Indian Institute of Science

Variational Methods in Mechanics and Design

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Hello we will continue with decay to variation today to establish the necessary conditions for the minimization of a functional so this lecture will reinforce some of the concepts we discussed in the last lecture that is the Gateaux variation and variation derivative pressure differential and pressure derivative but the focus will be on the Gateaux variation okay so let us start with a functional okay.

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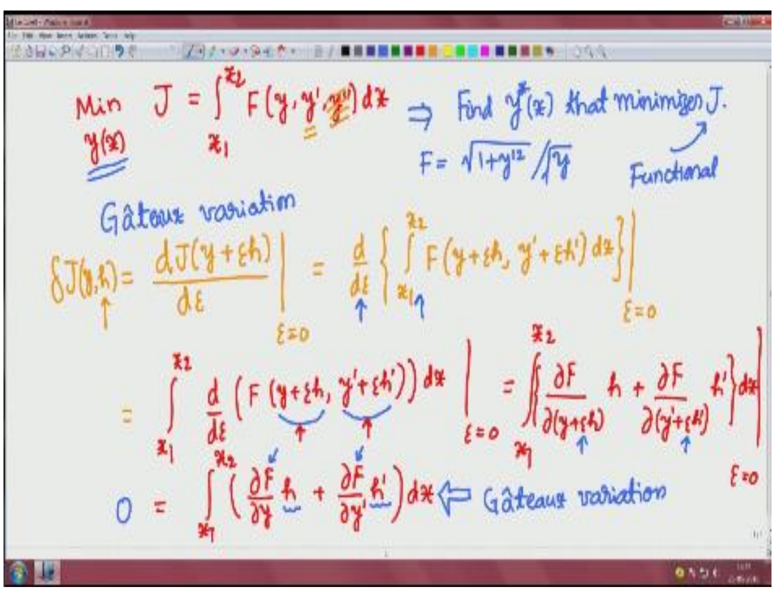

Let us take a problem that is I want to minimize a functional J which we take it in the form of an integral now going from x 1to x 2 integrand f which depends on a function y and y´ and if you want we can also put y ´´ as well and dx and we are minimizing with respect to this function y(x) which we need to find to minimize this so what this means is that this implies that we want to find a particular y(x) which we can call y * of x is unknown function that minimizes that minimizes the functional J so if you recall we call this thing a functional which is a mapping from the function space to areal number space we want to minimize that real number J is now a functional so it will be a real number a scalar we want to find this y * (x) that means that you have to search in the space in which this y(x) is okay.

And we said this minimization problem requires a necessary condition for that we need to take equivalent of the derivative or gradient equivalent meaning consider with unknowns are x1 x2 xn then we took the gradient and said that the gradient should be equal to a zero vector now we take Gateaux variation and set that equal to zero that becomes a necessary condition okay so in order to take that Gateaux variation so what we defined in the last class teaux Gateaux variation if you recall we had a limit definition also we had normal calculus-based definition the latter is useful for us to do the calculation.

So what that meant was that we take the derivative of this functional J not defined in terms of y but defined in terms of y+ ε h and take derivative with respect to ε and substitute ε = 0 if we do that then we get the variation of J with respect to y okay to begin with we will do this without considering the y´ ´ later will consider we take only y and y ´ okay so we have a functional J whose integrand depends on y and y ´ so an example that we have taken last time for that this F can look like this √ 1 + y^2 ´ / x√ y now it is a function this F is a function of y and y ´ that is what we have written if you have something like that based on this definition of got a variation let us do for this problem okay.

If you take this we have d / d ε (J y+ εh) so we had substitute into this integral x1 to x2 f now wherever why is we substitute y+ εh so we say y + ε h likewise for y ´ which is here we have to do derivative of y ´ the greater y y ´ ε h ´ okay in fact I forgot here that I should write this thing as variation of J I have to put y and h because in the definition of Gateaux variation this arbitrary function h that is that h(x) will be included okay today we will try to get rid of it but until now it is there so variation of y including that h is how this is done so we have this now and then we have to also put $\varepsilon = 0$.

