Indian Institute of Science

Variational Methods in Mechanics and Design

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NPTEL Online Certification Course

We will continue with where we left off in lecture 10 let us look at the problem that we are considering that led us to.

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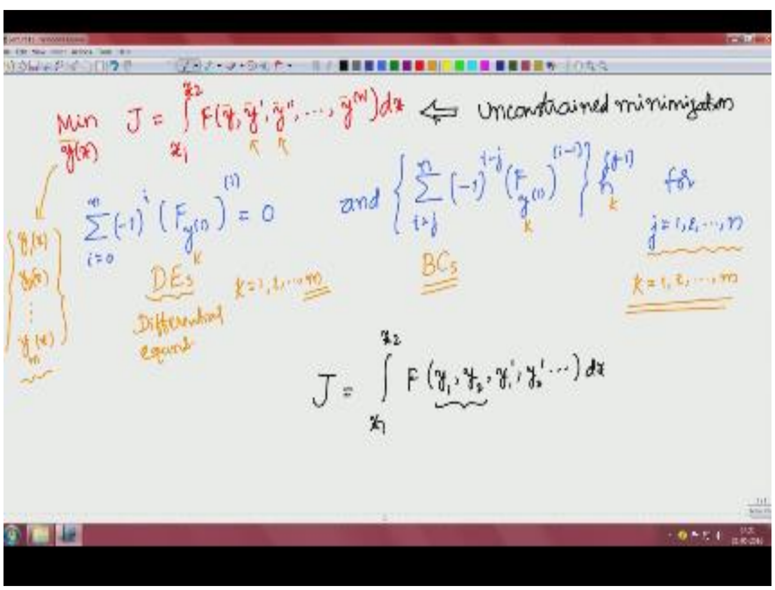

What we need to discuss today first we considered a functional that is j whose integrand depends on the unknown function y and its first derivative second derivative and up to n derivatives and we wrote the general differential equation and the boundary condition so this is the differential equation and these are the boundary conditions this is the most general form when you have only one unknown function y of x and integrand depends on n derivatives of that unknown function y f x.

We were thinking about a case where there can be more than one unknown function we were specifically looking at a problem where the integrand is a function of y1 and y2 two functions when you have that the differential equations and boundary conditions remain the same in other words if I say there are multiple functions I will put a bar on top of it to indicate that this one will be like an array where there will be y1of X Y 2 of X and so forth let us say YM of X so there are M unknown function show do this equations change.

So then we will say that each one of them will be bar over that that means that there will be M functions y 1 y 2 up to y m then all we need to do a change here is to add a subscript k there and say k goes from 1to up to m likewise we have to put K there and this also HK there and it will be K 12up to m so nothing else changes so when you have multiple functions that are unknown you will get as many differential equations as the function so differential equations now will be more so these are differential equations differential equations.

We have m differential equations now corresponding there will be M sets of boundary conditions I say sets because the boundary conditions themselves will be n if you have n equal to 1 meaning that you have only up to y prime there will have only one boundary condition if it is y ∂ble prime will have two of them j equal to one and two because now n that is number of derivatives that integrand depends on there will be that many boundary conditions so when you have m functions all that changes is that there will be M sets of differential equations and then there will be m, m differential equations m sets of boundary condition.

Sets apply two boundary conditions because there will be as many as the number of derivatives that are there okay so this is what we were discussing and what we try to consider was a functional where the integrand was supposed to contain two functions y 1 y 2 and then y 1 prime y 2 prime and, and so forth in a number of derivatives DX okay so in this context we should note that when you are dealing with two functions you can still have an unconstrained minimization problem meaning that whatever we have considered this type of problem this is unconstrained right.

So there is there are no constraints here unconstrained minimization right but then when there are two functions we can have a constraint also between them that is the example that we were looking at but you can also have with two functions unconstrained minimization also let us briefly look at that before going back to the example or formulation that we were looking at let us find an example of a problem where there are two different functions y1 andy2 and there are no constraints so such a problem.

