Indian Institute of Science

Variational Methods in Mechanics and Design

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NPTEL Online Certification Course

Hello in the last lecture we discussed how to derive the Lagrange equations when the functional involves in an integrand which depends on the function y and its first derivative as well as a second derivative the first derivative we did a small example which is to prove that in a plane if you are given two points the curve of least length going through those two points is a straight line that is an obvious result but we were able to verify that using calculus of variations framework and then we considered a functional of this form that I will write is minimizing a functional J.

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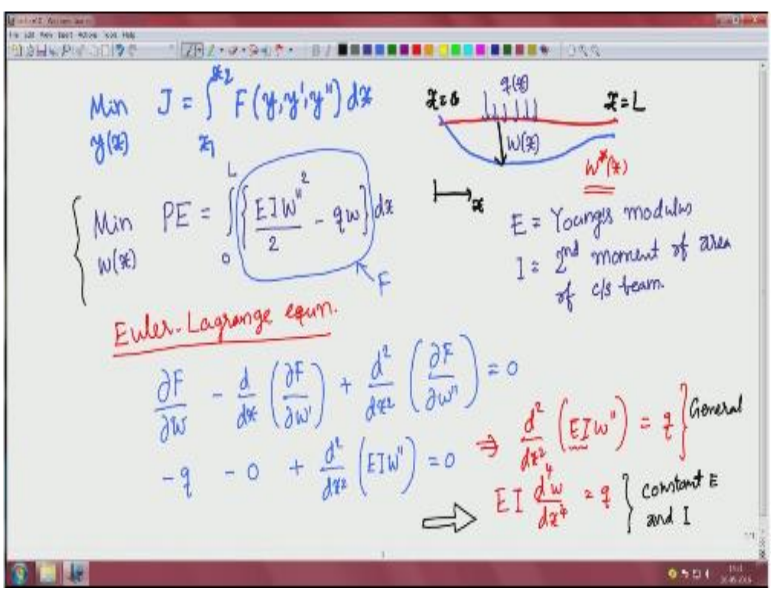

Which is now in the form of an integral x1 to x2 where the integrand depends on y y ´and y ´´ dx and this is minimized respect to y(x) and then we wrote a Euler Lagrange equations for this problem and we also considered an example that looks like this the example that we considered was a beam where the boundary conditions we do not say what it is right now when it deforms let us say it deep reforms to some form like this where this thing is w(x) that is if x goes like this let us say x equal to 0 at this end 0 and x equal to f at this end we want to find this function we want to find this function W (x) let me raise it and write it properly.

So this function W (x) okay that is unknown so they are like we have written this minimization problem now we say minimize with respect to W (x) that is the function that we do not know instead of J we have the functional which is the potential energy as you may recall that variational methods are also called energy methods so here we have potential energy PE which is defined as an integral 0 to L and the integrand we had written this EIW''^{2} /2 - qw dx okay q here is the force that acts on the beam okay.

Some force like this and on the beam which is a function of x q (x) and E is young's modulus of the material of the beam and modulus I is the second moment of area of the cross section second moment of area of the cross section of the beam okay we will discuss later where this comes about okay first let us take this problem and try to minimize it okay.

If you remember the Euler-Lagrange equation so this is recondition for this w-what when you say W want to find we actually want to find the W* (x) that deformation so this beam under some boundary conditions which will come to later under the given loading q (x) it should give us the minimum value of potential energy local minima potential energy and such a thing W* is what we want to find in other words that beams a deform knowhow to solve calculus of variations problems so here we have Euler Lagrange equation okay and that one if you recall it was $\partial f / \partial y$ in this case it will be W - d / dx of $\partial f / \partial W'$ + d^2 / dx^2 of$\partial f / \partial W''$equal to that is a differential equation.

Our F is this is our F we see that the F depends on w so ∂ f / ∂ w is going to give you - q and this F does not have dependence on W´ so that is equal to 0 + d^2/dx^2 (∂ f / ∂ W´´) which is there that 2 and the square to the cancel so we will have E I W ´´equal to 0 which gives us if I take this q the other side because minus q I will have d^2/dx^2 EIW ´´equal to q now if you assume that E and I that they do not depend on the x that is their constant cross section then we get EI fourth derivative sorry ´´second derivative and then you are two day two more derivatives d^4w by dx^4 = q this is for this for general this is for general where you have E and I vary with x and this one is for constant E and certain amount of area I.

