**Indian Institute of Science**

# Variational Methods in Mechanics and Design

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**NPTEL Online Certification Course**

Hello again, so we discussed in the previous part of today's lecture the application of taut variation in orde r to establish necessary conditions for a functional involving the function y and its derivative y′ or dy/dx okay. Now we will take a particular example to use those Euler Lagrange equations that we derived and the boundary conditions to understand that better, okay. So for that let us take a geometry problem for which we know the solution already okay, the solution that you know already when I state the problem you will know.

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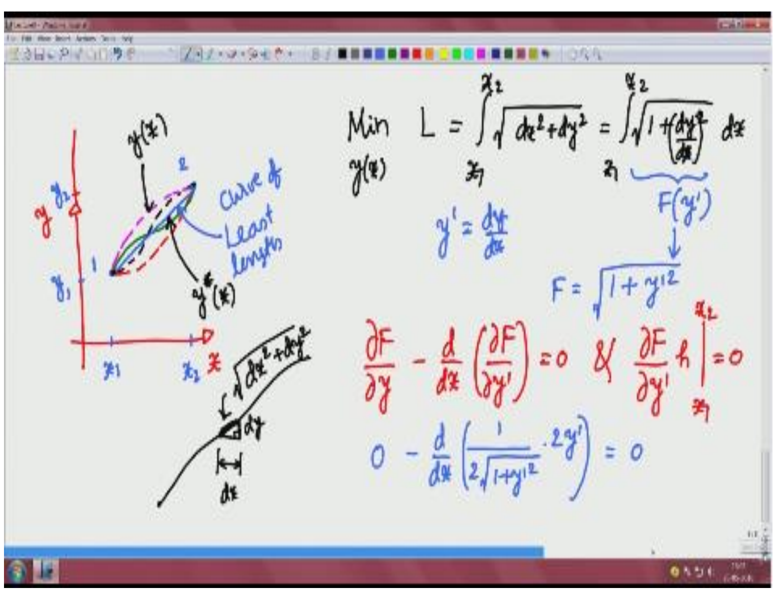

So let us take a coordinate system let us say x and y and I choose two points one point here one point there okay, so let us say this point is $x_1$ this is $y_1$ that means the point coordinates x1 y1 and this is $x_2$ and corresponding y if I extend this further somewhere here I will have $y_2$ so i have a point $x_1y_1$, $x_2y_2$ the question that I ask is that which curve I can that goes through this point 1 let us call this point 1 and point 2 between point 1 point 2 we can draw many curves that pass through this, right.

So we can draw a number of curves I can draw this, I can draw that I can draw this and a number when this is really a lot of, lot of different curves we can draw right, among all of them there will be one that minimizes the length of the curve okay, you know the answer if I give two points in a plane and somebody asks you which curve we have the least length immediately your answer would be that it is a straight line.

Because we can see that here if I draw a striking between point one and two that will have the least length, so least length so this will be the curve of least length we have taken a very simple problem for which we know the solution, right. Now we will solve this problem using calculus of variations to convince you that among all the curves that you take the straight line is the one that is going to have the shortest length of the least length between two points given points $x_1$ $y_1$ $x_2$ $y_2$.

How do we pose the problem we say minimize the length of the curve l what was J which is functional now L becomes the symbol for the functional with respect to y(x), right each of these that will be y(x) we do not know actually we know the solution that this blue one is actually y*(x) meaning that this one satisfy the differential equation of the Euler Lagrange equation, right. We have to establish that now mathematically using calculus of variations so I have this minimum with respect to y(x) I have to find that y(x) or y*(x) which is a straight line L the distance.

Once I know a general y(x) if I want to get the length so normally for anything if I have a curve like this I take a small bit okay, I take this small bit and use Pythagoras theorem to find that length of that right, so why Pythagoras theorem because this part here is dx and this is a right angle okay, and this is dy for that curve everywhere curve has X for every X there is a y so if I take this one I get this Pythagoras theorem so this distance is going to be $\sqrt{dx^2 dy^2}$ okay, and that is what we need to take and add up we want to total length from here to here or our curve from here to here we are do integration.

