Indian Institute of Science

Variational Methods in Mechanics and Design

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NPTEL Online Certification Course

Okay, so we discussed what taut variation is now let us complete the example that we took in the last part of a lecture. So again I would take this functional which is.

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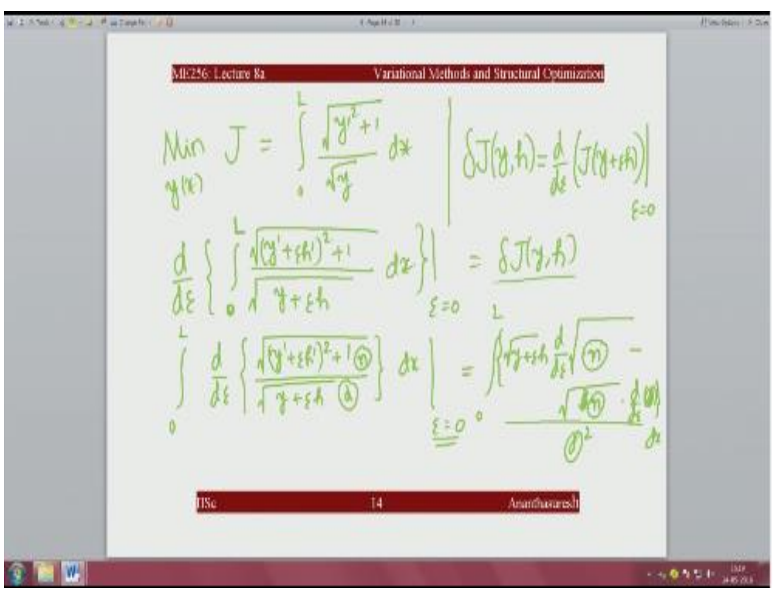

J=0 to l $\sqrt{y'^2}+1/\sqrt{y}$ dx so that is our functional we want to minimize this functional which is the bracket around time with respect to y(x), right and our definition of got a variation is this δJ y necessarily we should have an arbitrary function h which is what we said is d/dε(J) instead of we

have $y+\varepsilon h$ and then substitute $\varepsilon=0$ let us do this where I want to take the derivative $d/d\varepsilon$ of the integral 0 to l that is what our functional is and wherever y is I have to put $y+\varepsilon h$ and likewise instead of y' and put $y'+\varepsilon h'$ because we are replacing y with $y+\varepsilon$ net so y' should replace with $y'+\varepsilon h'$, ε is a constant with respect to which we are taking derivative square plus 1 dx, okay.
This is our Gateaux variation so this is when we substitute $\varepsilon=0$ that becomes our Gateaux variation, okay that one does not have ε because ε tends to 0 right. Now if you see we are taking derivative of an integral we can interchange the integral in derivative operation other words I can put the integration outside and take the derivative inside of the derivative integrand, so I can say $\sqrt{y'+\varepsilon h'^2+1}/\sqrt{y+\varepsilon h}$ okay, and dx and then a substitute $\varepsilon=0$ so you can see the order of differentiation integration is interchanged here and that is valid when the limits here we have 0, l if they do not depend on this ε okay, then we are allowed to do this, right.

Now this is if you say whatever y' h' yh are we can definitely derivative of this ratio right, so this is the Audrey derivative that you take respect to ε you do that and then substrate $\varepsilon=0$ then we will be left with this go to variation which will still be in the form of an integral that is something we need to take note, so if I take derivative of this now I would be doing using the quotient rule I have to do 0 to l, so I would do $\sqrt{y+\varepsilon h}$ and take derivative the numerator then that will be divided ε if I expand all that I will get something right.

Now the strike $d/d\varepsilon$ of square root of whatever we have over here and then we will say this is all the numerator minus and then square root of whatever we have over there time to derivative of let us say this one I will call this numerator let me call this new this is called denominator d here I will have the numerator and we are all alert and denominator, denominator so this will be now numerator minus derivative of the denominator right, the d and then divide by denominator square you do all that dx, okay.