This is a familiar relationship arise from the Euler-Bernoulli beam theory which we got it from minimization of potential energy okay this much we had done in the last class now we will try to write the boundary conditions also.

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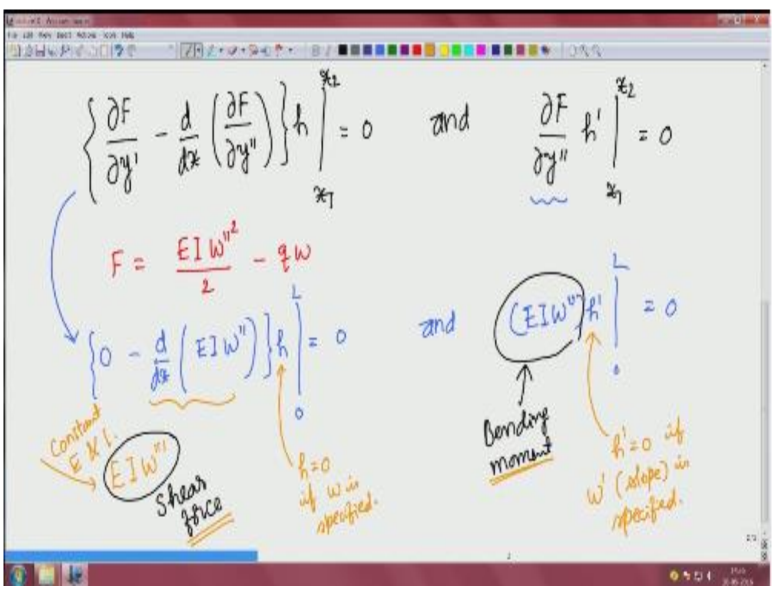

The boundary conditions were there were two sets of boundary conditions one was ∂ f / ∂ y´- d by dx of / general integrand that have why I am writing ∂ f/ ∂ y´´ times h evaluated at the two ends x1 x2 equal to 0 and we also had this ∂ f / ∂ y´´ into h ´again evaluated the two ends equal to zero this what we had okay now again let us see what f is for this problem f is and y instead of why we have W in this problem that is EIW''^2 by 2 - q W right based on that let us write this boundary condition there is no W ´so that is 0 - d /dx of ∂ f/ ∂ y ´´that is ∂ f / ∂w´ ´that gives us EIW ´´hence h is equal to 0 that is from 0 to L that is at zero as well as L and hand this one is EI W´´ which we just did this thing over here based on what F is instead of y we have w x h´ 0 to l equal to 0.

Now let us see what this means if w is specified his equal to 0 h equal to 0 like we had done in the case where integrate depends only on y ´are not y ´´h is equal to 0 if w is specified okay if it is not specified then this thing has to be 0 and this particular thick- does not matter is equal to 0 this is if you assume constant I and constant e that will simply become EI triple derivative okay because this already derivative and then we are taking t/dx assuming that EI and I are independent (x) that is their constant then this one we are writing again this is for constant E and I.

So for that we can actually interpret this again from beam theory if you recall that is nothing but the shearforce the shearforce a bending moment diagram we do that the shearforce likewise this particular thing here is bending moment because w´ ´is proportional to curvature and curvature is proportionate bending moment that is our Euler-Bernoulli beam theory this is bending movement okay so if you look at this h ´this h ´is equal to zero if W´ that is the slope of the beam is specified okay.

So you can see the connection between the boundary conditions which are essential that is h or h´ being specified means that the RW RW ´are specified that essential boundary conditions are deliciously boundary conditions when they are not specified the corresponding ones in this case the shear force and bending moment R0 whenever h is not specified YW is not specified at h is not equal to zero around the boundary condition EIW ´´´ the constant E and I must be zero likewise w’ the slope is not specified corresponding bending moment should be equal to zero so what we are saying here is that.