Now we said that we can interchange the operations of differentiation and integration okay that we can do when these limits are not affected by this parameter okay if we do this then it will become integral x1 to x2 we take the differential differentiation inside $d / d\varepsilon$ (F) which is a function of $y + \varepsilon h$ and then $y' + \varepsilon h'$ dx okay and then we substitute $\varepsilon = 0$when we are done with differentiation okay so now if you do this integral x1 to x2 we are taking derivative respect to ε but this function f depends on $y + \varepsilon h$ and then depends on $y' + \varepsilon h'$

So at use chain rule and this as ∂f by $\partial y + \varepsilon h$ because that is what is there in this function okay and then we are take derivative of this that is derivative of that one with respect ε that gives us a h okay and then we have to do for the second part that will be $\partial f /\partial y' + \varepsilon h'$ and then take derivative of this which is with respect ε which is h' so we get h' and then dx now in this we had substitute = 0 when we do this we get the integral x1to x2 ∂f by if substitute $\varepsilon = 0$ here okay that will simply become $\partial f / \partial y$ times h + again we look at this ε here we substitute equal to 0 then we get $\partial f/ \partial y'$ times h' dx okay.

So now what we have here is what we call Gateau variation okay that is equivalent of the gradient as we discussed in the last lecture now the necessary condition is that this Gateau variation should be equal to zero so we have an integral that says this integral should be equal to zero is a necessary condition for any function y that we take where is y is in this F right that you will get an integral and say the integral should be zero okay now you see this we still have this unknown function h and h' actually why is also unknown but in this h is an orbit function that you can choose whichever way you want.

And we are we have that and also its derivative and we have to somehow get rid of it by using an argument that this h is arbitrary okay so now if we proceed further.

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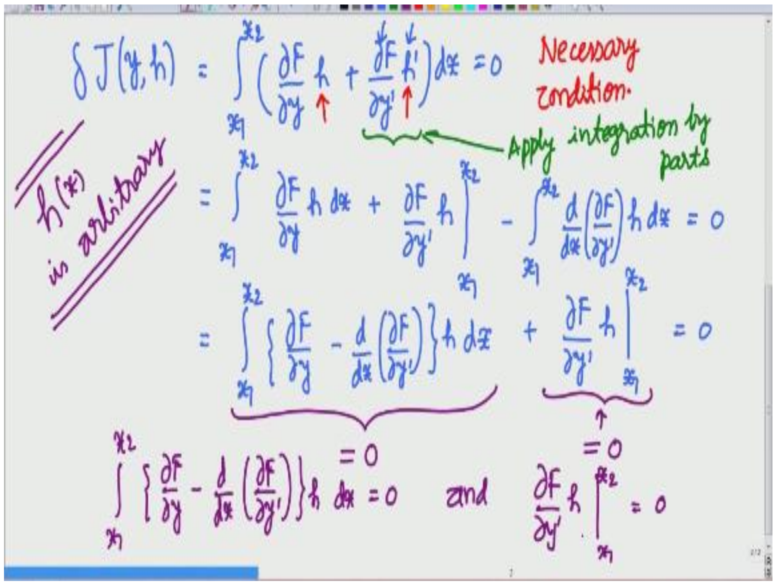

I will write what we had at the end of the last slide so what we have is this variation of y and h and for the J that we consider what we had is integral x 1 to x 2 $\partial f / \partial y$ times h + $\partial f / \partial y'$ times h' dx this being equal to 0 is our necessary condition this is our necessary condition that being equal to 0 now we have to do a little bit of work to somehow get rid of this h and h' okay for that we will use a famous lemma that is a basis for calculus of variations when there is h that lemma applies when there is h' also it applies but it is convenient if we convert this h' to h okay that also gives us something more.

So what we do now is to use integration by parts to integration by parts to convert this into h and some boundary condition okay so this one to this term we apply integration by parts by integration by parts just normal calculus everybody would be familiar with integration by parts so we leave the first term as it is integral x1 tox2 we leave the first term $\partial f / \partial y$ h dx as it is for the second one will take this as the first function this as the second function meaning normally what we write the first function into integral of second function.

