Indian Institute of Science

Variational Methods in Mechanics and Design

Prof. G.K. Ananthasuresh
Department of Mechanical Engineering
Indian Institute of Science, Bangalore

NPTEL Online Certification Course

Hello just continuing what we were discussing the general form of Lagrange equations for an integrand containing y and its derivatives up to n then we wrote the a Lagrange equation the boundary conditions.

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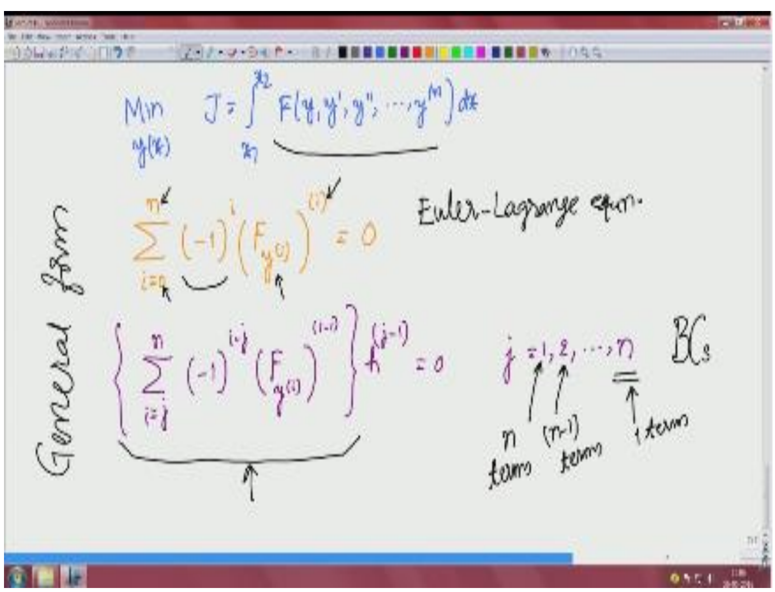

Let us look at that, so we have now an integrand that depends on y, y′, y′′ up to ynth derivative then we got this equation Euler Lagrange equation that is a single equation one equation will be there and you see how we have alternating sign coming because i goes from 0 to n in steps of 1, 0 1 2 3 up to n and the sign will be alternating i=0 it will be a positive sign i =1 negative sign i=2 positive sign and so forth and then if you look at this, this is how many times you take the derivative and this is with respect to what you take the derivative of the integrand yf.yi that means that i=0 that will be fy itself y 0 is y itself.

y1 is like y′ y2 is y ″and so forth and same number of times you have to take the derivative that so we get the a Lagrange equation by summing them all up and here we have the boundary conditions. How many boundary conditions are there, there are n boundary conditions because you have to do j=1 to n so if you have up to n derivatives you will have n boundary conditions if you have n=2 then we will have two boundary conditions and so forth.

So if you look at this, this is a little intricate because we are doing i =j to n summation okay, and if j=1 they will do i=1 to n right when j=2 2 to n and so forth the number of terms will be n when you take j=1 that is number of terms here will be n terms when it is too will have 2 to n so n-1 and so forth here you will haven terms in the boundary condition and here it will be j-1, j=1 means it will be just h okay, and then when this is there we will have n-1terms okay, finally when you have j=n you will go from i=n to n itself will have only one term right. Accordingly if you go to j=n this become h n -1that is you have to take h derivative up to n-1okay, and that is how all these terms will come.

If you do it for one and two that we checked just till while ago whether three we suggest that you work it out so that you can verify that this is valid, okay. But in engineering normally you will not have things beyond second derivative or even physics for that matter unless you go to very complex problems you will not have more than second derivative so understandable second ability is important but you can also go ahead and understand up to in derivatives what happens.

Once again this is our most general form, this is the general form of a character serration problem involving only one function y but n derivatives okay.