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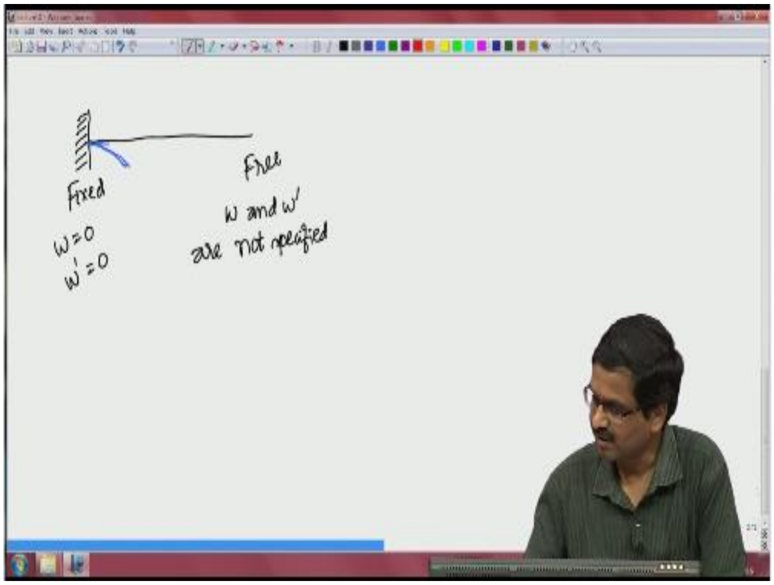

If I take a beam whenever talk about beams of us we will talk about is cantilever beam it is fixed here and then free here when you say free here is what W and W ´ are not specified they can be anything as per the loading applied on this cantilever beam and fixed means that w is equal to 0 w ´is also equal to 0 that is why when this beam bends we have to make sure that W is 0 and the slope is also 0 we have to go with slope 0 there at this end it can be anything okay so here we have W is equal to 0 w´ equal to 0 that means that if we go back and look at what we have.

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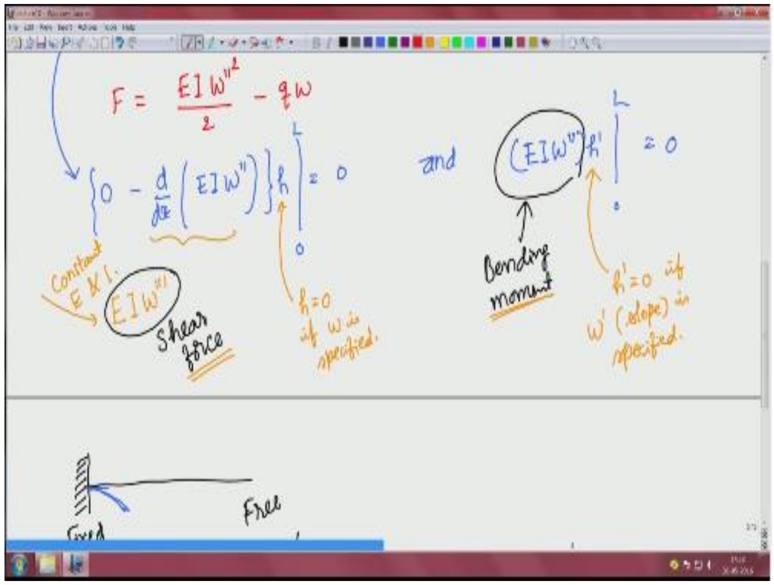

When W specified surface is equal to 0 in this case the shear force need not be 0 it can also be 0 in special cases in generally not be 0 similarly w´ slope is specified by cantilever beam over here then the bending moment need not deserve it can also be 0 but need not be 0 okay but if it is free w are specified then the shear force should be 0 at this point and show so should be the bending moment both shear force and bending moment or 0 here okay.

