Indian Institute of Science

Variational Methods in Mechanics and Design

Prof. G. K. Ananthasuresh
Department of Mechanical Engineering
Indian Institute of Science, Bangalore

NPTEL Online Certification Course

Assuming we say that we have to take to perturbations.

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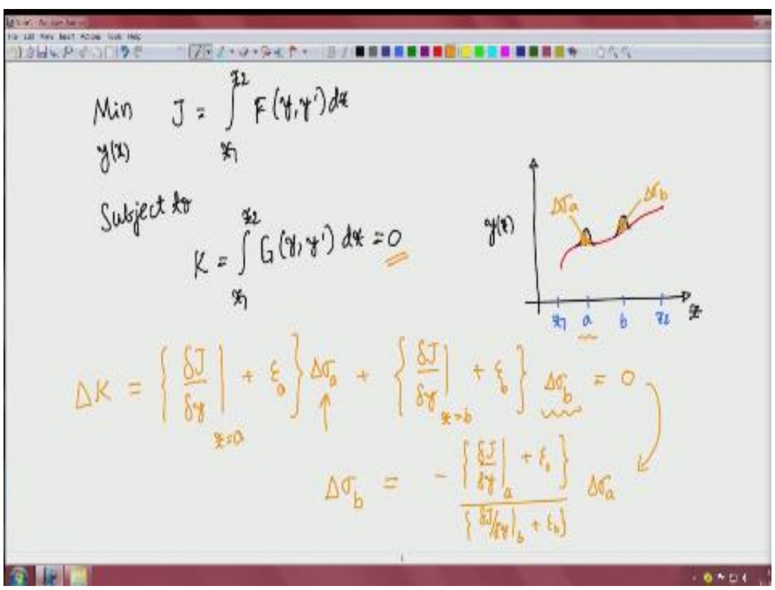

So that equality constraint remains satisfied and now we express Δσb okay all the errors ε *δ* either go to zero anyway Δσb can now be written as - *δ*J *δ*y derivative at a + ε a/ this one *δ* J x *δ* y derivative at b + ε b times Δσ a that is what we get from here to here right.

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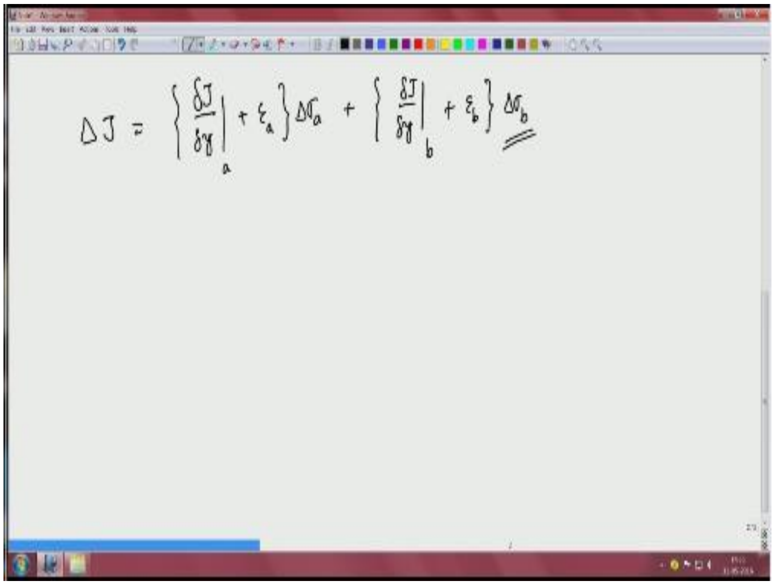

Now what we do is consider the change in first-order change in our functional because the same to perturbations we have taken a function y unperturbed at two points a and b then this one will also be there this will be if I looked at it you know here what we have we have to make a small change when you have K this is actually K here or not J right.

So let me change that this is yeah so that is δ K let me take the same thing δ K δ K δ K and δ K okay not J because you are doing in K now we will try to do this in J that will only change will become this is δ J xδ y at a + ε a into $\Delta\sigma$ a the same perturbation that we had area + this will be δJ / δyb + εb into $\Delta\sigma$b that is the chain that is not equal to 0 what we need say well when we take this in terms of only a single perturbation σ Δ σ b.