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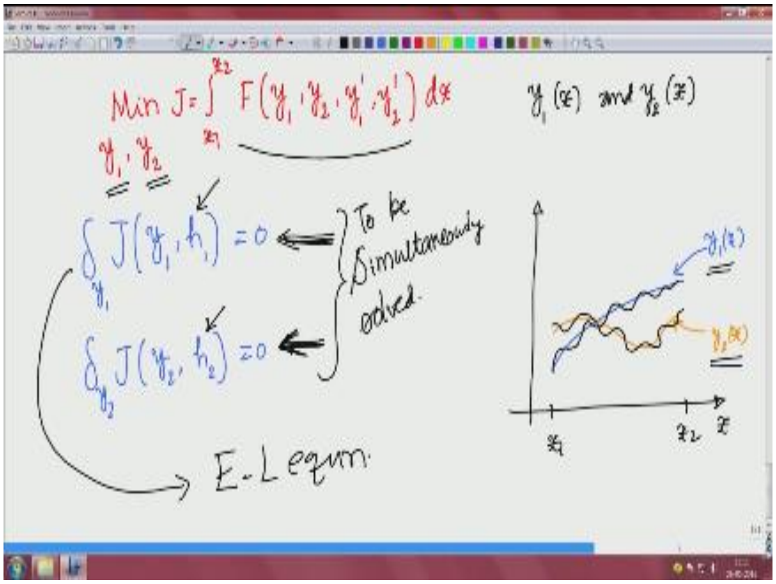

Now let us go to a case where we have our functional which goes from x1 to x2 whereas this integrand now depends on two functions let us say y1 and y2 so both of which we have to make them as variables there could be problems where which we will see definitely instructural optimization where there could be multiple function that you need to minimize something with respect to then we have y1 y2 and there could be first derivative of y1 there could be second derivative of y1 will stop there for now, okay.

So we still have one independent variable only x that means that our unknowns are y1(x) and y2(x) two unknowns okay, and both of them have up to first derivative only better can also be n derivatives when you have two functions what happens right, now when we take variation okay, we have the gateaux variation so normally we write y1,h and that right, y1,h1=0 sometimes we also write y1 over here to indicate that we are taking variation with respect to y1, we have to do that and we also have to do with respect to y2.

So variation of that y2 and its variation h2=0 in other words if i have this situation only one independent variable X now I have to find out from the same limits x1to x2 I have to say that there is this function let us say that is y1(x) and let us say there is another function this one let us say y2(x) and both of these together define our integrand or our functional right, in that case we have to take variation of both of them that is I have to take a variation that is small perturbation of y1 which we indicate with h1, okay.

And this also needs a small variation okay, that is our h2 both of my head would simultaneously do but then we get two conditions because there are two functions if there are two functions that are unknown we need to differential equations to solve for them, okay. So you need two differential equation that is what we do so we take one function keep the other one constant and take the variation that is you take perturbation only in y1 first then you arrive at this differential equation then you perturb respect to y2 then you get this differential equation both of these differential equations we have to solve simultaneously.

Because just because variation taken respect to y1 does not mean that this equation will have only y1 it will have y2 also similarly this equation it does not mean that it is going to have only y2 it is going to have y1 and y2 both of these differential equations have to be simultaneously solved, simultaneously or to be simultaneously solved okay, we get two differential equations both are simultaneously solved, okay. So it is possible to come up with some problems example problems, where you will have two functions we will consider one in geometry as an extension of the bracket upon problem okay.

So again when you say cultivation is 0 once we have this we know how to write Euler Lagrange equations from it but that will be same as what we did so far when you have only that you do for y1 and you do the same thing for y2 as well okay.

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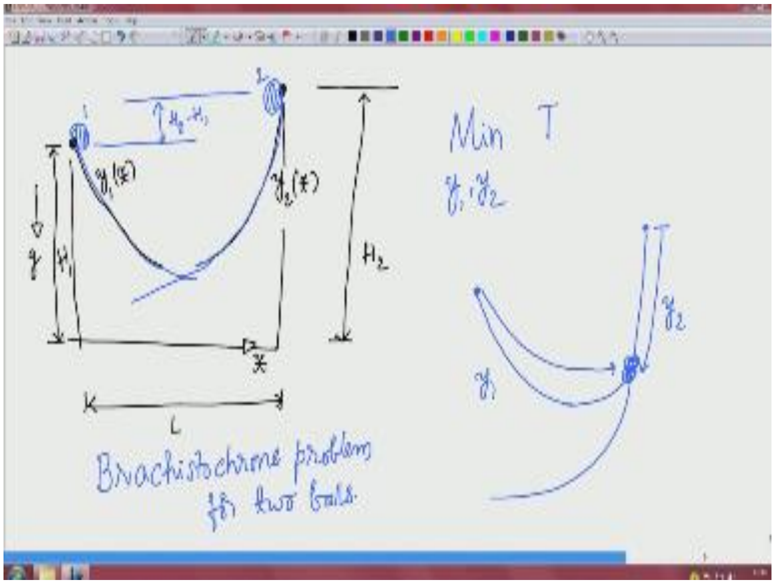

So let us look at such a problem let us say I have a point here and another point over there two points in a plane so I have a point there the point here, now there is let us say gravity because a brackish token problem right, we want to find a function for this let us say this is x this direction horizontal and direction is x vertical direction is y where we have the thing we want to find this function let us say y1(x) somewhere we do not know where okay and we have another function.