So we have to do integration from x1 to x2 2 for our curve of interest x1 to x2 $\sqrt{dx^2 dy^2}$ we had add it up, okay integration needs the dx we take the dx out right, when I take it out what I get is $x_1$ to $x_2$ integral $1+\sqrt{dy^2 dx^2}$ or $dy/dx^2$ I have taken out dx we got a proper looking integral, okay. Now this thing here becomes our integrand f in this case that integrand f depends only on y′ because in our notation y′ is dy/dx, okay.

So what is our F now f is $1+y'^2$ under square root so that is our F now, and let us recall our Euler Lagrange equations which was δf δy-d/dx(δf/δy′)=0 and then we had of course the boundary condition right, the Drichlet Neumann are essential and natural now we understand what is essentially what is natural in the context of this example δf/δy′ intoh x1 x2 =0 right. Now let us substitute over here so this one δf/δy there is no y now our F now this is f right.

There is no y there so this is 0 and then we have –d/dx of δf/δy′ y′ we have we are take derivative of this F with respect to y′ then we get 1/2 $\sqrt{1+y'^2}$ and then we had to use the chain rule so this respect to y′, $y'^2$ is there so if you take derivative if you become too y′, okay. We take great respectively y′ not as x but that is here d/dx of the whole thing this is equal to 0 okay.

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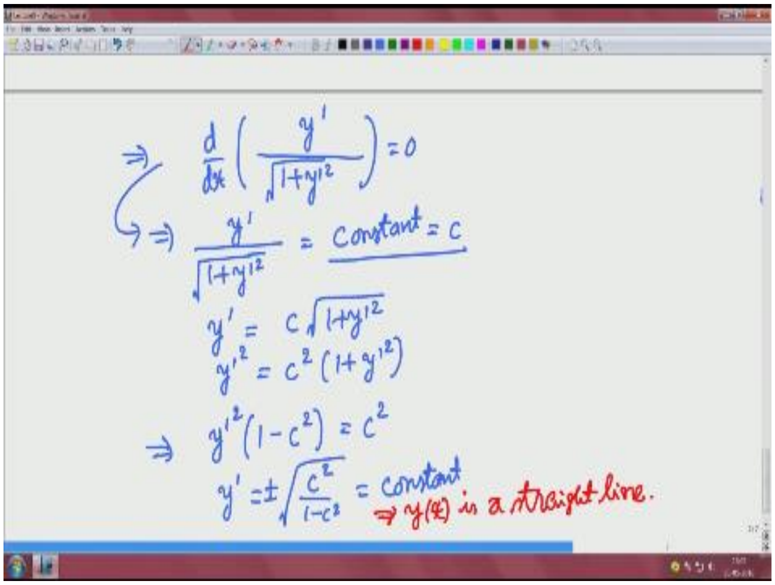


Now let us go there so we can see this so what we got from here is d/dx with the minus sign that we can take up it is equal to 0anyway d/dx into 22 here gets canceled so what we get is y/√1+y ′ =0 what does that mean it means that y′/√1+y ′ is constant because derivative of a constant equal to 0 that is what we have in other words we integrate it from this step to this step, right we get this.

Now if you think about this if you say this constant is some C then i can write I can square I can take this other side y′=c times $\sqrt{1+y'^2}$ I can square both sides I will have $y'^2=c^2$ into 1+ $y'^2$ that implies that if we bring that y′ that we have there it will become 1- $c^2$ or I get y′=c2/√1-z square plus or minus does not matter it is just there either sign but what we get is y′ is constant because c is constant right, that is what we had now it says that y′ is constant y′ is constant means what our curve y(x) is a straight line so that says that y(x) is a straight line that is what we wanted to prove mathematically we got it now right.