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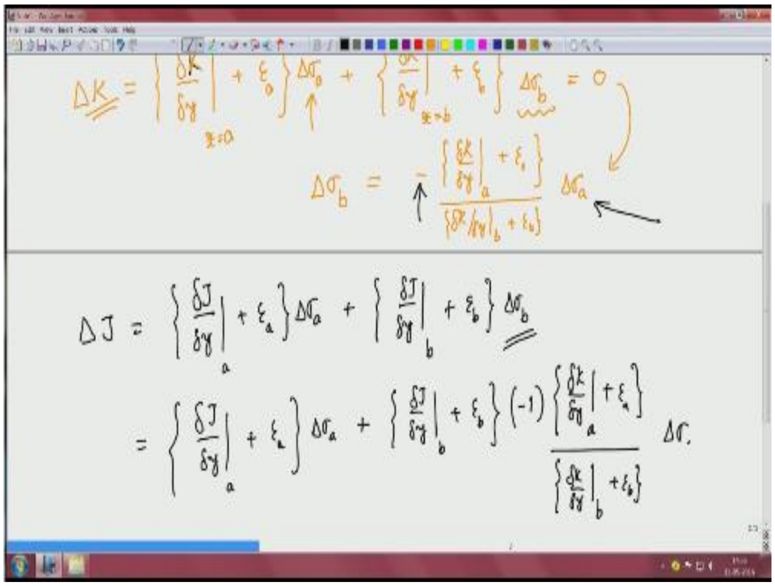

We have now written in terms of $\Delta\sigma$ a so we can write this this is a first order change in the functional objective functional because of the change in the function actually two changes to perturbation so we will have δJ δy at a + ε a into $\Delta\sigma a$ now instead of $\delta\sigma b$ will substitute what we have here so that will give us this thing δJ / δy at bε b and this will be the - okay I will just put it is – 1 do account for that times I am δK / δy εa/ δK/ δy atb + ε b this whole thing times $\Delta\sigma$ a.

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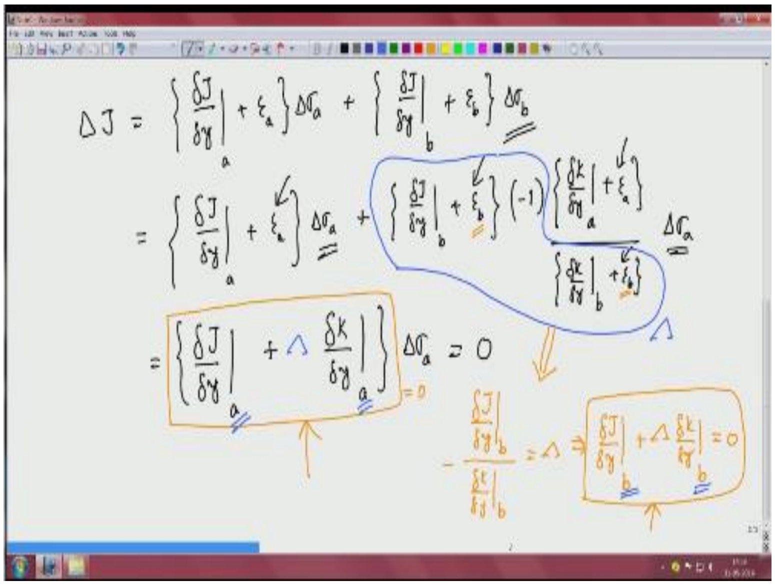

Now if we look at this is $\Delta\sigma$ Δ σ the perturbation there is only one that we can make other perturbation is dependent on the first perturbation to make the constraint satisfied the K equal to 0 or δK equal to 0 first order thing now if you collect all of these this one is going to be we will say that all of these ε that we have ε a in several places they all go to 0 if a perturbation really small the other this $\Delta\sigma$ a tends to 0 also right when you have that we can leave out all these things now what will be left with will be $\delta J / \delta y$ at a + we want to call this portion include in the negative sign so there is a negative sign which came because of this here right including that we want to call that as sum λ okay.