So h ´is their integral a second function becomes h evaluated but the two endpoints the boundary and then minus we are applying integration by parts to this second term integral of minus integral of limits from x1 to x2 derivative of the first function that is d / dx so now integrates respect to x you have to do that derivative respect to x d / dx of derivative of the first function into integral of the second function h ´ is the second function now you write h now we can collect the terms that are under the integral sign x1 to x 2 we have ∂ f / ∂ y there is h here this also h.

So I can take that minus d / dx of ∂ f / ∂ y ´ together multiplied with this arbitrary function h it is still orbiting is still with us we had to somehow leave it somewhere which will do shortly dx and then we have the boundary term ∂f/∂ y ´ h evaluated at x1 x2 okay that should be so whatever we have done that is equal to 0 so this is equal to 0 and this is equal to 0 right now we have an integral we have this integral and we have this boundary term right these two sum of these two should be equal to 0 there are different possibilities each one of them can be 0 or the sum can be 0 but a sum can be 0 is little bit tricky because this particular term is evaluated only at the two ends x1 x2.

Whereas this one if you were to do integration this is for the entire domain that you are doing as opposed to the second term which is only for the endpoints so even if you construct a special function h and some why that you find you probably can make it zero but will be applicable only for one particular h if you want it to be applicable for any h because so far let us recall that h is arbitrary it is our choice in the definition of Gateaux variation this h of x is arbitrary we can take anything that we want so you want to retain that spirit.

So what we say is that this should be equal to 0 and this should be equal to 0 okay so from here we write x1 x2 integral with limits x1 to x2 of this integral ∂ f by ∂ y minus d/ dx of ∂ f /∂ y prime x AH dx is equal to 0 and we write something that is 0 dy ∂ f/ ∂ y prime into edge at the limits x1 x2 is equal to 0 that is what we say because anything else that you do it you may be able to do for a particular edge function H of X.

But not for arbitrary H okay so we can write these two like this all right now let us take it further this one and that one.

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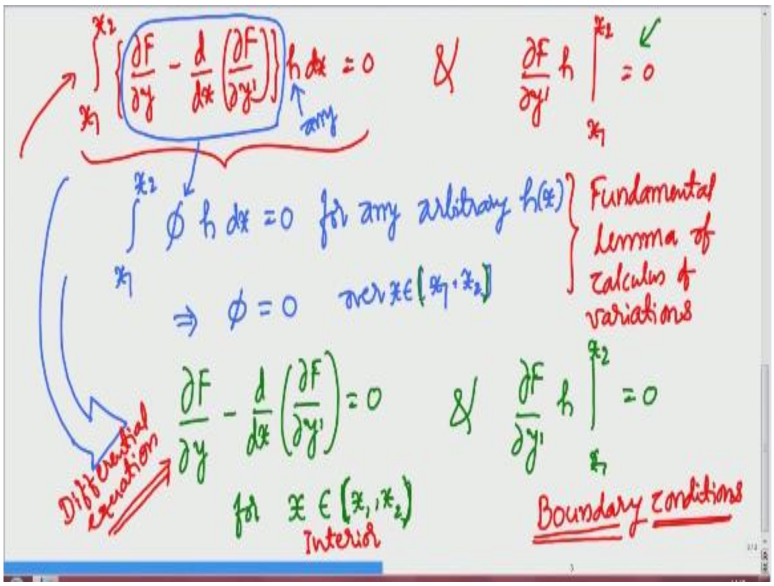

So I will write them again in a different color so we have x 1 x 2 ∂ f / ∂ y – d/dx(∂ f/∂ y’) time’s h DX equal to 0 and we have rho f by ∂ y prime into h add the two limits x 1 x2 equal to see that is what we had at the bottom of the last one now let us look at this okay so we want this to be true that is this integral being 0 we want it to be true for any H that is arbitrary any function H arbitrary that means your choice or anybody's choice then there is something called a fundamental lemma in calculus of variations which says that if we have let us say an integral such as this x1 x2 where I have something called 5 okay some function times h DX equal to 0.

If we say it is equal to 0 for any arbitrary h(x) okay let us say you have something like this what the fundamental lemma says is that this statement that F h by any h integral is 0 then it says that F must be identically zero or the entire domain X 1 to X 2 so this will be over your domain X 1 to X 2 it should be equal to 0 everywhere so far this is a fee which is a function of X when you do it or this integrand part of it.

