Indian Institute of Science

Variational Methods in Mechanics and Design

Prof. G.K. Ananthasuresh
Department of Mechanical Engineering
Indian Institute of Science, Bangalore

NPTEL Online Certification Course

So we discuss now how to deal with local constraints or function type constraints, let us solve one example to understand what this means. Let us go back to the example of what I had called Chatterjee's problem okay.

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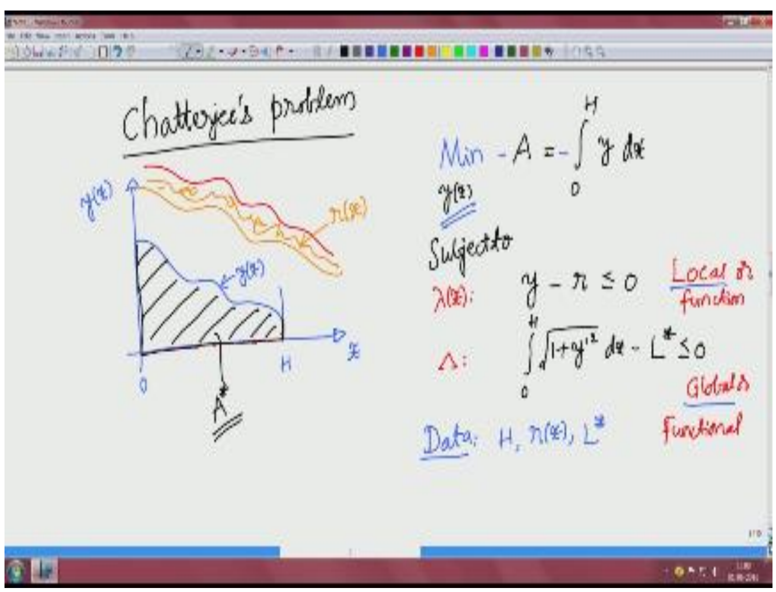

So let us write that Chatterjee's problem this after and in the Chatterjee is a professor in IIT Kanpur in my discussions with him this problem had emerged so I continued to call it as

Chatterjee's problem so that problem is that I am showing the roads that we had discussed earlier as access now there is x-axis there is a y-axis I need to find a function y(x)such that there is a river flowing okay, let us say this is the river r(x) okay, and there are some limits that is i 0 to some H here, okay.

Now I need to find a function y(x) okay, I need to find that function y(x) in the limits 0 to H such that let us say I say this function y(x) like that this y(x) has to be below or of x because we said there is a river flowing so let me show it like a river there is some muddy water okay, it has to below that and the former who is asked to take as much land as is given by the government let us say that is some A the land A this area under the curve is A that is given let us say that is A* that is given.

But this former can take A* land within this domain 0 to H but has to be below the river what should be the curve which will be the boundary of this there are already these boundaries this is a boundary is a boundary if there is this height that is also a boundary but the new boundary has to be chosen the former wants to take A* area land but he or she wants to minimize the fence length that is the length of the fence, so that problem we can pose as minimize length of the, minimize the length of the fence are for a given fence he or she wants to maximize area let us write it that way so not minimization, okay.

Let us actually pose it as maximization maximum A okay, not A* let us say this former says I have budget of only so much to spend on building of fence which is this y(x) function so here she wants to maximize area, so area will be ydx, area is ydx from 0 to H right. If we have such a thing here our unknown here of course is y(x), right so when we have this problem and we have some constraints here right, subject to we have subject to the fact that y should be less than r less than or equal to because the river is there why his or her friends cannot go into the river we have this constraint.

Which is a local constraint because this should be satisfied at all points in the domain at all points it is a local type constraint or a function type there is no differential equation right, so there is simply y(x)-r(x) is less than or equal to 0 right, that we discussed already how to deal with that there will be a correspond Lagrange multiplier function there we also have the constraint that former has imposed which is the length of the fence length of the fence 0 to H $1+y'^2$ $\sqrt{}$dx minus some L* that is chosen is less than equal to 0 this is a global constraint right, so this is a global constraint or a functional type constraint, okay.

So we have chosen a problem which has both a local constraint and a global constraint okay, again how does this constraint come up this because the fence length of the curve that is $\sqrt{dx^2+dy^2}$ we have taken dx out that becomes $1+y'^2$dx okay, and what is data for this problem we are given H and we are given this r(x) river profile and L* okay, these are the thing that are given to us we need to find y(x) okay.