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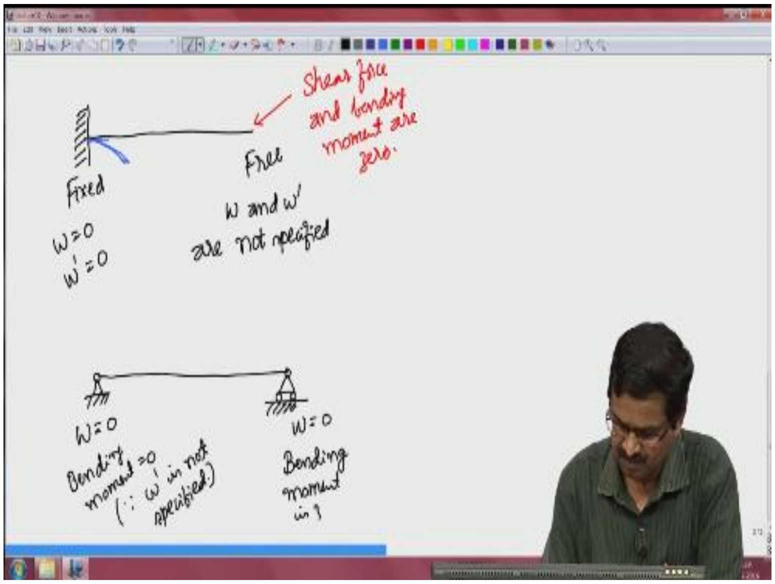

So that is how the boundary conditions also follow from calculus of variations we can take any other boundary condition so we can take let us say the simply supported beam so we have a pin joint here and we have a pin joint with a roller support over here this is what we call simply supported beam here w is specified whereas w ´ is not here also w is specified w ´is not what this automatically means is that since w´ is not specified the bending moment here is 0 bending movement is equal to zero because W´ is not specified because W´ is not specified okay are the same thing here again bending moment must be 0 here bending moment must be 0 here because that is our boundary condition.

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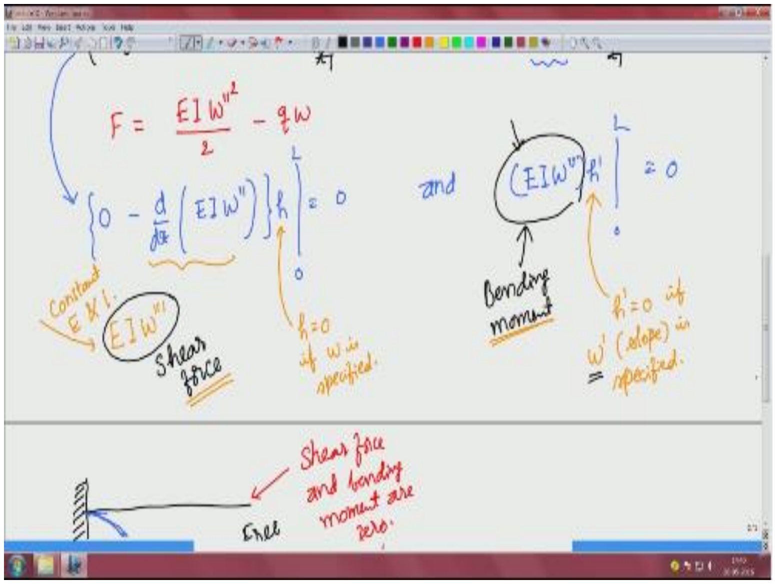

When W ´is not specified this must be equal to 0 okay.

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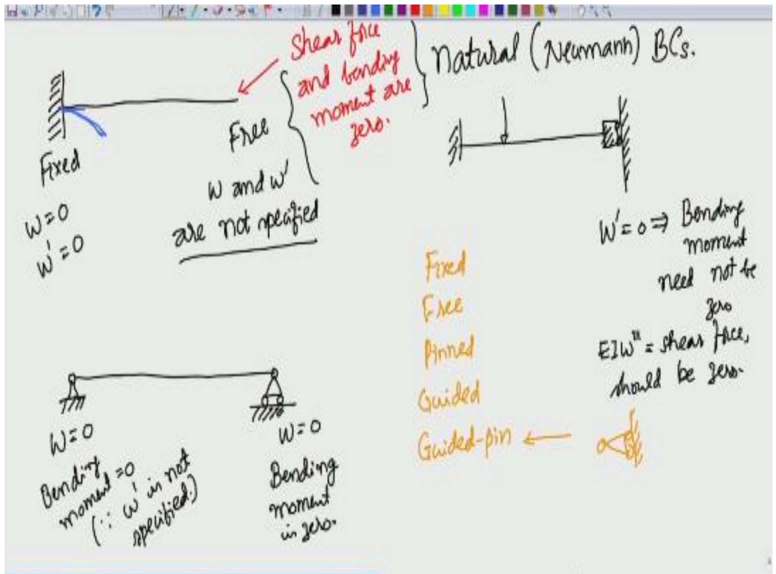