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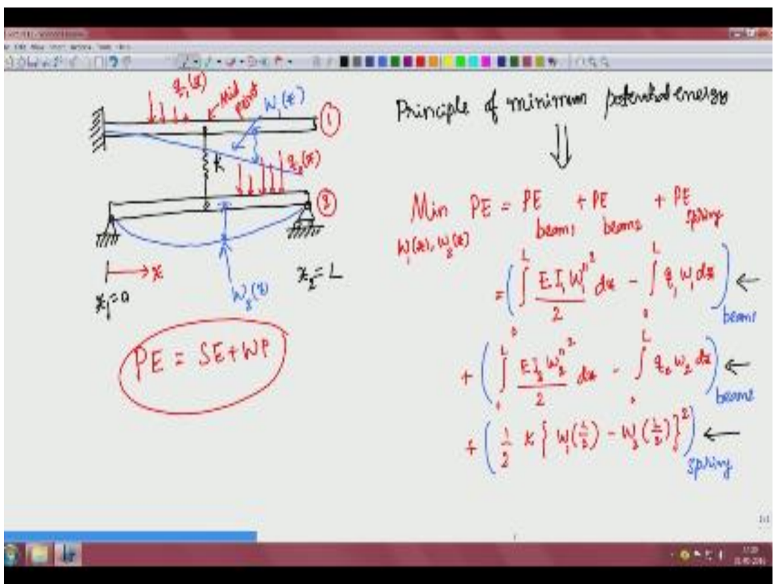

Let us look at it physically so if I have let us say a beam which is fixed at one and free at the other end let us consider another beam okay let us say this one may be supported with a pin joint over there and let us say pin joint with a slider or the other end okay there are two beams here and we can connect these two with respect so a spring of spring constant K so now we have here normal x 1 is 0 and then x 2is L so this is 0 and let us say this x2 is L that is lengths of both the beams are the same l so our domain for the problem is x 1 equal to 0 to x 2 equal to L.

And what we want to find out is how do these beams deform if there is some loading let us say on this beam there is some kind of a loading let us call that Q 1 of X let us say there is some kind of a loading q 2 of X on this second beam both of them X goes from here so this X goes from 0 to L and we need to get how does this deform let us say that deforms like that and this might deform like this let us say now let us call this function not y normally four beams we denote it with W let us say this is w 1 of X is its deformation.

And this deformation is w 2 of X meaning at any point from it is neutral plane to the default profile that is w 1 X and this one is w 2 X okay now if we use the principle of minimum potential energy for this problem so principle of minimum potential energy okay that means that we would like to minimize this so I will write minimize potential energy for this beam okay then we have to minimize its with 2things that is the top beam whose deformation is w 1 x and bottom beam who is the VW 2x both of these becomes our unknowns our variables.

So w 1 of X W2 of X 2 functions that is what we are looking at right so we are looking at a case where there will be two functions okay y 1 y 2 now that became w1 w2 so that we can write from 0 to l yeah so we have potential energy of the two beams and the spring we have to include so I would say potential energy of beam one that is this is beam one and then we have beam to we have to write potential energy of beam to and potential energy of the spring that we have there again we call that potential energy is strain energy plus work potential keeping it in mind.

We can write this potential energy here as for beam 1 it is 0 to L if I am assume X modulus is e and second moment of area is I then W ∂ Prime this is a first beam so W 1 ∂ prime square divided by 2 that will be the strain energy for beam 1 so that is strain energy and the work potential is a negative or the work done that will be 0to l q 1 is the loading acting on it and w 1 is the displacement this will be for beam one and then we will have 0 to L e I.

In fact we can call it I1 and I 2 will take the same material but moment of area could be a different function this will be w 2 prime square by 2 DX minus 0 to l q 2 w to TX okay so this is for beam one so this one is for beam one potential energy and this one discipline energy for beam two and then we have the potential energy our standard work potential spring does not have a force so it should be strain energy that is going to be half if we say this is a mid point where the spring is attached okay.

Let us assume that this is the midpoint that is L by 2 so the extension of the spring so k times extension of the spring square so that will be w1 at L by 2 okay minus anyway we are going to square which are we right does not matter w 2 L by 2 square okay so this will be for there is no work potential for the beam so this is for the spring if either string so there is spring so if you look at this potential energy now if you look at the terms that we have that is this first term second term and third term they depend on two functions w1 w2.