One thing that we would change is that whenever we maximize something our convention is always to minimize so let me change that maximization to minimization but I will put a negative sign, okay minimizing negative area is equivalent to maximizing positive area.

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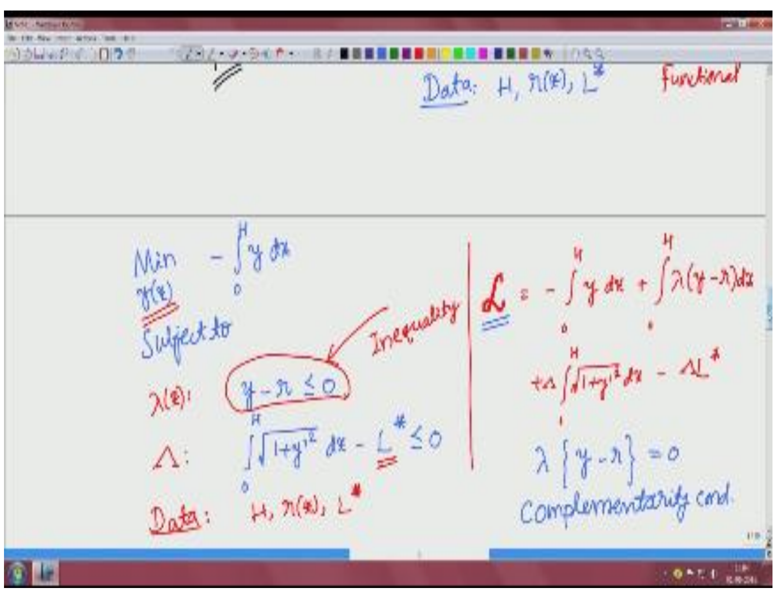

So our problem now becomes again to have continuity, right so we have that right so we would write the problem minimize minus integral ydx that is the area enclosed from 0 to H our unknown function is y(x) subject to y-r less than or equal to 0 and then 0 to H $\sqrt{1+y'^2}$dx-L* than or equal to 0 and do not forget to write the Lagrange multipliers λx and λ okay. When you have this the first thing we do is to write the Lagrange which is 0 to H minus is there ydx+ 0 to H λ times y-r dx.

Note again that this constraint this local constraint should be valid for all points in the domain so we have to integrate after multiplying the corresponding λ, λ is also defined everywhere this constraints also you find everywhere we multiply everywhere and add because ultimately our Lagrangian is a functional it should be a scalar okay, plus we have this other one zero to H λ $\sqrt{1+y'^2}$ dx-λ L* okay, this is not Lagrangian L* this is simply length* okay, this length here that is not the Lagrangian, Lagrangian we have slightly scripted L okay.

This is the thing now in order to solve the problem I have this function y(x) here what is given the data we should always right we had written above will write again we know H we know rx and we know this L* here, okay local constraint which is an inequality that is a difference you see. If we have equality with only one function this would have been any improperly post problem but what we have here is an inequality when you have inequality.

We recall what we had discussed in earlier in one of the earlier lectures dealing with finite variable optimization when you have inequality we write what is called the complementarily condition that says λ times y-r=0 again this is applicable for all points in the domain x1 to x2 this is called complementarily condition, okay that means that either λ is 0 or y-r is 0 at different points in the domain from 0 to H in this problem okay.

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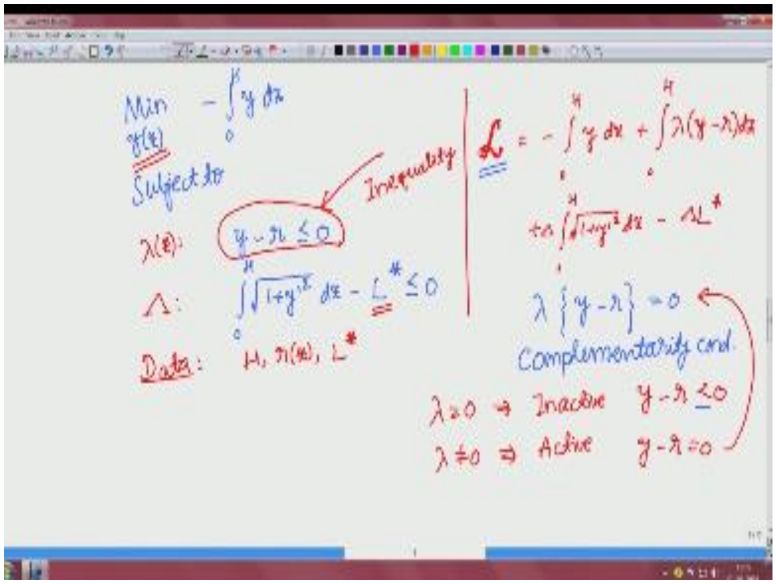

So if you go back to the problem if you say y-r=0 that means at some point the function may just follow the river okay, so here we say the constraint is active, active constraint right, in other places if it deviates from it like that in this portion this is active, okay. Let me put it in blue other portion let me write in red it is inactive right where y is less than r that river profile, right. Whereas in some portion it is just equal in that portion where it is active the λ that we have need not be 0.