So based on this you can consider various boundary conditions let us say we take this boundary condition let us say fixed on one side and this side I would let it slide along a vertical axis with a block moving so here w is not specified it can be anything that is I apply some loading here it is going to move right so W is not specified at x equal to L but W´ is specified because it is a block moving here and that block has to slide vertically so it does not allow the beam to rotate there w prime equal to 0 w' equal to 0 means that bending moment there need not be 0 so implies that bending moment bending moment need not be I should say in the movement need not be 0.

Since W is not specified then the e I w triple prime which we said is shear force okay that should be 0 because W is not 0 okay so the three Slayer Nyman you can specify only one either directly or not meaning either essential or natural the fact that here whenever W and W prime are not specified automatically these things becoming zero are called natural boundary conditions or Norman boundary condition naturally it happens natural or Norman boundary conditions okay.

So likewise you should try a number of beams because you can have number of barnacles for a beam so we have considered fixed free pinned guided like we have shown and you can have guided with okay so that is this particular one is going to be like this there will be a pin and this one is guided vertically okay you can have all different of boundary conditions and then write the appropriate things based on whether w specified or w prime is specified okay.

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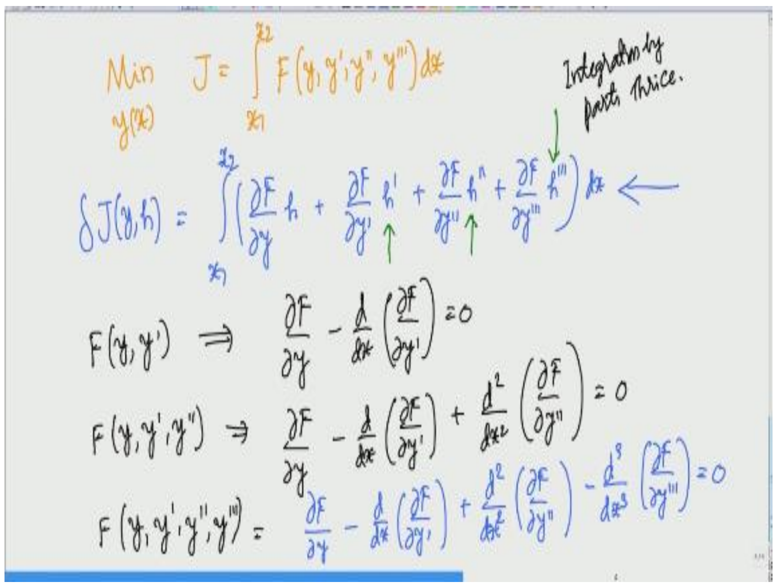

So this way we are able to solve the function that have two derivatives let us go one step further we had one first derivative second derivative now let us see if we have third derivative what happened let us say I take a J where I have x 1 to x 2 f y, y', y'' and y''' DX again if you recall how we dragons equations to convert that to first not this actually let us first do this necessary condition that is δ variation y and H okay that is an integral still x 1to x 2 ∂ f /∂ y times h plus ∂ f /∂ y' H'+∂ f /∂ Y''H'' ∂ f/ ∂ Y'''H'''DX.

So we have to do one integration by parts to get that 28 so that we can apply the fundamental lemma of calculus of variations and we have 2 integration by parts by twice to get this HW prime to become h and the boundary conditions and now for this one you have to do integration by parts three times so you have to do integration by parts thrice okay which you can do for one to get some practice but without that we can actually see some pattern so when we had an integrand which depends on only and y and y prime our Euler-Lagrange equations look like ∂ f/∂ y – d/ DX (∂ f/∂ y) = 0.

Similarly if I integrate integrand y, y' and y'' that gave us ∂ f/∂ y- d/ DX (∂ f/∂ y') + d^2 /dx^2 (∂ f/∂ y'') = 0 that is what we got now there is a pattern meaning that when we had a Y and Y prime that still has only two terms the same two terms because presence of Y double prime we got an extra term they a plus minus plus and derivative order is increasing so if we have three derivatives y, y', y `` and y``` then it is we can see the pattern we can also do the long-winded way doing integration by parts as many times as required.