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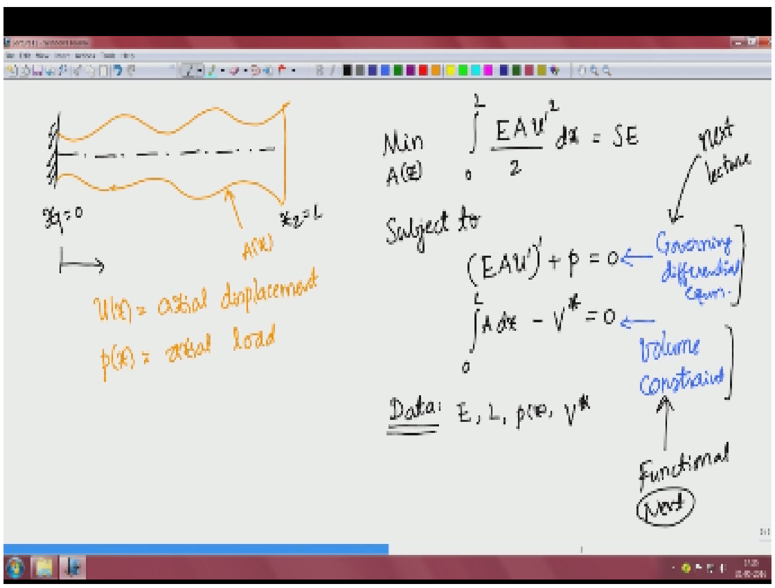

So that is what we were after so you can have problems involving two functions without constraints in mechanic's geometry one can think of problems of that kind so now moving ahead we go back to the problem that we are considering where we took a bar optimization problem where we had let us say a bar that is fixed and this is the neutral axis or axis of the bar and we want to optimize the area profile of the bar area profile of the bar which we said a of X that is one variable for us and then there is deformation axial deformation.

So U of X is axial displacement or deformation displacement is more correct because points will be moving so axial displacement function for some given force P of X this is the axial load or force applied here such a thing if you want to minimize the volume of this par or minimize the let us say strain energy in this that is a problem we were considering the last lecture so again if I go from zero to L it is X 1is 0 x 2 is L and X goes like this to define all these functions a of X u of X and P of X.

When you have that so we want to minimize strain energy which will be into a into u prime square by 2 DX a of X is our design variable here we want to come up with the optimal area of cross section to minimize strain energy so this one is strain energy of the power under the given load P of X now u prime we said is a state variable a state variable should be gone by the governing equation which we had derived in the last lecture so subject to we have this e a u prime crime that came from I Lagrange equation and p is equal to 0.

And we also had a volume constraint so we had a constraint that 0 to L a DX minus v star is equal to zero we can also write it as less than or equal to as if there is an upper bound on volume meaning you cannot use more material than v star so if you look at a problem like this, this has two constraints so there is a constraint here which is volume constraint and here is a constraint which is the governing differential equation governing differential equation okay there are two constraints and in all these problems we should very clearly right.

What is data for this problem what is assumed here X modulus is assumed length is assumed and the loading p of x is assumed and v star is assumed we have to write this data in such a way that once you say what is known or assumed the rest of them become variables so if you look if you scan through the problem k of x that is design variable and again appears here and rest of it is known right and then you of course we do not know because that is not in the data.

So a and you are two functions that we do not know that is what we are looking at when you have two functions they can be governed by the constraints there can be cases whether strains which we already considered in this lecture how to pose a problem that has two functions but no constraints now we are look at a problem where there are two constraints when there are two functions these constraints one of them is a differential equation other is just a an integral or functional.

This is the functional depower constraint is integral over constraints these are the two things that we need to discuss now when you have an objective which is a functional and then a constraint which is a functional so this is a functional constraint that is what we will discuss what happens if it is there and then we will discuss what happens if it is in the form of a differential equation that will discuss in the next lecture this is for the next lecture and this is what we will discuss next right now okay this is a structural optimization problem in its simplest form okay.

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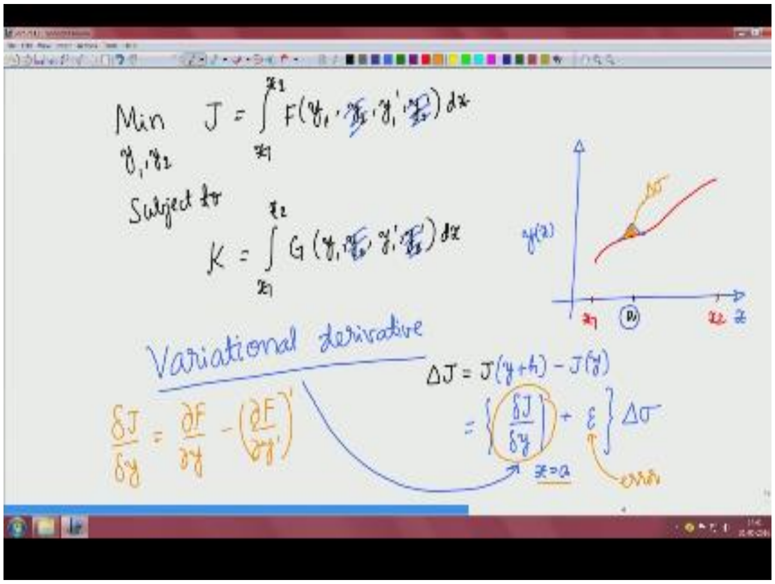