So either a grunge equation which is what we wrote here Euler Lagrange equation, gave us what looked like a formidable differential equation but actually turned out to be very simple that y(x) is a straight line, okay.

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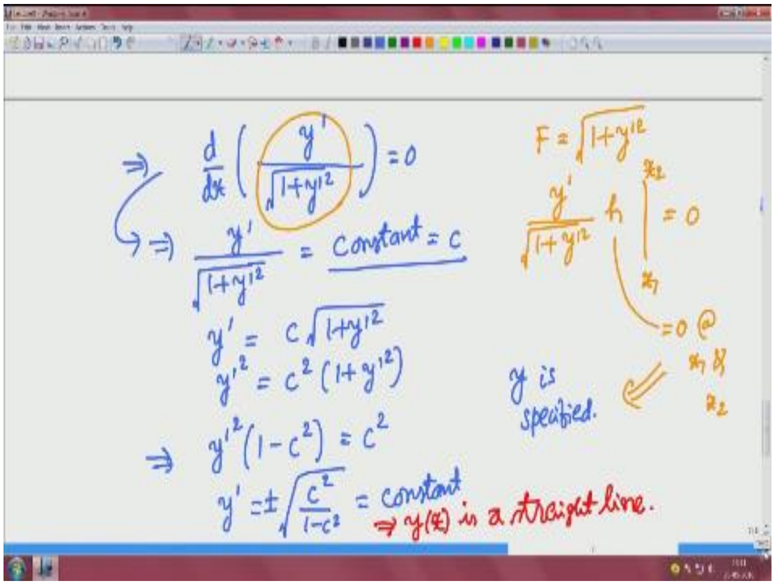


Now let us also look at the boundary condition that is this part let us consider again note that the f that we have here is $\sqrt{1+y'^2}$ sign now what this boundary condition is saying is $\delta f / \delta y'$ times h right, so if we do that what we get we had just done it here right so this $\delta f/\delta y'$ is actually this it this is a divide our prime so that basically is so let us look at this boundary condition which is over there where we have f which is this and what it says is that $\delta f/\delta y'$ which we had calculated already there so I will write $\sqrt{y'/1+y'^{2}}$ times this h should be equal to 0 that product at both ends that is x1 and x2.

Now if we recall the problem that we stated this points x1 y1 x2 y2 are given meaning that y was specified at both these points y1 was specified y2 is specified at those points we cannot take variation, because we are not allowed to perturb there it has to be that value that means that in this case that H is zero at x1and x2 because y is specified again, remember this is because y is specified at either boundary so here we have at both ends we have what is called essential boundary condition or deliciously boundary condition.

Later we can take an example where we do not specify why then there will be a natural boundary condition on a boundary condition, okay. Let us take an example, that actually gives us that natural boundary condition as well so as we do.

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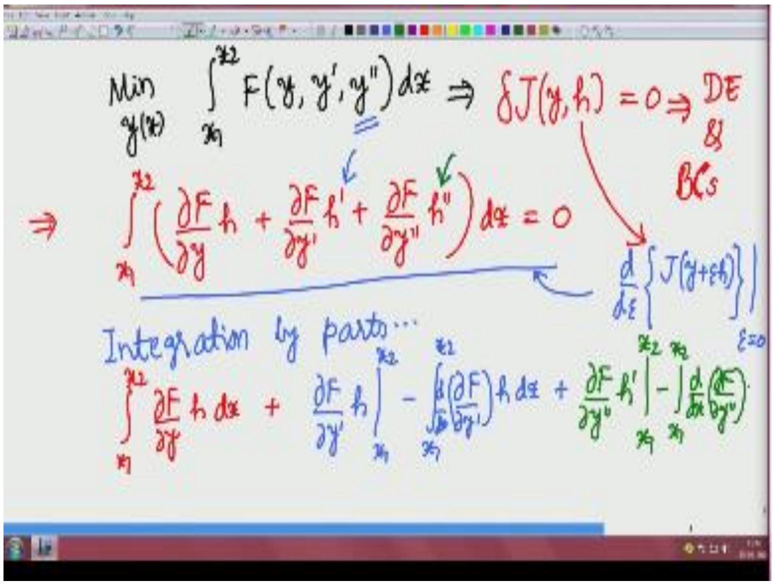