So I will write that λ okay what will be left out after make ε a tends to 0 is $\delta K / \delta y$ at a okay all of this x $\Delta\sigma a$ this we argue should be 0 this is the first order change and that should be 0 if y that you are considering is actually a minimizing function okay that is what we got so you got an equation now let us circle it yeah this whole thing because $\Delta\sigma$ is arbitrary this whole thing should be equal to 0 right let us remember that now let us also try to write what we have here we define something to be λ okay.

Now again these ε we will not right because they tend to zero what we have here is δ J δy that is a variation derivative at b okay divided by δK / δy at be with a minus sign that is equal to λ so what does that give us when we write it δ J δ y at b + because λ these are - sign let us say we take the - and the other side and we have δ K multiplying that that will be λ times δK / δy at be equal to 0 so we got another equation here we had one equation another equation if you look at it they are the same only thing is this is evaluated at a this is evaluated at b let us remember that the two points we chose for perturbations their arbitrary you could have chosen this a and b if you go back and look at.

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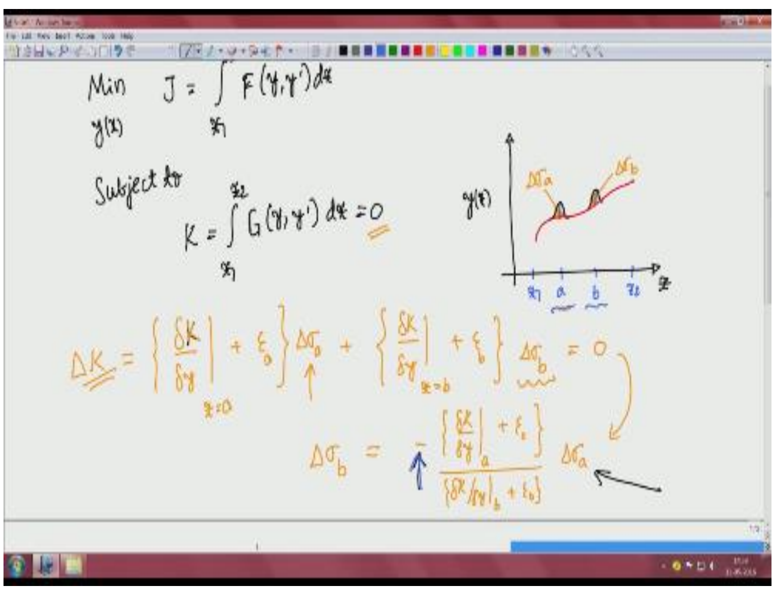

It this a and B because in wherever we want that means that.

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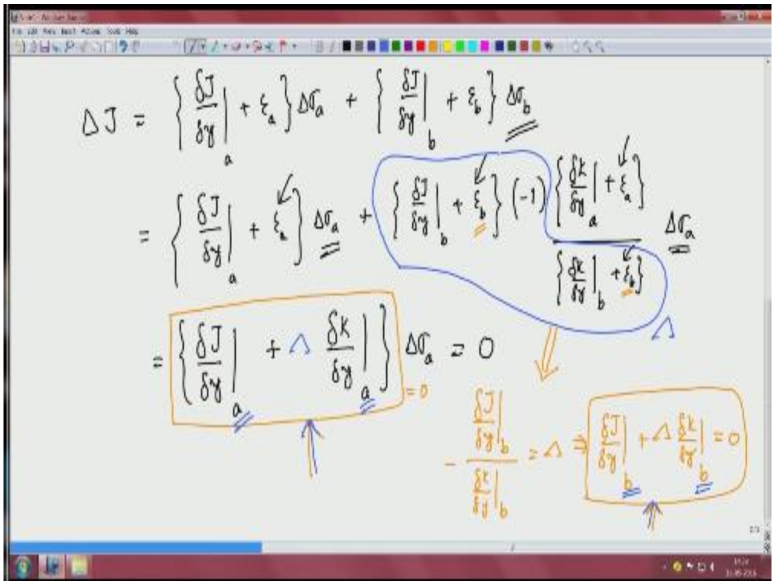

The two equations that we have here this is equation one and equation two are to be valid at every point in the domain fromx1 to x2 so that gives us a necessary condition.