So for x that belongs to this interval anywhere it should be are everywhere should be equal to 0 you have to think about this, this is called the fundamental lemma of calculus of variations fundamental lemma so lemma is a thing that is not good enough to be called a theorem okay in fact if you think about this it is even intuitively obvious why it should be the case because you have a function fee you are multiplying by another function H and you are calling that H to be an arbitrary function.

And if it has to be 0 you better have Ø equal to 0 everywhere otherwise you construct an F for a given H somebody will come and change the edge because H is arbitrary then this integral will not be 0 okay infact the proof of this also works by contradiction okay the proof is not so important for us because this can be intuitively understood so this is fundamental lemma of calculus of variations and that is what we will use now over here okay.

We will use that over there because what is our Ø our Ø now is this portion okay that is our Ø for any arbitrary H this integral V H equal to zero is satisfied only if this Ø is equal to zero everywhere so we write okay from here and our fundamental lemma together we can write our equation so we write ∂ f /∂ y -d / dx of ∂ f /∂ y prime equal to zero for this X that you have it will get some expression there for any X belonging to this interval X 1 X 2 okay.

But we should not forget the boundary condition so the other one that we have over here so ∂ f by ∂ Y prime H X1 X 2 equal to 0 it may be actually wiser to keep them as these brackets rather than square brackets because this interior this is the boundary right so this is on the boundary of the domain that is X 1 X 2 this is interior everywhere okay so for a given F this thing is actually a differential equation differential equation that is what we will get we will do an example but what we have here is a differential equation and the boundary conditions for any differential equation whether it is initial value problem or boundary value problem you need conditions at the ends it is a boundary value problem you need boundary conditions with initial value problem initial conditions here.

It is a boundary value problem at you as we call BVP boundary value problem so here is a differential equation okay and here are the boundary conditions and that is what calculus of variation gives again if you recall what we started out with.

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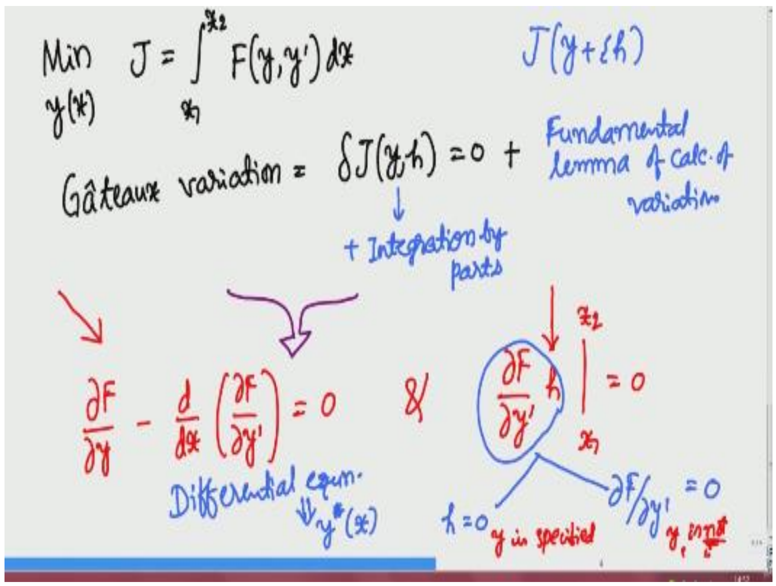

Is a problem statement was minimize with respect to Y of X an integral j from the limits x1 to x2 where the integrand depended on y and y prime DX and then we to the get variation and said that being equal to 0 is the necessary condition so we got the equation ok so again let us remind ourselves that we took the two variation which is the turn j y + h equal to 0 and then we used what we called the fundamental lemma fundamental lemma of calculus of variations okay.

And in between we also added to this integration by parts okay I am just recalling or summarizing the derivation that we just did integration by parts all of that gave us all of these things give us what we call the differential equation which is ∂ f /∂ y – d/ DX(∂ f /∂ y prime) equal to 0 in the interior and the boundary condition ∂f / ∂ y prime into this edge x 1 x 2 equal to 0 the edge disappeared from the differential equation there is no edge here.