And a substitute for ε, because ε is there we substitute $\varepsilon=0$ you will still be left with an integral okay, that integral will be your first variation so first variation is also an integral okay, as we take up more example then you will see how we can do this routinely.

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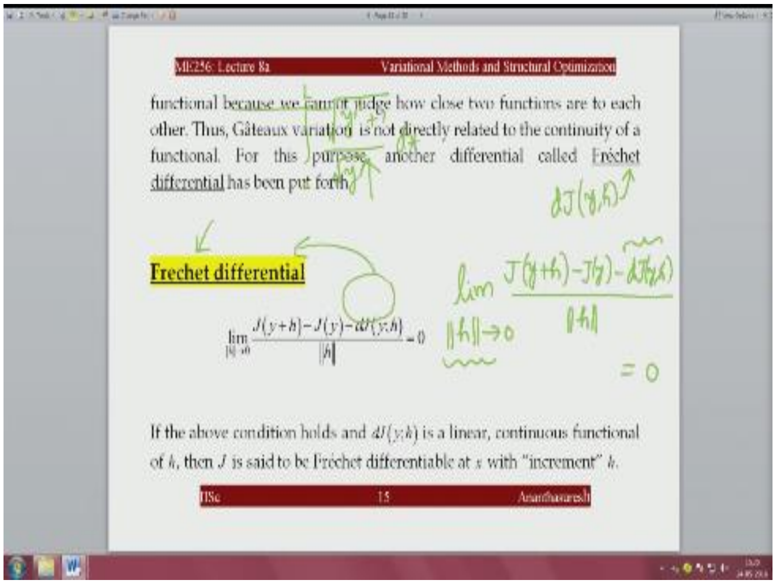

And be able to use it to find the minimum function okay, now as we just mentioned in the last half the lecture we have to talk about this second one called Frechet differential, Frechet differential is defined slightly differently from the Gateaux variations so here let us write it down in case you cannot see the small font here we do not have ε instead we take this arbitrary function that does not go away we say that that arbitrary functions norm goes to 0 and what we have in the numerator and the denominator are the denominator we make that to be the norm of that function h and in the numerator we put down y+hε-JY then - dJyh and say if this limit is equal to 0 then this dJ so that the one that we write dJyh we say is the Frechet different, okay.

So when this Frechet differential exists meaning that this limit exists and it is equal to 0 and we are able to find it dJ then Gateaux variation is also guaranteed but other way may not be true so Frechet differential is more demanding because we are making the norm of the pre-function into

0 we are dividing by that norm and trying to make a limit vanish that is become 0 if you can define such a J then you have Frechet differential, okay.

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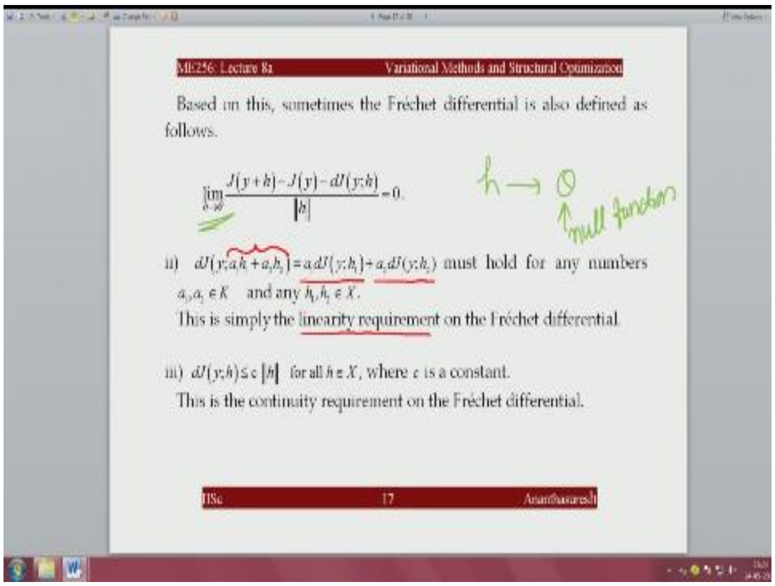