So let us write out this problem in a general way let us say minimize a functional J in our usual way will not c 0 to L will say in a general form X 1 to X 2 where this F depends on let us say y y1 and y2 let us say and its derivatives also to begin with you can say y 1 prime Y 2 prime and so forth DX and there will be two functions y 1 x2 subject to okay we have a constraint of the form of a functional, functional constraint k say zero not 0 to L we will say x1 to x2 x1 to x2 integrand g that can also depend on both functions and its derivatives okay.

If you have a problem like this how do we solve right so in these cases we have to recall what we had considered earlier which is called the concept of variation derivative okay so for concept of variation derivative let us write down what that means this is something that we had discussed I think it was in lecture eight variational derivative okay what that meant was if I have a functional like J or K does not matter.

Which one if you want to let us say I have a function x and y of x it can be y1 y2 let us say one y of X let us say that function looks like this in an interval X 1 to X 2 now if we make a small perturbation at some point that it is called this point A if I make a small perturbation that point that is there I pull it up here right what that means is that there will be a little change in the function the function becomes like this so because of this changing at a particular value A in the function y what will happen to j that is change in this.

So if I say j in fact we do not need to take two functions right now so we can take only one function let us say one function I have taken so we can knock off this for now for our discussion for our discussion this does not it is not needed so if I take this change because I have made some change in y what happened to change in the functional so what we do is we have this y plus we have this h at j with y and then this little age which we call perturbation unlike the previous perturbation of variation we called which is effective everywhere in the domain.

Now it is effective only at particular value A if you have that and you subtract out the original one without that thing this we had said can be written as the variational derivative Δ j x Δ y that is how we write this variational derivative which is evaluated for this X at a particular value right so it means that if I were to change the function y of X at a particular value in the domain let us say A what is the change in this Δ J is what we are writing.

This plus let us say some small error times Δ Σ is what we had written right so this Δ Σ is this area so this little area that we have just imagined that this red curve is like a rubber band at X equal to a you have pulled it up right then there will be a little extra area that is Δ Σ that is Δ Σ this is some kind of an error because we put an equal to sign here error term because the first variation is like a first order term that first order term we wrote it in this form then this becomes a variation derivative this will be an expression X equal to A.

They will be some value to it x equal to B let us another point I take there will be different value as I move along the domain from X 1 to X 2 this expression Δ j x Δ y will keep changing what is this Δ j x Δ y this is nothing but as we had discussed in lecture 8 nothing but a Lagrange equation in the Lagrange equation there will be something equal to zero that something is an expression true for every value in the domain that is what it is so if you have only the function y and y prime a Lagrange equation will be ∂ f by ∂ y minus ∂ f by ∂ y prime.

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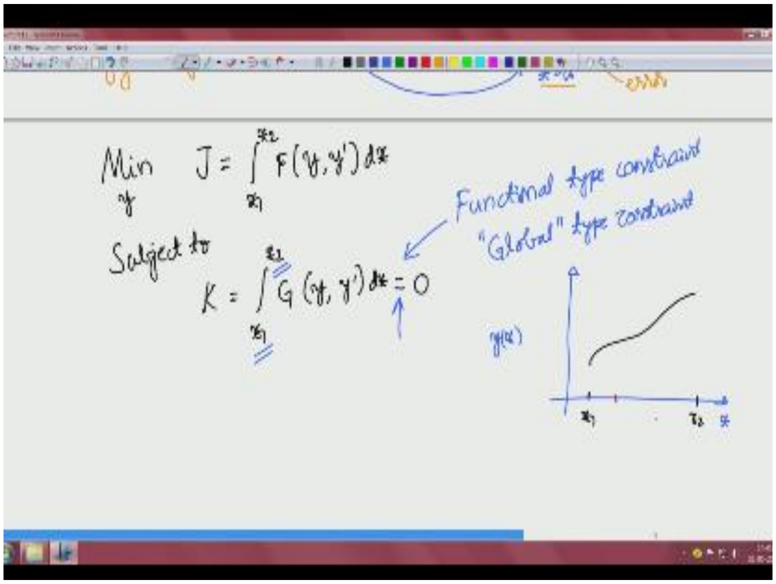

That is what is this variational derivative that is what we will use now in order to do this problem so let us rewrite this problem in a way that is convenient for us with only one function because where there is a functional constraint we can optimize the problem even with a single function meaning I can have an integral which is x 1 to x 2 where my integrand is dependent on why only one function y prime they give me more derivatives also for now we will write y will be there and then I have this subject to AK which is also from x1 to x2 integrand is J G.