Let us actually consider the next problem, where we minimize again with respect to an unknown function y(x) but this time we will take F to be dependent on y, y′ that is the first derivative as well as y″ okay, a little extension or generalization instead of taking only first derivative we also take the second derivative right, this is the problem it is this problem again the necessary condition would be the that the variation of this respect to y and the arbitrary function the arbitrary function does not bother us because the fundamental lemma gave us a way of getting rid of it to get the differential equation and then boundary conditions whether y is specified or not tells us when to ignore h when not to okay.

When y is specified h is 0 when y is not specified h is still arbitrary then what multiplies the boundary condition should be equal to 0 right, that is the thing that we will try to get here so that should be equal to 0which by the way gives differential equation as well as boundary condition this gives us differential equation I am using de for that differential equation and the boundary conditions at the ends okay, that is what calculus of variation gives us, right.

So let us do this problem so if we say that go to variation is 0 again we know what that gives which is that we have x1 x2 δf/δy h+ δf/δy′ into h′+ δf/δy″ into h″ dx equals there is a first step in the gateaux variation this from the definition again definition of the two variation to recall for you is d/dε of the J where you replace y with y+εh and then substitute ε=0 that is what we did for one now we have done it for the case where there is also double prime.

In addition to y and y' that is what you get when you do this okay there is no trick there this is the definition there is limit definition which is equivalent to this and that is what we have done okay now earlier we did integration by parts to get rid of this H prime now we have to do that two times we are to integration by parts twice to get rid of HW' H' HW' x H.

first HW' becomes H' and then becomes H right so let us do integration by parts twice so integration by parts once integration by parts if we do we get we leave all on the first term as it is we do not touch it because that already has H which is good enough for us to apply the fundamental lemma DX let us put that and let us take the second term let me write second terms using blue color.

So you are clear where it is going so that I will do integration by parts the first function ∂ f by ∂ Y' integral of the second function which becomes H evaluated at the boundary x1 x2 - integral of x 1 to x 2 ∂ f / ∂ y' we have to do derivative of the first function into integral the second function DX okay that takes care of the second term now let me use green color for the third term will do integration by parts for that you get ∂ f / ∂ Y'' first function integral a second function second function here is H'' okay.

Now if it integration once that you get h' this get evaluate at the two boundary points x1 x2 minus x1 to x2 derivative of the first function that is d by DX of ∂ f /∂ x'' integral of second function and then there is TX that should be equal to zero very integration by parts once.

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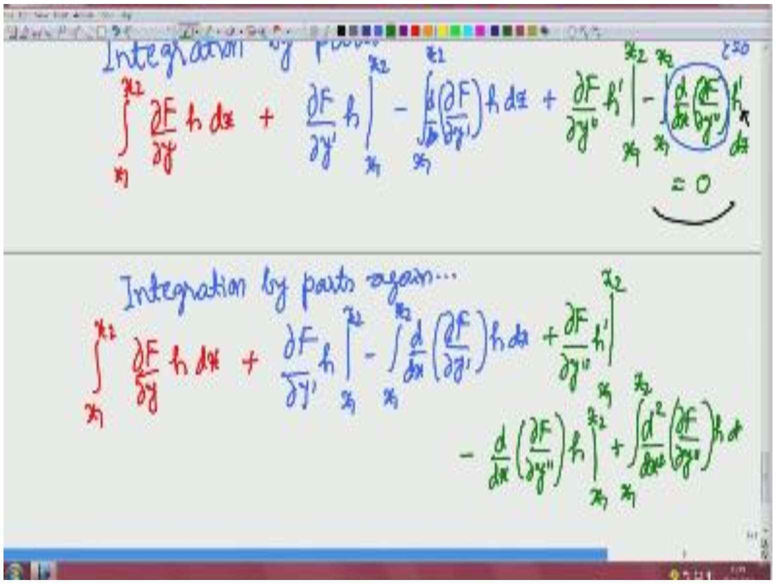