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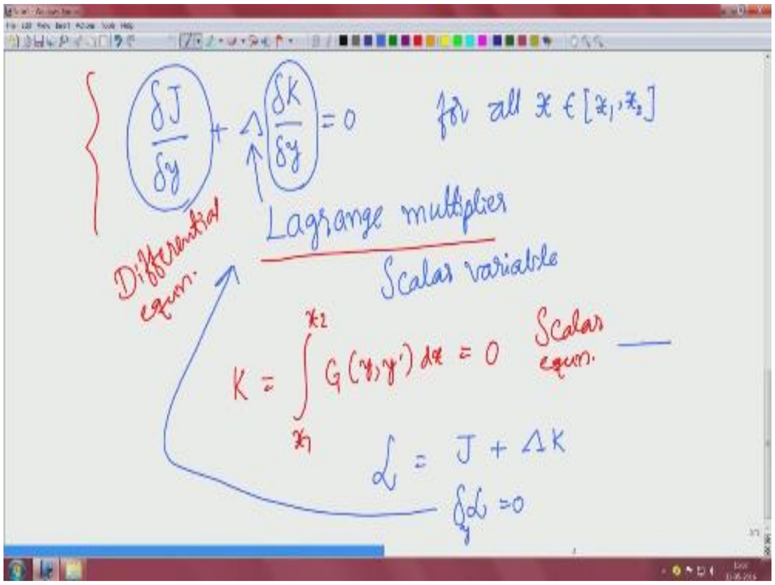

Now which is to say the variational derivative of the objective functional okay + some λ times variational derivative of the functional type of constraint equal to zero this should be true for all x belonging to the interval x1 to x2 everywhere they should be true and that is the necessary condition for a global constraint problem now this thing like we had done the finite variable optimization problem this is called the Lagrange multiplier okay this multiplier is a scalar variable a scalar variable okay.

So we have to find that how do we find it because this is a differential equation because this a Lagrange equation will be an expression this will be an expression this is some constant that we do not know as yet this is essentially the whole thing that we have here is a differential equation right this is a differential equation okay and then we will boundary conditions also that we can write for it but in order to solve for this Lagrange multiplier in equation fortunately we have it what is the equation to solve for that we have this x1 to x2 G that is k equal to G y y´ ´ dx equal to zero.

This is a scalar equation this is a scalar equation because if you remember you recall a functional is a scalar it is value because function is a mapping from function space to a real number space it is real value scalar so this is a scalar equation so we can solve for a scalar unknown which is λ okay so when this is the what we have here is the necessary condition we can also write the Lagrangian here just like we had done penetrate variable optimization you have the J + λ K we can write a Lagrangian then if you take variation of this Lagrangian with respect to this function y equal to 0 what you get is essentially this differential equation okay and we have the constraint which is this which we can solve.

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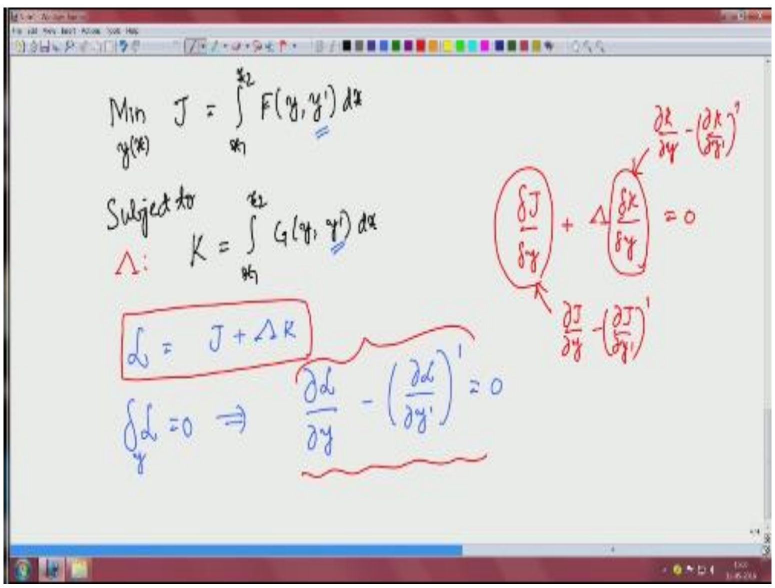