Because complementarily condition says that either λ is 0 or Y - R is 0 their product is 0 right so when λ is zero it becomes it makes the constraint inactive in such a portion in general y - R I < zero it can be zero in some special cases in general it is not other hand if λ is not equal to 0 implies that the constraint is active that also means that Y -0 R = 0. Because of the complementarily condition, okay.

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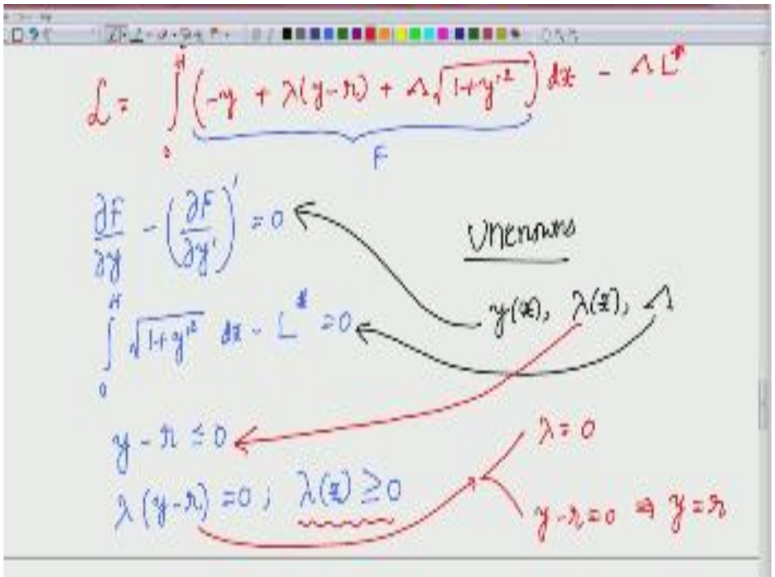

So for this problem we can write the necessary conditions okay we can write the necessary conditions in the manner of KKT conditions for calculus of variations that is Euler-Lagrange equations + KKT conditions that is what we mean by necessary conditions for this problem if we do that we have this Euler Lagrange equation if I right we have in the Lagrangian we have this entire integrand let me write that integrand let me write the Lagrangian here the Grange in 0 to H - y from the objective function λ times y - R okay.

And we have capital λ time $\sqrt{1+y'^2}$ DX and then we have this λ L* basically I have collect the terms that where here we wrote we wrote this one we wrote this one we wrote this one and then we also wrote that which is immaterial that mean yeah this hour lagrangian so now what is f the integrand is this when I write Euler Lagrange I Lagrange equation I have to write $\partial f / \partial Y - \partial f / \partial Y'$ whole thing prime d / DX of that equal to zero we only have Y and Y' here in this problem we do not have Y'' and other things okay.

That is true and we also have the constraint when I read KKT conditions I have to write the constraint also 0 to $H^2 \sqrt{1+y'^2}$ DX – l* = 0 and we have our inequality constraint YX - R X is less than or equal to 0 and then we also have the complementarily condition which is λ times y - R = 0 and as complementarities condition arises with inequality we also have this other thing that λ of X is greater than or equal to 0 it cannot be negative for the same reasons we had stated for finite variable optimization okay.

So now we have some equations to solve for unknowns are again let us see what our unknown sir our unknowns are clearly Y of X and also we have λ of X that is not known right and we have this upper case λ these are the things that we do not know in order to solve for y of X which is a function then we need a function equation or a differential equation we have a differential equation here okay now in order to solve for λ with it is a scalar unknown we need a scalar equation which we have and for lower case λ that is this one we have a function type constraint which is this and this complementary condition gives rise to two case is either λ is 0 or Y - R = 0 and then this automatically gets insured so this gives rise to two cases case 1 case 2.

One is λ =0 other is y – r = 0 where y - r easy to deal with we get a y already why is r that means that you follow the river profile whenever you do not follow the river profile then λ = 0 the condition becomes inactive okay.