Or we can write this as ∂ f /∂ y d/ DX(∂ f/∂ y') + d^2 / dx^2 (∂ f / ∂ y''') then minus so here we had double prime not triple prime this is double Prime this is ∂ ∂ Y'' now if you minus third derivative D^3 /dx ^{3} (∂ f /∂ y ''') = 0 we have seen the pattern now we can do the long-winded way and find that this what will happen okay.

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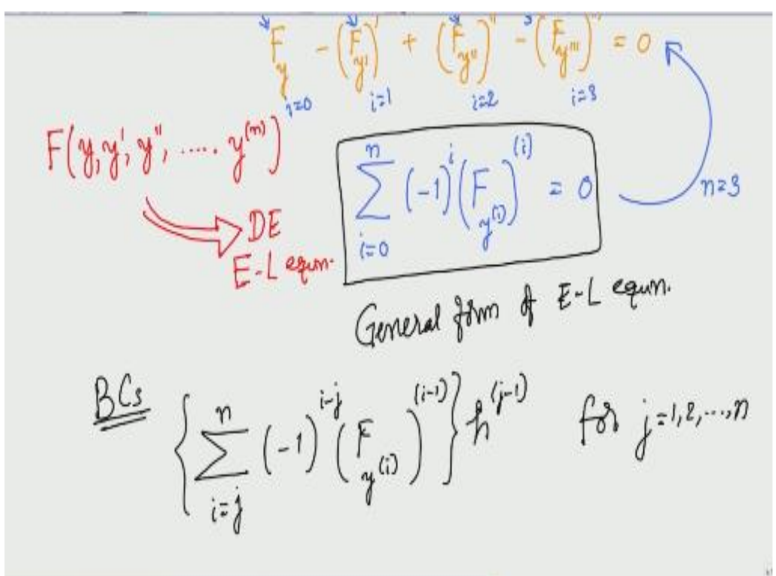

So in fact when you take n functions the same pattern will continue plus minus plus minus plus minus ∂ f /∂ y, y' Y'' x'' and so forth so let us do a shorthand notation so that you know we do not have to write this F sub y means that it is ∂ f /∂ Y okay so this is FY and then next one we will write F Y' for ∂ f/ ∂ Y Prime and then we will put a crime there okay plus we can write f y double Prime and then we will say double prime of that d square by DX square minus F Y triple prime.

And then triple prime of that okay so each time whatever you take you take that derivative Y prime Y double prime what will prime triple prime it is a consequence of how many times you do the integration by parts okay based on that if we have a functional whose integrand has let us say why y prime y double Prime and so forth up to why nth derivative okay I am writing it as in brackets n if we have the Euler Lagrange equation for this that is differential equation or Euler-Lagrange equation.

Fort his turns out to be basically I can write all of these one at a time or I can do it in the form of summation where I have to go if I have n I will have up to n derivative because of when I had only one derivative I had right I had d/ DX ∂f/ ∂ Y prime okay now we have to then there will be 2 3 and so forth right so we can write this general equation in the notation that we have right so you can say f we are to take the derivative where we have let us say i equal to we have to see because there is why there is nothing so I will put 0 to n, n equal to 3 this is this I equal 0 here I equal to one there I equal to two there I equal to three there.

So I have to go 0 to n 0 to 3 and let us also take care of the sign okay let us take care of the sign which is minus 1 raise to I because the first term has positive signs so minus 1 raise to 0 is 1 and then when it is I equal to 1 that is a minus and so minus 1 raise to 1 which remain minus 1 then I equal to 2 that is this term that is second third term here that will become positive negative we have that and then we have fy indicate this as we are taking derivative Y prime by W Prime and so forth.

And all of this where to take derivative I x equal to 0 okay so if you take this from n equal to 3 then we get this in this one if I put n equal to 3 then I am going from 0 1 2 3 so I have 0th and then the first term here and then second term here and the third term there and the sign of it is a minus 1raise to 0 1 minus 1 to 1 negative and then positive and then negative right so that is how we get so this will be the general form of the Euler Lagrange equation for integrand that has n derivatives.