So we can right now the necessary conditions for the problem where we have minimize J which is from x1 to x2 with an integrand fy and y ´ dx where y is the unknown subject to or constraint k equal to x1 to x2 where integrand is G which depends on y and y ´ dx let us denote let us make it a habit to indicate the corresponding Lagrange multiplier right there okay with a colon there to indicate Lagrange multiplier.

Then we can write the Lagrangian for this problem as J + λK then we take the variation of the Lagrangian or write Euler Lagrangain equation for the Lagrangian directly meaning if I were to write what I would do is $\partial L / \partial y - \partial L / \partial y'' = 0$ zero because you are taking up to the first derivative if there are more derivatives you keep on adding as we have already discussed.

so the concept of writing the Lagrangian comes out as we have derived today by taking to perturbations and making sure that the constraint remains active due to perturbations then you got a condition that looked like this of course Lagrangian included means that what we had was $\delta J / \delta y + \lambda$ times $\delta K / \delta y$ these two things being what we get when you write the Euler Lagrange equation so this will be this part will be $\partial j / \partial y - \partial J / \partial y''$ equal to 0 likewise this portion going to be $\partial K / \partial y - \partial K / \partial y''$ well not equal to 0 right suggest the expression yeah.

So that is what we have and what we have here going to take variation that is what it means when you take Lagrangian this L has J+ λK that is what we will have ΛK coming from this and total this is what we get okay this is how we do when you have a constraint let us take a simple problem from mechanics geometry or mechanics.

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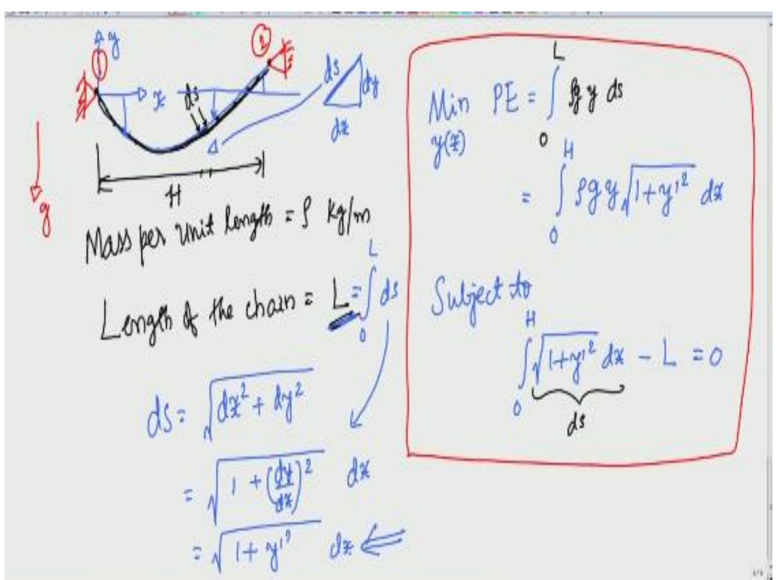

Which is the famous problem again if I have two points and there is gravity this time I am NOT going to talk about brackish different problem there are two points and that is gravity we take a problem of a chain or a rope which has some mass okay when you are going to hang it between two points this let us say it is fixed over there and fixed over here okay there is there are two point seven point one coordinates are given point two coordinates are given what shape does a chain take under the effect of gravity this is the constrained minimization problem because we know principle of minimum potential energy.

So this chain let us say that the chain has mass per unit length mass per unit length equal to let us say a rope some kilogram per meter right this is some kg per meter I have this ρ that is given to us and the chain has a length also length of the chain length of the chain or a ρ heavy ρ let us say this is length is l okay and let us say the separation between these two is some let us say H okay horizontal separation is H vertical separation will be there you know one and two points coordinates are given to us.