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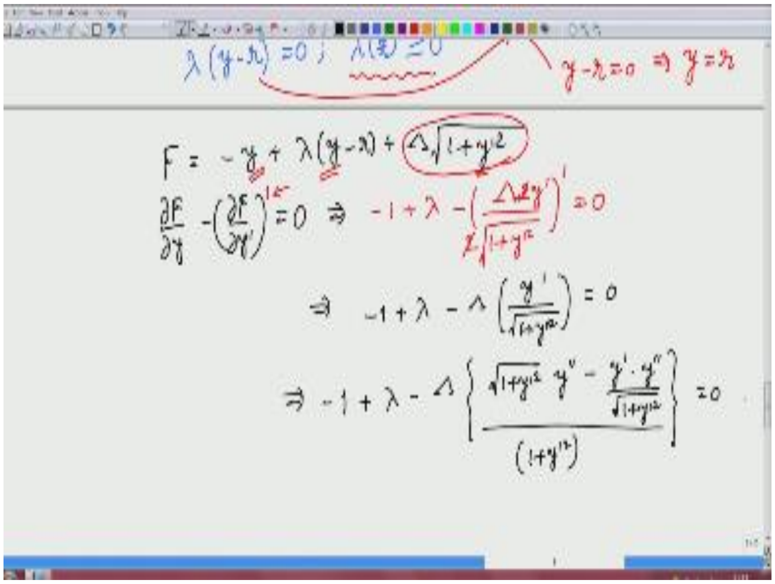

So let us first write this Euler Lagrange equation that we have here for F remember our F is - y λ x - y + λ is y - R + Δ times square root of 1+ y prime square okay let us finish up this problem so I will write f -y + small λ times y - r + Δ x 1 + y'^2 now what I want to write is ∂ f by ∂y - ∂ f by ∂ y' = 0 that means if I were to f by ∂ y' is over here gives me - 1 but also get why they are so I will get +λ 0 k - 1 + λ and that is it there is no why there is y prime here so I have to write - ∂ f by ∂ Y' that will give me λ I am taking derivative of this one with respect to Y' that will give me 1+ y'^2 root there will be a half here there will be a 2 y' let me write it anyway + 2 y'.

Because Taylor is respect to Y' is y prime square there that gives 2Y' and it is to get canceled okay and this whole thing derivative that is what we have here right that prime right that is equal to zero the words what we are getting here is - 1 +λ - λ times y' / $\sqrt{}$ 1 + y'^2 0 let us expand this is a necessary condition let us see if there is a solution lying in front of us so let us take this derivative of this okay the derivative I take λ we have we use the quotient rule here.

So that will give us 1 + y prime denominator derivative of the numerator and then – the numerator time derivative of the denominator that will give us we already have that. So that will give us divided by square root of 1 + m square half and this will go we will have y '' there at a derivative of that you know y' thing divided by square of the denominator that you become 1 + y'^2 = 0 okay.

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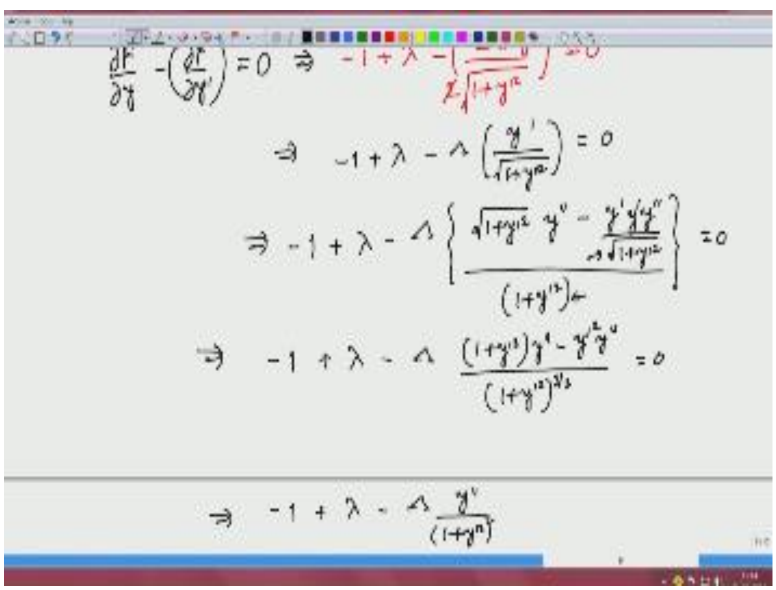

Now if you further proceed - 1 +λ - λ if I take LCM of the op portion that will give 1 + y'^2 square and y'' let me write that so it will become 1 + y'^2 x y'' okay. Y' okay here I will get YY' there will be another Y' take derivative that becomes y'^2 y'' I suppose yeah and in this here we will get 1 + y $'^2$ raise to 3 / 2 because you already have this and then this is coming together that is equal to 0 so if I do this further - 1 + λ- λ here y' y'' will get cancelled what I will be left with will be y'' / 1 + y'^{2} t / 2 = 0 okay, that is what we get.