Let us say this is y2(x) we have to find this y1 y2 their separation is given okay, so their separation along the horizontal axis is given let us say this thing is l okay, their vertical heights let us call this from some datum this is h2 well let me put H2 so we do not confuse that with our variation h2 and this one it says h1, so if you have this h1 and h2 and l given we want to find these curves again this is some curve y1 this is some curve y2 such that they both hit each other that is at the same time.

If I leave a ball at point A let us say there is a ball here that I leave and there is a ball here that I leave this is the ball 1 and ball 2 if I leave them at the same time due to gravity they are going to come down we restrict them to come down along these curves y1 and y2 at some point these two will meet okay, some point we do not know where they will they will meet at that point we want it to be the minimum time it is a bracket token problem with a little twist earlier problem we had only one curve one ball and it had to reach a certain height.

But now actually this h1 h2 are just given for you and their difference is more important so between these two now that will be h2-h1 and this L when they both are left simultaneously when do they come and that time has to be the least okay. If we consider this problem this is Brachistochrm is h token problem for two balls Brachistochrm problem for two balls one and ball one and ball two where do they meet we do not know in fact we won't be able to solve this problem based on what we have learned so far we will come back to it.

But I want to give an example of a problem where there are two functions x 1 and y2 the time taken for this type of this to be the same because when the hit this ball let us say would have come on the way I have drawn the thing this comes here this comes here they are going to hit each other there but if you think about it both our bracket second problems that is for ball one and ball two both are not going to follow psychlloyds right the psychlloyds the answer for this problem where they hit is what we need to find okay.

But the time taken by the balls before the heat needs to be minimized so if I can say minimize that time taken okay with respect to two functions y1 y2 okay y1 andy2 are different let us say I take this point at this point I take y1 let us say properly like a cycloid let us say y 2 I take a curve that goes something like this when will they hit if we are lucky by the time this cycloid let us say it goes here okay.

If that time taken for this ball to come from there to here and for this ball to take time to go to their then they will meet over there are two balls otherwise they actually go somewhere else okay, is a tricky problem but the fact of the matter is that time taken that you want to find has to be minimum they both have to meet at a point and that is where this y1 and y2or two functions that are there okay.

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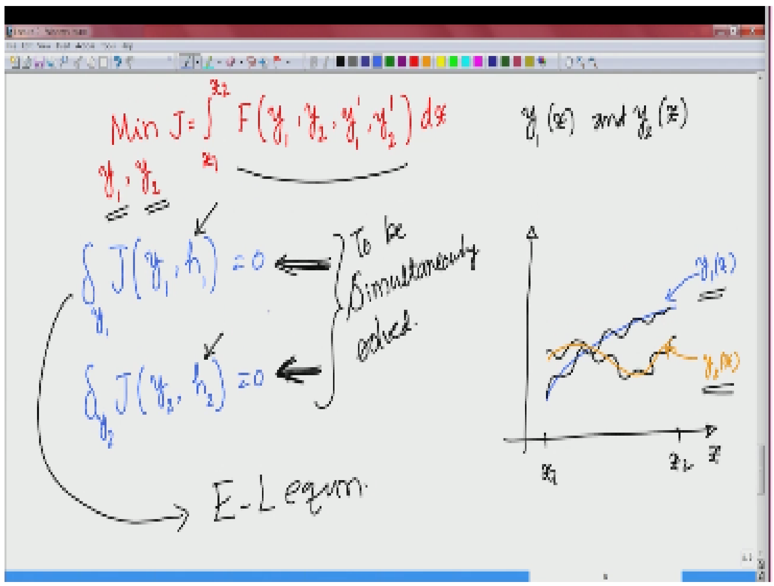

When you have such a thing we will get to differential equations that are shown here from the got a variation and they have to be 0.