How do we pose a problem because it change all these little points on the change every one of them wants to minimize its potential energy as a consequence everything wants to go down as much as they can right so but then they cannot all go because it is held fixed or the two points because they are all this is in extensible chain we have to consider change usually when you take you cannot pull it you cannot apply tension and then try to elongate it.

Because you are considered inextensible ρ here then what shape does it take, we pose the problem using minimum potential energy so I will write minimum potential energy let us indicate with PE itself potential energy under the gravity we take some reference where we measure this let us say I take this reference right here I put a coordinate system this is my X and this is my Y the shape of that chain is my function y(x) right.

Taking from reference coordinate system that we have taken now petitioner G wherever the beam you know from here we measure this how much ever this will have some Y value here at certain x and as I extend it there will be some y there y there and so forth a different values of x where x goes from super energy we can write from because I have take warrant system here 0 to an on horizontal separation h 0 to H.

If I take a little piece of the chain here okay with the DX there are let us call this D s a little piece over there okay let us say that is ds what is its mass it is going to be P ds how much is it come down at that point it has come down xy and because there is gravity we also should put P G because that is the force gravitational force ρ gy ds will be the petitioner g due to that little element d s that we have shown there.

And this d s we have to integrate over the entire thing right now it is not let us say zero to H we have to do it over the entire length of the chain that is given to us that is 0 to L because L is the length of the chain we have to go along from here all the way there by taking small pieces of d/s everywhere okay, that is the objective function that we have this we can rewrite okay because our X is not there yet.

But I can write because the ds that we have we can write it as $\sqrt{dx^2 + dy^2}$ that is basically Pythagoras theorem right I said this is d X let me write it here little DX that the ds that is DX and dy okay so if I enlarge it this is ds DX and dy this is ds, this is DX this is dy how do you get ds Pythagoras gives us this I can rewrite this as if I take DX outside okay DX square if I take outside becomes DX and when I do that this will become 1 when I want to take outside I have 2 divided by DX here that becomes dy/DX square are in our notation.

This will be $1 + y'^2$ dx okay so I can write this as now d s is turned into DX so I can say zero to H because that is our limit for X as we have taken that will become P G Y x into $\sqrt{1+ y'^2}$ DX because that is what we have for D s over here right now this is subject to a constraint right if there is no constraint you will make this Y as negative as possible to minimize the potential energy that is all the chain links are not attached to one another if all of them want to minimize potential is it together they will all fall down to minus infinity.

But we are not allowing it because links are connected there is a constraint on the length of the chain length of the chain how do you take this will be integral ds right so we have this little length you take and you do it from 0 to L again for Ds we have this so I can write this constraint as the length of the chain 0 to h s$\sqrt{1+y'^2}$ DX again remember that this is nothing but ds okay so this 1 minus length of the chain that is given to us should be equal to 0.

Now if you look at this problem this is an example of a calculation problem where there is a functional type of constraint okay we had that already in the problem statement of a bar optimization but we first wanted to understand how to do these things if I have a problem like this.

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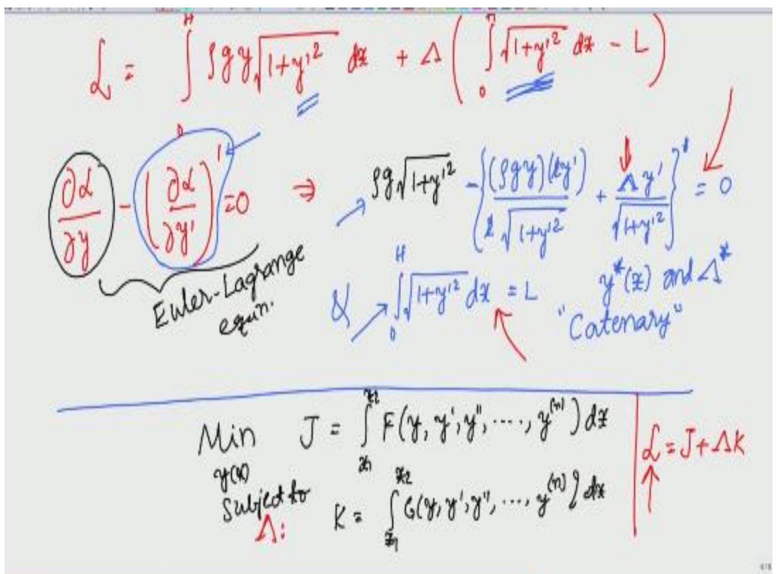