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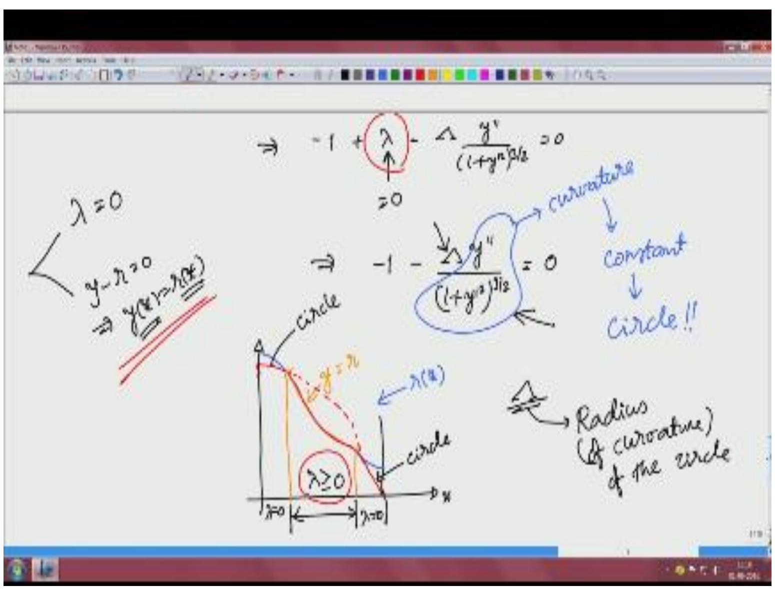

Now we have two cases right so one case is $\lambda = 0$ other case is y -R equal to 0 or Y of X = R of X so in the domain 0 to H we have two possibilities one is the function y of x = to R of X solution is known we follow the river profile other is $\lambda = 0$ if I substitute for the second for the first case $\lambda = 0$ right then - 1 – λ X y'' + 1 + $^{y'2}$ 3 /2 = 0 or in other words if you notice what this is this actually curvature right in your beam theory you would have used this result many times $y''/ 1 + {}^{y'2}$ h 3 what is curvature what is saying is that whenever you do not follow the river profile you should choose a curve which has constant curvature.

Because it is constant because there is λ and there is - 1everything else is constant here so this curvature is not a function of X right curvature is constant when curvature is constant what should be that curve that curve should be a circle, right so the answer to what we call the Chatterjee's problem 5x and let us say this is the river function this is our of X and we have to be from here to here only so this is fenced in now if you want to solve this problem for a given fence length okay you, you have to let us say whenever you are not touching the curve you should be a circle if the circle goes like this that is you are going to enclose maximum area.

That we know that is a parametric problem instead if you were thing comes out to be in such a way that let us say River profile let me change the river profile little bit to make our point okay let us say the river profile I do like this all right then let me erase what I have chosen earlier okay so our thing this was our answer but that will not be allowed now because we are going above we should be below the river so this is not a load so in this portion what happens is this portion is not allowed we just have to follow the river there we have to follow the river okay.

So in this portion from here to here are in terms of this here, here y that we want is equal to r other portions it is a circle okay so here it is a circle and here it is a certain because that would enclose a maximum area for a given length of the curve that is we know as is parametric problem so by having a global constraint we get this Lagrange multiplier λ or we get this lower case λ so lowercase λ is zero in this portion here it is 0 and here also it is zero in between this region this λ because the constraint is active it is following the river profile give it greater than equal to 0 okay.

And what about this capital λ that simply indicates the radius of curvature of this red curve the Y so Lagrange multipliers always have some physical meaning this radius of curvature of the circle radius of curvature of the circle meaning radius of the circle itself radius of this circle that we have that indicated by this λ that what comes from the equation that we have here so we safe a Chatterjee's problem circular arc is a solution but whenever circular arc goes above the river profile like here we have to follow the river right.