Let us do it one more time okay so now we do integration by parts again integration by parts again a second time we want to do again let me use a color coding the first term that we have it remains as it is ∂ f / ∂ Y times h TX plus let us go to the second term which is already there is nothing because there is H here is H there so I will just write that ∂ f / ∂ Y' H at X 1to X 2 we do not read integrate by parts for the second term as well x 1 x 2 d by DX of ∂ f / ∂ Y' into h DX now for this term the boundary term part is fine that is already H' is there that does not bother us because in the boundary condition ∂ f / ∂ Y'' H' X 1 X 2.

What we will do is for this term that is for this to get rid of this H prime we will do integration by parts one more time that gives us now we take this one as a first function that is we take this one as the first function so I can write minus because the minus n is there D / DX of ∂ f / ∂ Y'' first function integral second function first function is that integral of the second function is H' that becomes H.

That will be the boundary condition X 1 X 2 then minus there is a minus already so minus of minus plus derivative of the second function second function that we divide is the first function but we already had d by DX another derivative $D^2$/ $DX^2$ of ∂ f / ∂ Y'' integral second function that is h DX and the whole thing should be equal to 0 okay.

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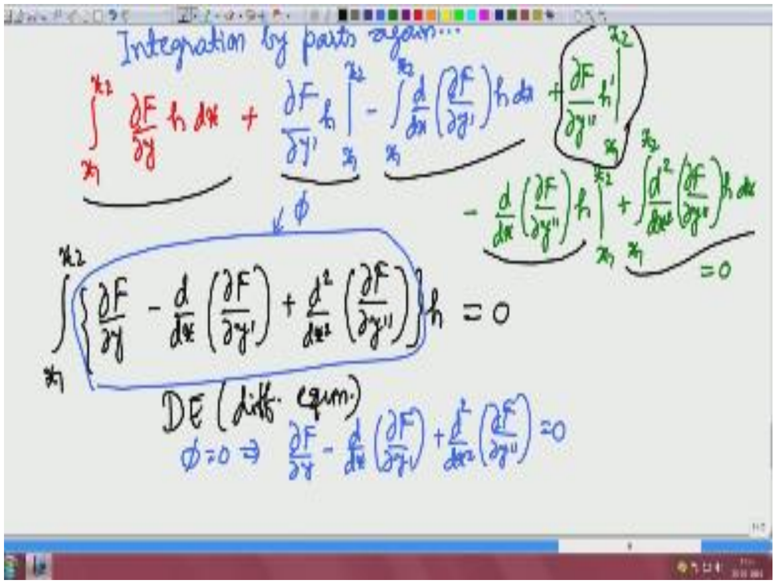

Now we can collect all the terms there are integral terms so this is an integral and this is an integral this is an integral let us collect all of them as 1 X 1 to X 2 what I have ∂ f / ∂ Y this as H this has H this has it so we will take H their out so minus d by DX of ∂ f / ∂ Y' + $D^2$ / $DX^2$ of ∂ f / ∂ Y'' all of this times H okay that is what we get integral part which we say must be 0 because integral boundary terms both being 0 will be true only for some specific age but H is arbitrary.

So we actually separate out the boundary terms and the integral term so we got our differential equation so this is our short form for differential equation right differential equation will come if you apply the fundamental lemma which is that whatever we have here that is our fee okay we had used that argument that fee times H should be equal to the overall arbitrary H the fee should be equal so our that is fee equal to 0 meaning we have ∂ f / ∂ Y – D/ DX ∂ f / ∂ Y' + $D^2$ / $DX^2$ ∂ f / ∂ Y'' = 0 we got our differential equation.