Then I can write for this one I can write the Lagrangian right so let us remember our objective function which is in this form the constraint which is in this form I can write the Lagrangian for this so Lagrangian is the objective function 0 to H okay which is P G y 2 $\sqrt{1+y'^2}$ DX plus where to put it Lagrange multiplier set we have to make it a habit to put that functional type constraint uppercase Greek letter a scalar unknown ρ times the constraint which is 0 to H $\sqrt{1+y'^2}$ DX minus length of the chain okay.

This is our unconstrained problem now because we wrote a Lagrangian for this we are at a Lagrange equation for the Lagrangian itself right that means that I have to do $\partial l / \partial y$ minus $\partial l / \partial x$ prime equal to 0 this will be the necessary condition for the constraint problem where there is a functional type of constraint in this case that functional is an integral Euler Lagrange equation with the Lagrangian okay.

So we can do this now so we have $\partial l / \partial y$ we have Y here and now here l see so we can write it as P G 2 $\sqrt{1+ y'^2}$ that is $\partial l / \partial y$ okay so let me insert this in black so you can relate to the black color now let us write this part in blue color okay- we have to write $\partial l / \partial y$ prime it is over here and it is also over here more terms will come in so if you do this so y is now kept as it is because you are doing partial derivative respect to Y prime so this will have P G Y derivative of this will take it downstairs $\sqrt{1+ y'^2}$ there is a half that comes.

And then y prime square derivative will be to let me write it half because this is you know square root half will come and now this will become 2 y prime okay and this 2, 2 gets cancelled that is for this portion we will also have plus ρ times this part right there is a y prime there that will be again to will get cancelled this will be 1 plus y prime square and then y prime 2 y prime 2 gets cancelled to get this whole thing we are to take derivative like this one okay.

That is equal to 0 this will be differential equation for the chain so to simplify it now you have $\sqrt{1+ y'^2}$ it goes but then there is a prime right we have to do this basically we have a differential equation with which we can solve for the change shape in fact when you solve this you still do not know this ρ right how do you find the ρ for that we have the constraint so this and our constraint which says that from 0 to H $\sqrt{1+ y'^2}$with a square root DX minus L equal to zero or that is equal to L.

So this differential equation and this constraint have to be solved to find our solution Y star X and our solution value of ρ star if you find that you would have solved the problem and that happens to be what is known as a cat in every say famous geometry or mechanics problem it is both geometry mechanics problem it is called a catenary a chain takes the shape of a catenary in fact many bridges that are built suspension bridges will also take this bridge if you take this shape called catenary okay.

So what we have done now is we have taken a problem where we are minimizing punish energy subject to a functional type constraint there is a integrand for this integrand for this both are defining to functional and we have solved the problem so by using this concept of Lagrangian so just before we finish let us say what we have done today we have done a general problem where you have minimized there is a functional which will have x1 to x2 an integrand that depends on a function infact it can be any number of derivative.

Even though whatever we have discussed we did not say it but variation derivatives applicable to any number of derivatives okay we have a function like this subject to a constraint which also can depend on any number of derivatives so integrand here can be y y', y'' and y nth derivative DX if we have something like this so here we should not forget to put this ρ which is the Lagrange multiplier we write the Lagrange so Lagrangian if you say again is l is j +ρk and write a Lagrange equation for the Lagrangian directly on the boundary conditions if you write all of them you get the answer.

Answer here is arriving the differential equation and the constraint like we have a catenary problem that is the differential equation this is the constraint to solve for this ρ okay, now we know how to deal with a functional type constraint in the next lecture we will deal with a differential equation that constraint, once we do these two we can solve any problem in calculus of variations then you can do any problem in mechanics or structural design optimal structural design, thank you.