And then YY prime DX minus well that is equal to 0 we can take let us see what we wrote earlier so actually we did not write anything is a constraint means that we have to say this is equal to 0okay likewise here we will write this is equal to 0 constraint violation of an equation or in equation or inequality let us say we take equal to 0 if you have a thing like that we want to find out what are the necessary condition for this if it is only unconstrained we know what it is we have to read Oleg range equation.

How do you do with the problem where there is a any quality constraint so this is n equality constraint of the functional type constraint this is the functional type and strain it is also called global constraint global type constraint it is global because this applies to the entire domain 2x2 not just a point which will look at a set in the next lecture because in our bar optimization problem let us go back and look at it when we wrote the bar up tension problem we have this which is a global constraint whereas this is a governing differential equation.

Which should be true for all values of x in the domain X 1 to X 2 that is a local constraint low it should be valid whereas a volume constraint such as the one we have here is a global constraint of Burton's the entire domain from 0 to L okay so now let us look at a global type constraint when you have that whenever we have to say we are minimizing J an objective function here to find a certain y star of X that is our unknown function how do we know it is a minimum we have to check around that function in the function space to see if it is a minimum.

Now how do you check whenever you perturb you were constrained which should be 0 here equality when a part of why a little bit this integral x 1 to x 2 G might not be0 with your perturbation in fact usually it will not be if you put a bone at one point right then you cannot say that J for that why perturbed y is less than this and hence y star is not a local minimum right because it is not a feasible point meaning that whenever constraint is there and you do not satisfied in a perturbation we are not doing it right.

So what we will do now is not have 1 perturbation but we will have 2 perturbations so if I have let us say our function y of X here okay let us say the function is something like this at x1 and x2 we choose two points let us say there is an A and there is a B two points okay.

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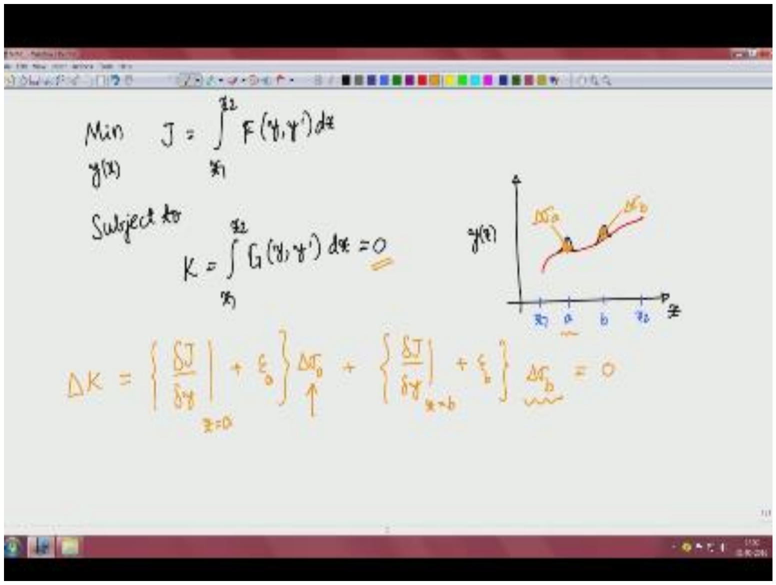

Okay now we have two points A and B we take two points because even after perturbation let us say we write this change in ke it is Δ K there are two perturbation so where we have to write this variation derivative Δ jx Δ y this is the perturbation done at A plus we had let us say error in that epsilon a times Δ Σ a that is Δ Σ is this one so this is Δ Σ a plus we have a second perturbation of the duration derivative defined at x equal to p plus error there epsilon P times Δ Σ P.

And this area is Δ Σ be the total thing should be equal to 0 right we need to do to perturbations so that the constraint remains equal to 0 that is it is satisfied when we do this just like we had an infinite variable optimization will be able to express one perturbation in terms of the other that is what we will do so we will say Δ Σ be here can be expressed in terms of Δ Σ a then we will take the similar one for J and try to come up with the concept Lagrange multiplier for this we will take it up in the next half.