And the boundary terms are also there let us write them so we can see them also take a look what we have there is a boundary term here which has H there is a boundary term here also that has H there is another boundary term here that has H' minute okay so we will collect the ones that have H and H' separately okay so remember this is ∂ f / ∂ Y' H this is -D /DX ∂ f / ∂ Y'' times H so the boundary conditions.

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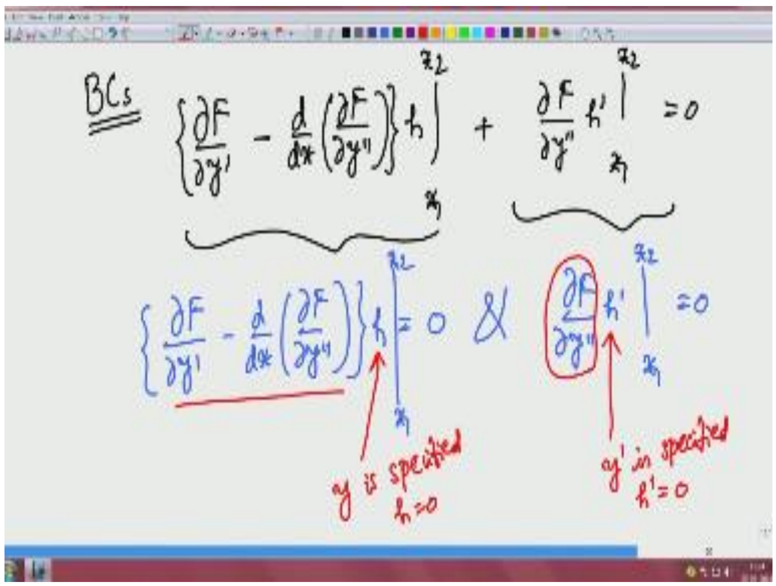


The boundary conditions now if I right so I have ∂ f / ∂ Y' that had H and then we also have D / DX of ∂ f / ∂ Y'' both of these x h at either n x1 x2right so now we wrote me to care of this do to care of this then we have this also like that let us write ∂ f / ∂ Y'' H' so plus ∂ f x was it plus or minus that is plus minus was only here ∂ f / ∂ Y'' to h' here it is H it is H' X 1 X 2 = 0. So we separated out the differential equation integral part and the boundary conditions now boundary conditions to you can have the sum that we have written being equal to 0 in some cases and those cases are not rare against it will be for boundary condition certain boundary conditions Hand H' together can give you are you can make them separate that is more common.

The other one is also possible but it is less common where the sum of these two that we have written is equal to 0 but most often individually these two things being equal to 0 is common so we will write it that way but remember there could be cases where both the terms summing 20 is also possible as a boundary condition at X 1 and X 2 if you separate our which is the more common case what we say is that ∂ f / ∂ Y' – D / DX ∂ f / ∂ Y'' times H = 0 at either end okay and ∂ f / ∂ Y'' x H' at either end x 1 x 2 = 0 is possible separately.

Or together sometimes together also will consider later but right now let us separate out so you get the boundary conditions okay now when is h0 when y is specified if y is specified h is 0 similarly if y' is specified h' is 0 when it is not specified this other part is equal to 0 when h is not specified that is zero when H' is not specified this is zero like what we consider earlier okay in order to make sense out of this.