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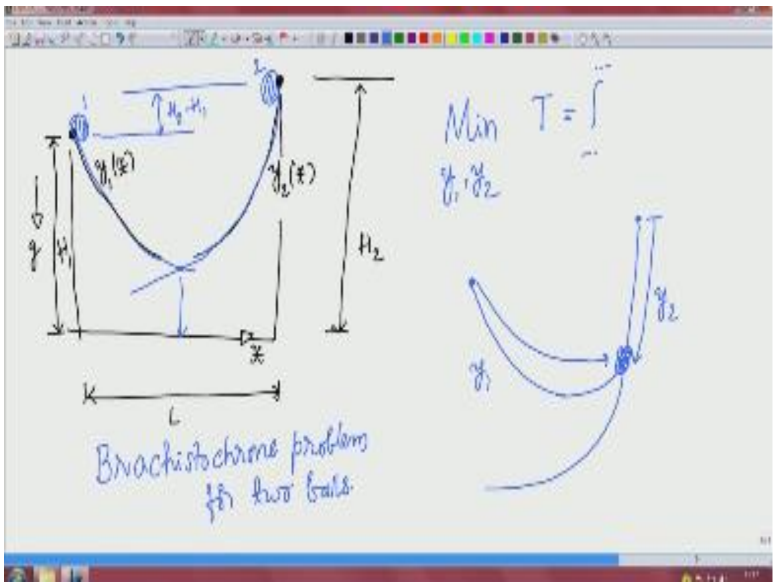

But we can fire we can pose for this problem we cannot do right now because we have n known here where the meet we do not know we know X 1 for this problem why one but not intimated point for a second one we know X 2 where you do not know the starting point right that is why it is difficult to write as you know normal way that we write something to something is not possible so later on we will come back to this problem where we have variable end conditions this x1 x2 that we have they themselves are not fixed okay but for now say that we have two variables and such things are there in structural optimization.

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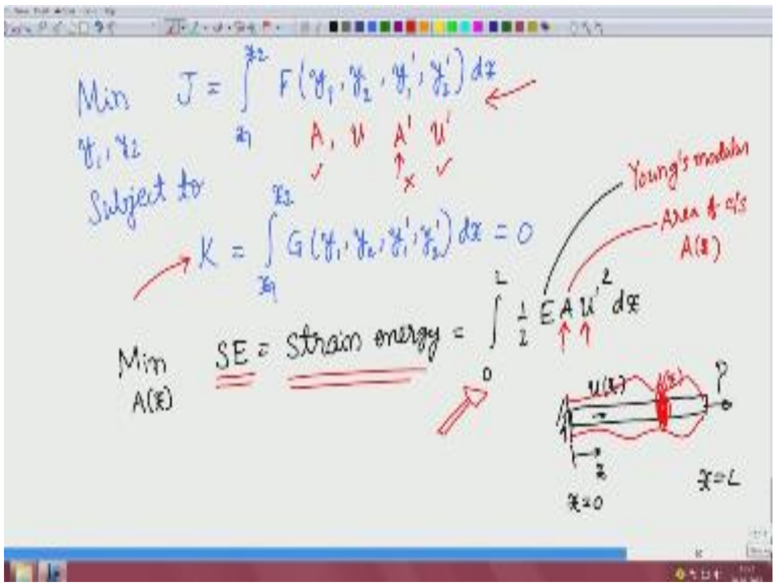

Where we can take a small problem but only thing is that that will also involve a constraint which you have not discussed but let us take this as a motivating example to talk about constraints so far we said minimize one functional J but we can also consider problems such as this let us write that minimize J in terms of one independent variable x1 x2 fy11 function another function and then let us say first derivative of this second derivative of that first derivative of the second function sorry that can be there and two functions y1 and y2.

And let us say that there is also a constraint okay subject to which we are not considered to be considered so far we can have a constraint let us say G which is also another functional okay, so let us say that is G so I will use some okay let us call this k and call this g let us say k where this g it can also bey1 y2 y1'y 2' DX let us say they should be equal to 0 okay, if you have such a problem then we have two unknown functions y 1 y 2 we can find it by satisfying this constraint that is given as well as to minimize the objective function.