And there are some new answers for this that is sometimes this Frechet differential instead of making if you look at the what we have the limit here we are not saying the norm of h tends to 0 instead we are saying h tends to θ here, so if you recall in our vector space definitions this θ is the null function that is the additive identity in the function space or vector space okay, here we are saying that h(x) tends to the null function runnel vector a null function whereas the definition the ratio we are using norm of that, okay.

So that is the another way of defining Frechet differential and this Frechet differential unlike the two variation they are linear meaning that if there are yh1 h2 if you take we can make that as let us say we have a Frechet differential with me change the color okay, so here we have if you notice we have not just one h we have a1 h1 then a2 h2then you can write it as a1 times Frechet differential of h1, h1 is arbitrary function a2 times which is of h2 so this holds that means that linearity holds for Frechet differential and so it is for the gateaux variation because gateaux is being a first-order thing it is linear just like when we have a function f(x) when I take the gradient are multiplied by ΔX that is a first order term that is a linear term.

Similarly the functional that we get which is Frechet differential or gateaux variation they are linear okay, and there are this continuity requirement also how do you define continuity of a function in a for a functional we say that for the functional dJ defense less than or equal to C times the norm of the arbitrary function h(x) if such a C exists then we say the Frechet differential satisfies the continuity requirement. So it basically there is a parallel to normal gradient that we define that same thing we can do for Frechet differential and gateaex variation.

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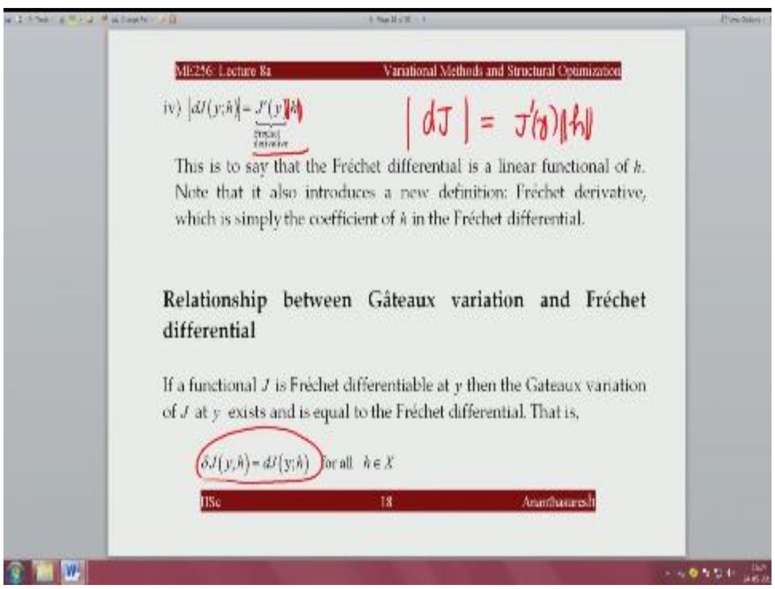

There is another thing called Frechet derivative this is another little term basically what you say is that we just define the Frechet differential dJ you take the absolute value of that because you remember J is a functional and Frechet differential is a functional gateaux variation functional you can take absolute value because that is basic real number that absolute value if you take if I can write that as J′(y) that is called the Frechet derivative and we have to take this norm of this h which will also be a number so here they should be and norm then that becomes what is called the Frechet derivative okay.

And as we said that when fresh a differential exists then that is also equal to Gateaux variation okay, we can have gateaux exists but not Frechet differential but when Frechet differential exists that automatically implies the gateaux variation exists and then there they both are equal.