There the inequality constraint is active then there will be a corresponding Lagrange multiplier coming in into the problem then the curvature becomes will not be true because this λ then k λ here will not be 0 that will be variable and that is why we follow the river profile which we have from here okay so we have solved a problem that involves a global constraint as well as local constraint in mechanic's this kind of subsidiary constraint has another use to solve contact problems.

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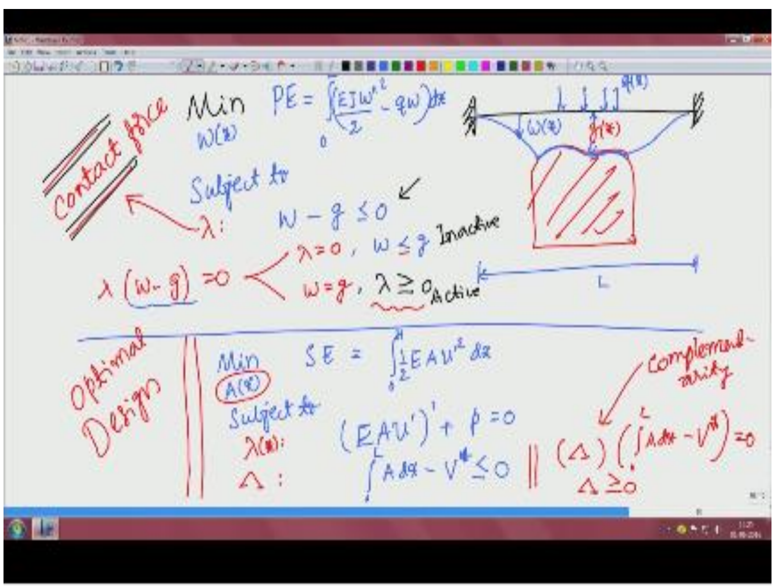

Let us I have a beam which let us say is fixed at both ends and I have a rigid obstacle of some kind here okay there is a rigid obstacle okay now when this beam deforms it cannot go through it so there it has to follow the profile when there is some loading that is applied on this say some Q of X okay then it has to follow it cannot go into this obstacle so it where it has to follow this so that can be post here so minimize potential energy of the beam here w transverse displacement everywhere this w of x is the minimize potential energy0 to L.

Let us say this length of the beam from here to here is LE second moment of area w mm square minus 2 and then Q times W this is the total potential energy of the beam which we had also mentioned earlier subject to subject to midway pressure energy will find W of X to minimize that that gives a statically solution but then we have the problem that the W that we get has to be less than the gap function so here we can define a gap between the beam and obstacle okay.

We say that that is G less than or equal to 0 right W minus whatever gap that is different at different points that will be specified and hence there will be a corresponding λ here this λ turns out to be contact force here because Lagrange multipliers have physical meaning okay it is a contact force and that also makes sense when we write the complementarily condition that λ times W minus G equal to 0 has two possibilities either λ equal to 0or W of equal to G when λ equal to0 then W is less than G strictly because this need not be 0 when λ equal to0 you can also be 0 in again in some special cases which you do not need to worry about right now.

When W equal to G this already satisfied then λ we say is greater than or equal to 0 okay then in this case it is active where as in this case this is in active so contact problems can be solved using variational framework when you have function type constraints okay so we have solved the problem shortages for example to show that how we can deal with inequality constraints which are function type constraints we also see the same thing in contact force problems in mechanics and that brings us to our structure twinge problem.

Which we solve in the next class which we noted at the beginning of this lecture minimize strain energy which is 0 to H EAU ‘ square DX at half and area profile is the design variable subject to governing equation is a function type constraint or a differential equation local type constraint EAU ‘, ‘ plus P equal to 0 and then 0 to L A DX volume constraint which is a global constraint which we can put it now as a inequality constraint right so whenever put inequality corresponding multiplier here λ X and here it is we use lowercase Greek letter for a function.

And we use uppercase letter for a unknown constant whenever you have this here also we can write complement at the conditioned λ x the constraint 0 to L a DX minus v star equal to 0 and λ is greater than equal to 0 just like the previous problem we wrote this lower case λ greater than equal to0 at every point in the domain it unknown constant one constant is greater than or equal to 0 and this is the complementarily condition complementary clarity condition okay.

This particular problem we will solve in the next lecture and also do some computations to see what are the optimal area of profiles for this that way we see calculus of variations with constraints now in geometry we saw Chatterjee's problem contact problem in mechanic's now we also see this in optimal design that is the demonstration that variational methods are useful in geometry in mechanics as well as in design thank you.