Let us write such a problem where I would say minimize here we have let us say strain energy se2 take a physical problem related to structural optimization strain energy of a one-dimensional system which will describe not a beam is also one dimensional system but we will take a bar 0 to L half e ay u'^2 DX okay so here the variable that we have such as pensions a of X which is the area of cross section of a bar so what we are considering here is a bar normally constant cross section bar let us say if it is fixed here and a force is applied a p you know how to calculate its deflection that is if is our X X = 0 here X = L here.

And there will be at every point some deflection which you denote as u of x and e is Hanks modulus of the material and this a is the area of cross section and that needs to be found so this is not known this a of X what is the FX at some point I can say that is the area of cross section at the location X I have shown it to be uniform cross section but when you're optimizing it need not be it can have some shape like that okay and that is area of cross section that we need to find at a at a point that is your unknown okay.

Now we have now any an integrand in this case strain energy half EA u'^2 a we do not know u of x whose first derivative is involved here that also do not know that is like this one we have two functions which is why one I have a instead of Y to have you okay I do not have a’ but actually I do not have you here so far but I do have u prime I do not have a’ this is not there in this problem but this is there this is there this will also come once we write the governing equation for this what we want to do here in structure optimization we want to minimize the strain energy forgiven volume of material that will become this constraint.

And there will be a governing equation there will be another constraint that we will get okay let us formulate that problem first actually explaining how we get this particular expression for strain energy okay, so let us talk about that first strain energy.

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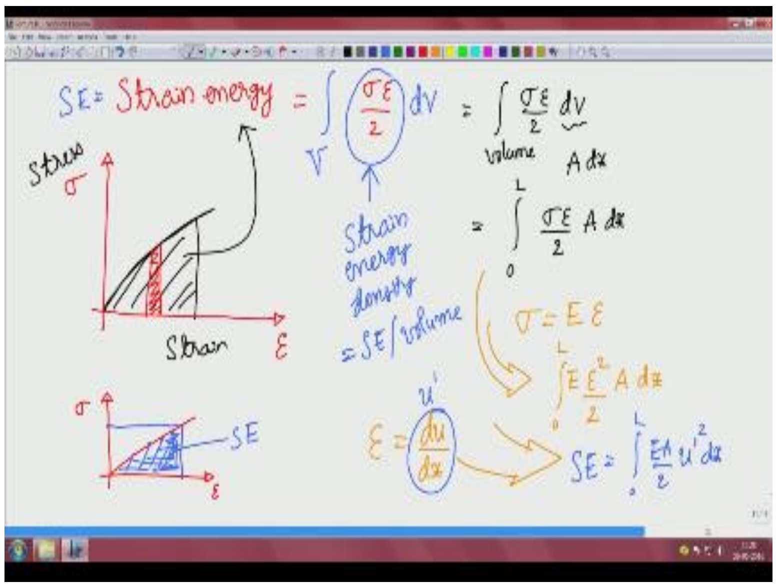

Of an elastic system strain energy strain energy of another system is nothing but if I plot stress and strain okay so this is stress this is strain will be some curve the area under this curve is our strain energy have deliberated on a nonlinear one you can have linear also but whatever it is that is a strain energy so if you want to write it mathematically you would say you have to integrate right so if you take a particular place here that is the strain energy for that portion okay that I can take as a small thing that I accumulate okay.

If it is linear okay instead of taking non linear that I have taken if I take linear that strain energy is going to be simply $\sigma\epsilon/2$ right instead of taking this way nonlinear I have to accumulate and do that if I take it to be linear so let us do it over there I have $\sigma\epsilon$ it is linear like that area under this anywhere is going to be whatever σ that is there at that point what our epsilon that I point half of that this is my strain energy area under the tab that this is my strain energy this is as a okay.

But this is strain energy per unit volume which we call synergy density so this thing is actually strain energy density strain energy density meaning it is s e per unit volume so you not to get the total strain energy I have to integrate it over the entire volume okay so what we have written here over the entire volume of the bar in this case we have σϵ/ 2 DV right we are taking a bar as you can see in the figure which is going from X = 0 to X = L of Aryan cross section so I can write this DV as a times DX.

If I do that then I can make it X going from 0 to L σϵ / 2 into a TX why do we do that across the area of cross section we have we assume that for a bar stress and strain or constant that is if I take a particular cross section that we have shown here on that entire cross section we have σ and ϵ the same so I can write that volume integration as a one-dimensional digression where x varies from 0 to l and then replace d with a times DX that is the small volume DV for this problem okay.

