decides what λ x should be okay, our necessary conditions are to do with only the function that we do not know what is design variable another is state variable okay, so in fact the governing differential equation is the one that lets us compute.

The state variable provided we know a of x okay now we take let me write what is he integrand for this problem that is what is the integral sign so that is EAU'^2 by 2 + λ times EAU ′ + P plus λ times A that is what we have and then we have of course what is this integral right this whole thing is to integrated and then we have λ times V star that is just a constant that we wrote need to worry about okay, this is what we are interested in let us expand this term before we proceed so that we can see U' U'' explicitly.

So let us write this as EAU'^2+ λ times if I expand it will become EA ′U you were A′U both are functions of X when you take this derivative we do normal product rule EA′U + λ EAU ″+ λ P + A is our integrand now we write the electrons equations are we just say variation of the Lagrangian with respect to area of cross section being = 0 that gives us the Euler-Lagrange equations with respect to A that is we write do f by δ A -we have if you see EA′ there is no A′ so we stopped there δ f by δ A ′= 0 okay.

And then we will write the boundary conditions later first let us write the Lagrange differential equation part okay, if we do that here A is here and here and here okay let us write do F by δ A the first term gives us EU'^2 by 2 λ EU″ + uppercase λ that is this δ F δ A - then we have to see where A′ and A′ here that gives us the second equation second term which is λ EU and actually this is not A′U′ okay .

Let us see so when you wrote δ F by δ A should be λ EA″ λ e so these ′ that the whole ′alright so actually this looks like ′ let me rewrite that that is hopes okay, we go there AU′ okay, that is λ AU′ and then whole prime because of what we have over here that should be = 0 right let us expand this that is right $EU'^2$by + λ EU ″+ λ- now if we take this derivative okay ,we will get two terms λ ′ EU′ and we get –EU″ okay, that is =0 now we notice that this term and this term the cancelled leaving EU'^2 2+ λ - E λ ′ Q ′ basically thus that term i am writing in a different order E λ ′U ′=0 let us leave this equation .

Let us call this 1 in fact this equation is what we call design equation okay, design equation and this is something that may look amusing now because there is no A here when we say we are designing a bar we are actually determining a of x function but this equation .I do not find a but yet as we will see this enables us to find area of cross section optimal area of cross Section okay, that is equation 1 design equation.

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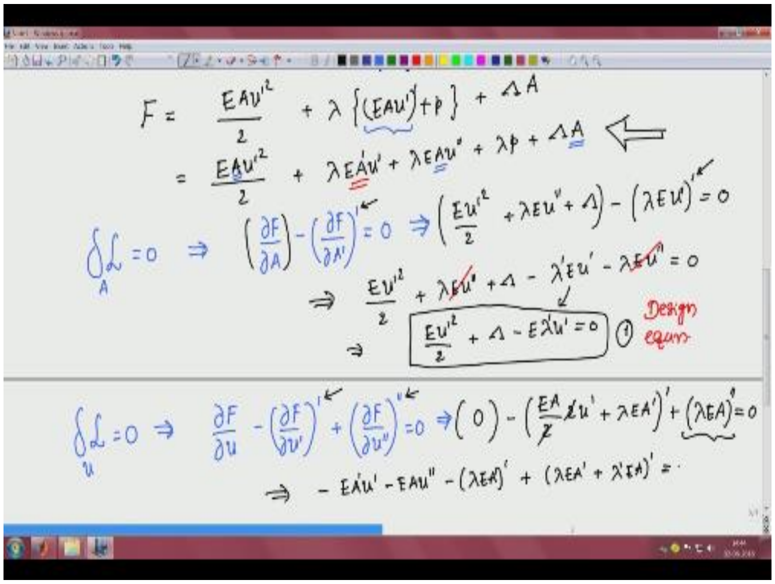

Then we go ahead and take variation of the Lagrangian with respect to the state variable ok because that is also a function let us see what F is which you see at the top of the screen If you take that then that will be δ F by δ U because you is present in integrand F –δ F by δ U ′ and then there is U ′also so we have to say to get include the third term δ F by δ U′ ′= 0 we will also have boundary conditions which will come to later for now let us take this δ F by δ U where is you U is over actually you is not there right.