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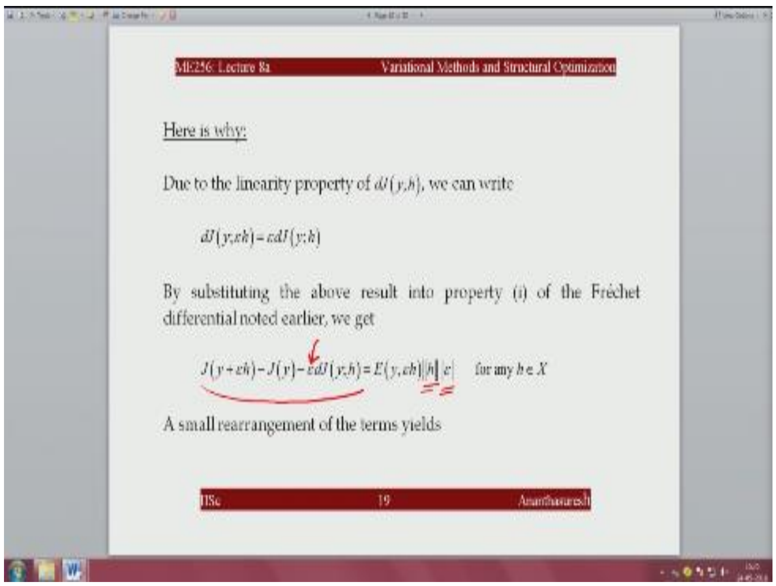

And there is a little reasoning as to why it is so, so all we are trying to say is that when you think of that limit that we used in the Frechet differential expression Jy+εh-J(y)-ε dJyh okay, then that is equal to the error term that may be there in it that e yεh times the norm of h and this absolute value of ε this is true this is one of the properties of Frechet differential, so in the definition of the Frechet differential between how the ε.

Now if you say included ε then that thing will not be equal to 0t when ε tends to 0 then we said divided by ε tends to 0 we said that is Frechet differential now we are saying differently this is one of the properties of Frechet differential where this quantity on the left side is equal to the quantum the right side that is this little error that will be there times norm of h times absolute value of ε okay.

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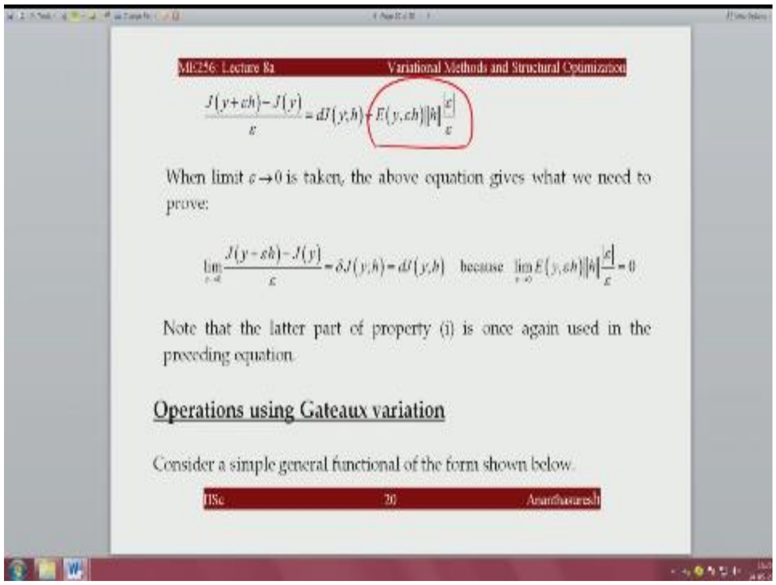

Now this one if we rearrange slightly so we got that ε that was there in the right hand side here and take this dJ over there mean then we get something like this, now as ε tends to 0 this particular thing goes to 0 that is this whole term goes to 0 then what we have this is nothing but ε tends to 0 is gateaux variation equal to Frechet differential okay, there are some operations that you can do with gateaux variation.