If you also assume the stress strain relationship that is σ = x modulus times ϵ then these two will lead us to 0to l epsilon square by 2 with an e into A DX okay so what we have is EA half EA epsilon square what is epsilon for a bar epsilon for bar change in length d you the original length DX that is if I take any little DX over there this is DX that is going to deflect by an amount du and that is how we write the smalls linear strain do x DX with all of these things we get the strain energy of a bar as 0 to L e a by 2 epsilon square.

This is U prime in our notation u prime square DX that is exactly what we wrote over here half EA u prime square okay that is how we get the strain energy all right now let us talk about the governing equation so whenever we say under a given loading a bar has to be made as stiff as possible meaning strain energy is minimized to the extent possible we also need to get the governing equation okay which will govern the behavior of you here we had.

If you remember we are doing a problem where there are two functions y 1 y 2where I said a and u, u right now is not restricted but u is to be restricted because u has to be the solution as a displacement of this bar under a given loading okay.

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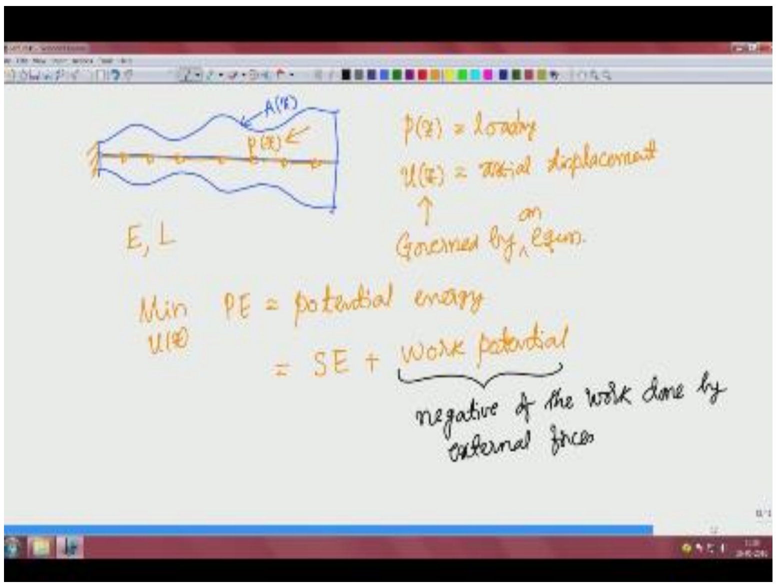

So for that let us take a bar let us get this be the axis of the bar where area of cross section is varying in some fashion which we have to find by solving a problem let us say this is the bar where this is our area of cross section let us assume that there is some loading that is given on this every point there is some loading which we say is a function of X so P of X under P of X there is also going to be a displacement so P of X is loading due to that loading will also have axial displacement view of X.

We will have axial displacement when you have excelled that is what we need to find and this axial discipline should be governed by a differential equation see if I give a bar let us say it is fixed and free bar have loading A I will give you x modulus okay and the total length of this if we have that we should be able to compute u of x so that should be governed by an equation governed by the equation which is the static equilibrium equation in order to find that we can go back to our minimization of potential energy okay.

So for that I would say minimize potential energy let us define what it is now we had minimize this potential energy which is with respect to that U of X of all possibly of X is going to take particular U of X which will mean vaporize energy that is strain energy plus work potential this is something that we must note the potential energy strategy is due to the energy stored in the bar work potential to applied force we have this P of X that is going to do some work that we contribute to potential energy and that part is defined as the negative of the work done by external forces negative of the work done by only the external forces.

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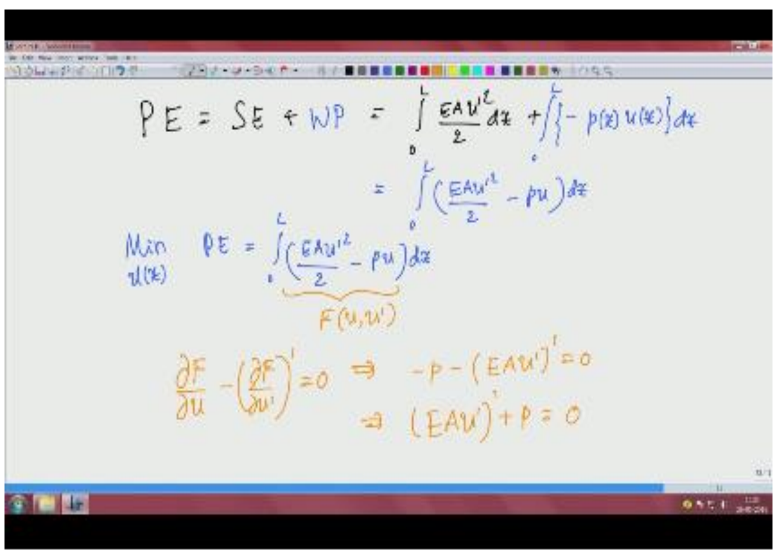