This general form can also be written for the boundary condition so this is the general form of Euler-Lagrange equation for integrand involving n derivatives okay similarly the boundary conditions can be written okay again you have to work it out one by one and see the pattern but once you find that pattern you will realize that it will look something like this you have I going from j to n us why we write it like this $(-1)^{1-j}$ f y (i) raised this whole thing derivative up to i- 1 and all of this okay.

Including summation x what we had h that is also taken a derivative j-1 for j equals 1 2 up to n okay so J goes 1 to N meaning that if I have only y prime that is n equal to one and only one boundary condition okay which if you recall okay let us keep this the boundary condition.

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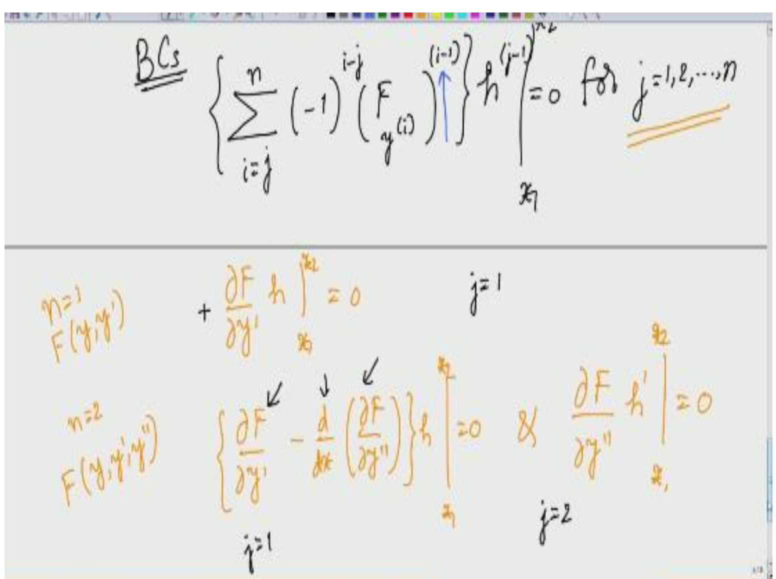

When n is equal to 1 meaning that my integrand was fyy' if I have that our boundary condition was ∂ f /∂ Y prime into H at the either end x 1 x 2 equal to 0 so I should actually put equal to 0 there and this is evaluated at x1 and x2 that boundary conditions after all that is what we have so what do you have here when n equal to 1 you are going from I equal to j equal to 1 so this is n equal to 1 here so I have to go I equal to how I will first of all I will have j equal to one to n.

So here I have j equal to 1 only because that is the n also here sell only one boundary condition which is this so now I should go from one to one with a j equal to one that goes up to n equal to 1 to 1 meaning only one minus 1 raise to I minus j both I and j are 1, 1 – 1 0 so I get a positive sign minus 1raise to 0 and that is a positive sign and then f y (i) right that I equal to one now because I goes from j 2 n-1 so I have y prime ∂ f /∂ y prime that derivative should go to I - 1 I equal to 1 that is 0 we get this right.

Now if n is equal to 2 you can verify this then I have f y and y prime and y double prime then we recall we had ∂ f/∂ y prime minus d/ DX (∂ f /∂ Y'') multiplied by H at X1 X2 equal to 0 that is what you will get but now j equal to I equal to j to n now j we have to try for j equal to 1and j equal to 2 as written here so we had that j equal to 1 we get 1 j equal to 2 we get another one that other one is ∂ f /∂ Y double prime into H prime again at the two ends x 1 x 2equal to 0.

So here I have to do j equal to this is for j equal to 1 this is for j equal to 2 right give j equal to 2 you have to go from I equal to so TPB yeah so j equal to 2 here so now I goes from 2 to n so let only get one term here j equal to 1 means that you will have two terms coming so we have the first term and then second term right I – 1 that is the first one I -1 that will be no derivative there whether we have one d/ DX as this one is showing okay.

So the general form of the equation differential equation which we have here as well as the boundary conditions if we understand that boundary condition be as many as the number of derivatives, so we have covered all of the general thing in terms of many derivatives of the function next we will consider what happens when you have multiple functions and then we will do a couple of problems.