But here we have h okay that is okay because what we say here as the boundary condition is that at X 1as well as X to this product that is product of ∂ f/∂ y prime and H should be equal to 0 how do we take care of that let us say at X 1, Y is specified y is specified we are not allowed to take variation there after all H is the variation right so it is a little perturbation we take around why if we recall our graduation definition which had j y + ε h okay.

H is a function that defines the variation right so if Y is specified we are not allowed to have h different from 0 there 0 should be assigned to H at that point that automatically satisfies this likewise if X 2 if it is y is specified then H will be 0 right if Y is not specified then we said ∂ f/ ∂ Y prime should be equal to 0 then that is how we get the boundary condition either y is specified if Y is not specified something else should be equal to 0 that is what ∂ f /∂ Y prime okay.

So this leads to two kinds of boundary conditions first is the differential equation okay that is important and using this differential equation we can find okay this enables us to calculate the solution function y * X okay for that we also need the boundary conditions we get two kinds of boundary conditions here okay one is H equal to 0 other is ∂ f / ∂ Y prime is equal to 0 h equal to 0 as we just discussed it is when y is specified y is specified the problem specifies what y should be at the end okay in which case it should be equal to 0 the boundary condition is satisfied if not if Y is not specified why is not specified okay why we will go to the next slide why is not specified then that should be equal to zero so that is go to next page.

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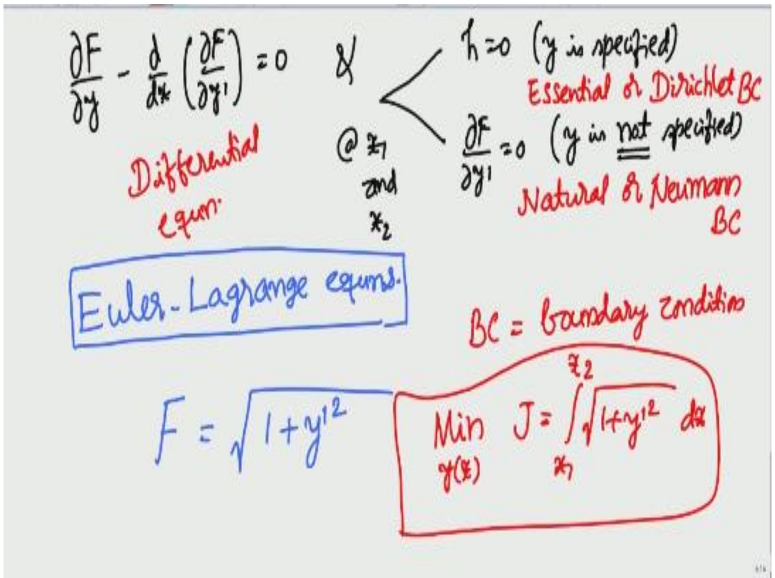

And try to rewrite this we have the differential equation that is not a problem ∂ f / ∂ y - d /DX(∂ f /∂ y prime) equal to 0 and the boundary condition has two possibilities right at both x1 and x2 okay so one is that H equal to zero when y is specified ∂ f /∂ y prime equal to zero when y is not specified okay both at x1 as well as x2 okay and there is a name for it now when y is specified we call it essential boundary condition or named after mathematician deliciously boundary condition.

Deliciously I will write BC for boundary condition so BC stands for boundary condition okay when y is not specified we call that natural boundary condition our name of a mathematician Neumann boundary condition so calculus of variation not only gives once again the differential equation for the minimization problem but also gives the boundary conditions right and these are called Euler Lagrange equations the differential equation that we have written is called Tyler Lagrange equation equations i say that there could be multiple functions that you do not know y 1 y2 y3.

And so forth so that is why you are saying I Lagrange equations in plural otherwise just Euler Lagrange equations okay so if you want to try let us take a particular why to begin with let us take 1 + y prime square okay meaning that the integral that I am considering the problem considering is minimize AJ where let us say from X 1 to X 2 $\sqrt{1 + y'^2}$ d x with respect to Y(X) then for this problem we can write the a Lagrange equation the boundary condition and try to find the solution which we will do when we continue this lecture, thank you.