If you look at this over there this is our integrand there is no U present there is only there are terms with U′ 1,2 terms with U′ one with you W′ so the first part will be just (0) δ F by δ U there are two terms the first term that will give EA by 2 and then we have U'^2 will give to U′ okay ,so 2, 2 gets cancelled so EAU′ and then the other one will be λ EA′ that is the other term and this should be ′ of that because we have that and then + U′ is λ E that is δ F by δ u ′ will be δ EA and we have to take ″ because of that and that should be = 0 so let us expand this so first one will get EAU ′where take another ′ of that okay.

So I will expand it EAU′ will be A′U′ - EAU″ okay EAU′ A′U′ and then EAU′ when you take it and the other one here I will leave it as it is in fact it is also -+ the other one i will not expand because I know that it gets cancelled with the thing that comes out of this term that has double ′

the derivative twice first time we will get this is a ′ by the way. So we have λ EA′ +λ ′ EA and then we have ′of that =0 okay.

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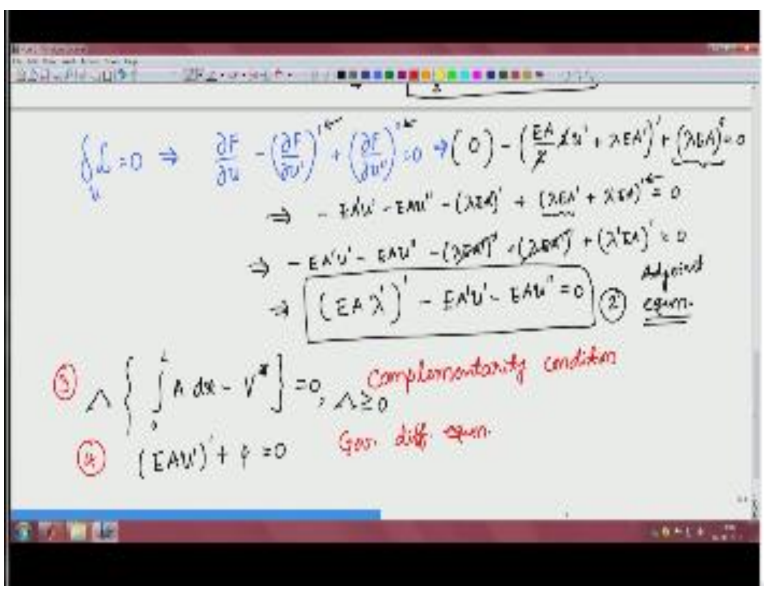

We further so E A′- EAU′ -λ EA′ that in want to expand because this term here is basically λ EA ′ and ′ because we have the ′ there and this one I leave it as it is that is λ ′ EA′ leave it as it is okay, now if we look at this these two get cancelled okay, so what we are left with in this case is λ ′ EA. I will write like this ′- we have EA′ U′- EAU″ = 0 okay you for algebra is correct this is our second equation which we got by taking variation of the Lagrange respect to U let us call this second equation this is called the ad joint equation okay ,yeah so this hour adjoint equation right.

So when we go back to this okay, let this the adjoint equation we have 2 equations and our unknowns are A(x) U(x) and λ FX and capital λ so we have in addition to this we have these equation that is 0 to L dx- V star okay and corresponding multiplier that should be = 0 which is what we called complimentarity condition whenever you have inequality condition inequality constraint we need to have this complementarity condition okay and then we have the governing differential equation $EAU''+P = 0$ which is our governing differential equation.

Another thing that we know should hold good is that in addition to this component condition this λ should be greater than or 0 λ can be 0 or positive but not negative okay ,so we have these equations so if I call this equation third and this equation for we can solve for this problem because we have four unknown that is A(x) U (x) λ F(x) and the capital λ we have one differential equation here equation 1 equation ,2 equation ,3 and equation and equation 4 you take a break now then come back to the solution of this equation in the next part.