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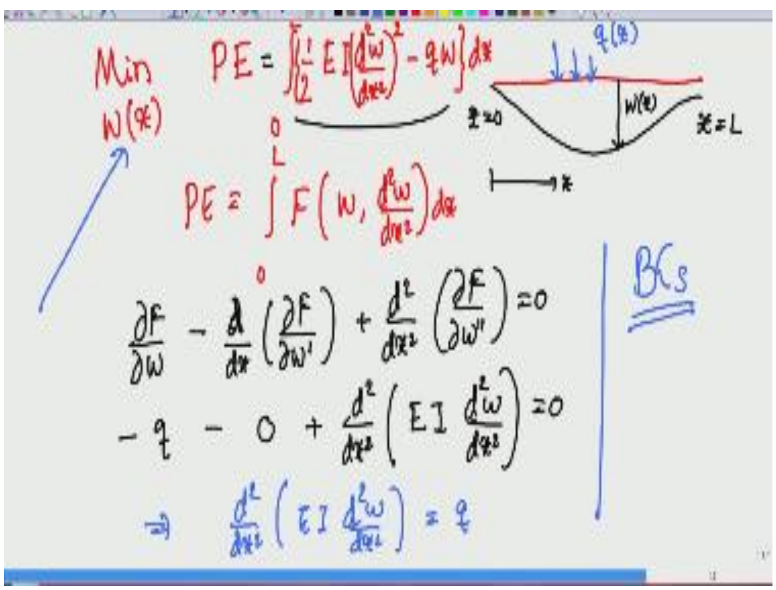


We should take an example of an integrand that has first and second derivative in it in addition to the function so let us take an example where I want to minimize this is what we call potential energy because we just discuss this minimum potential energy principle which means you static equilibrium with respect to a function here let us take that function to be w X which is a physical meaning which is like a beam if I have a beam and there is some loading on it let us say there is loading on it QX and let us say this d forms okay.

In some fashion there can be any boundary condition let us say that deforms in a particular fashion there is a different points will have deflection WX so X is like this X = 0 here X = L there span of the beam everywhere we need to find this function WX how do we find we find it by minimizing this potential energy the potential energy will write 0 to L so for what we are writing X 1 to X 2becomes 0 to L potential energy that will discuss later how we get the expression I am going to write for now let us just take it.

So potential energy here can be written as half what we call X modulus and second moment of area I for the beam okay into $D^2W / DX^2$ minus Q of x times w DX okay, that is what is potential energy we can discuss later how that comes about or maybe you already know from the in theory that potential chief of beam can be written in this fashion so what we have here this PE we have integral 0 to L like an integrand okay which depends on W the W is there we also have that this depends on $D^2W / DX^2$ DX there is no W' here.

But second derivative is there if you have this if you write or Lagrange equations that we just derived we get the equation okay let us do that quickly so we have ∂ f / ∂ Y that becomes w -D / DX of this is D / DX of ∂ f / ∂ w' we write and then plus $D^2$ +$DX^2$ of ∂ f / ∂ w'' sort of Y'' equal to 0right let us go up and see that that is what we had as the question right so ∂f / ∂ Y D - divided of p ∂y' $d^2$ / $DX^2$ over ∂ f by ∂ Y ∂'' now WR function so why is has become this if you do this what we get ∂ f / ∂ w.

Now if I look at our f here which is that w is over there that gives you minus Q and then second one is zero because there is no W' there w' is there $d^2$ w / $DX^2$ is there right so that is plus $d^2$ / $DX^2$ of ∂ f by ∂ w'' that gives us that square and to go away so I get e I into $d^2$ w / $DX^2$ =0 and this must be familiar to you so what we have here is what this gives us is $d^2$ / $DX^2$ of e I $d^2$ w / $DX^2$ = Q because it is minus q 1 other side and in fact if E and I do not vary with X like constant cross section beam they will become e I fourth derivative of W equal to Q that is what you would have learnt in Euler-Bernoulli beam theory where you have force balance and part in the solution.

Now we minimized potential energy current solution not only that we also will have boundary conditions okay if you write down the boundary equation which I will leave that as an exercise to you for this one you have F, F here and f + y becomes your w if you write it then various boundary I did not say what boundary it can be pinned fixed guided and so forth which will consider in the next lecture but you can try the boundary conditions here and make sense out of this deliciously boundary conditions and Newman boundary conditions are essential and natural so we will consider that in the next lecture. Thank you.