Here external force is that P external forces okay so what we have here if we write for this problem potential energy is strain energy plus work potential we just derived the strain energy let us actually right in two different colors so we can easily see which is which okay so they say strain energy we already wrote 0 to L EAU prime square by 2 DX and then work potential plus the work potential is negative of okay negative of the work directional is here the external force is P of X and that is going to move for a distance U of X everywhere that is that is going about potential together what we have potential energy is 0 to L.

So this also should be integral 0 to L DX because at every point we have P of X every point is going to move u of X right so together we can write EAU prime square by 2 minus p u DX that is our potential energy for a bar now we need to minimize this potential energy with respect to U of X which is 0 to L EAU prime square by 2 minus p u DX okay now this is an integrand that depends on u1 u prime so we can write the differential equation.

So if I call this f as a function of u and u prime then differential equation I Lagrange equation is ∂ f by ∂ u minus ∂ f by ∂ u prime, prime D by DX equal to 0 that gives us ∂ f / ∂ u is we have this PU that gives me minus P and then minus ∂ f by ∂ u prime gives me u EAU prime this 2 and square 2 will both cancel EAU prime, prime equal to 0 in other words what we got is EAU prime, prime plus P is equal to 0 that is the governing equation we get with these things.

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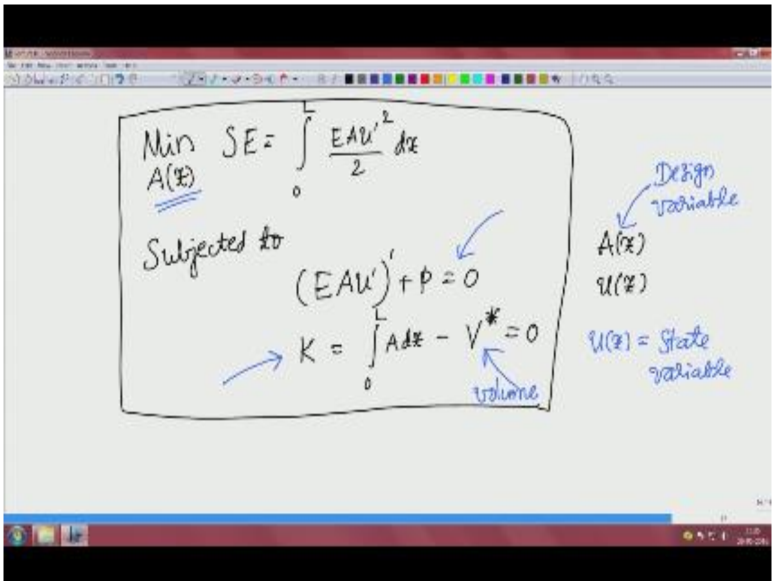

Let us pose this problem that we are after now when you have two functions here we have a first of all but the strain energy that we have taken0 to L EAU prime square by 2 DX that involves A and U and then we have to make sure that this you is are you prime is not something arbitrary it has to be governed by equation we just derived EAU prime, prime plus P is equal to 0 and then we also had this equation for the volume that K equals 0 to L A DX minus v star is equal to 0 that is the volume is specified for the bar and a DX is going to be the volume of the entire bar as you change the area of cross section.

That is minimum so now this problem if we look at this is dependent on two functions okay one is a of X and other is U of X which is not something that we are make up to Malaysian variable u here is called a state variable that is that state is the displacement whereas A of X is the design variable in structural optimization so we have now formulated a problem that involves two functions one is a design variable another state variable state variable is governed by a differential equation and there is a resource constraint which is on the volume of the bar it is a typical structuration problem.

We solve this we have to discuss what happens when there is a constraint okay which will consider in the next lecture and solve this problem and then move ahead with further generalizations of the calculus of variations problems thank you.