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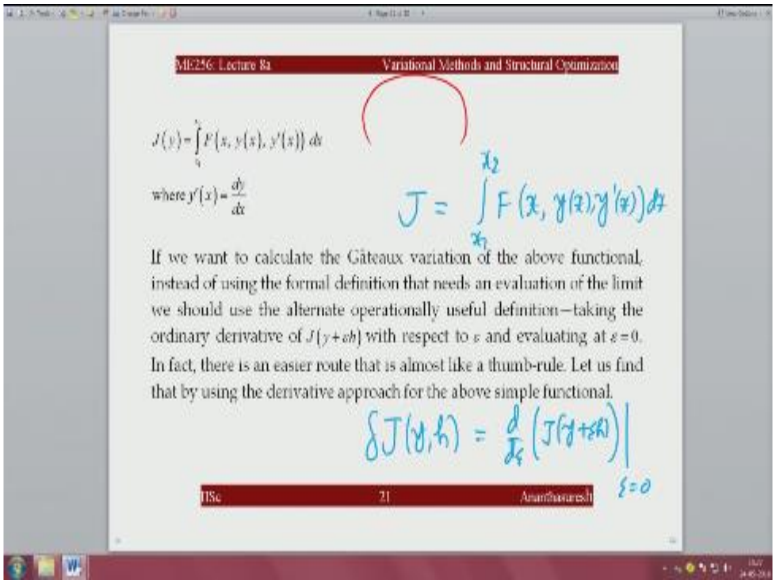

So here we are saying that when you take J a particular integrand f that depends on y(x) its first derivative and x okay, if you have something like that I have J I have this J I think it went into another mode looks like it okay, I have this J now it is from x1 to x2 this our functional F X also is shown but we do not have to show all the time y of x and first derivative of y dX okay, where y′/dx now we already said that know if you want to get go to variation that is δJ y and this h okay, this is y and h which involved that d/dε and all of that you know we say d/ ε of Jy+εh where ε tends to 0 okay.

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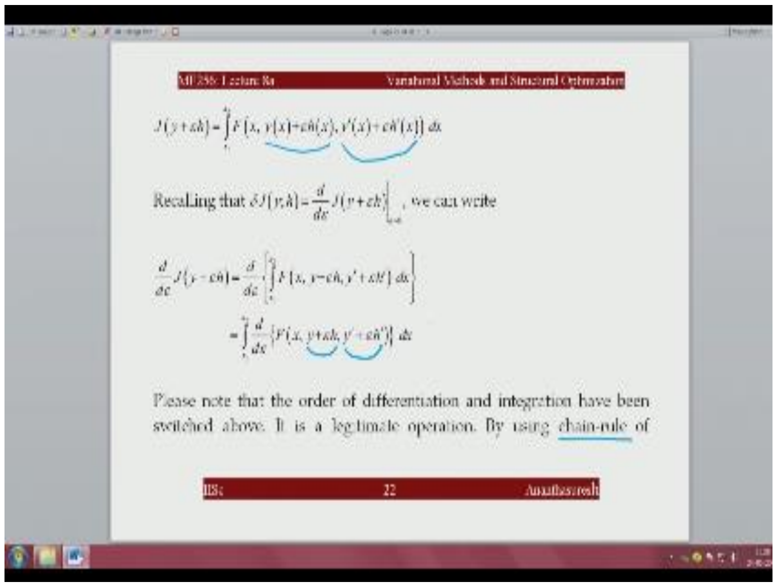

So if you want it more like a Monique or what sometimes people call Δ operation okay, you can just follow this now so where we had this function the f we have instead of y we replace with y+εh and y′ with y′+εH ′ okay now you need to take derivative of this and then substitute ε=0 with respect to ε now again what we have done here is we have interchanged the order this is differentiation and then we have integration what we are showing here is integration and differentiation when you do that so what we get here if you use the usual chain rule of differentiation over here because now this integrand depends on y plus epsilon H and there is also y prime plus epsilon H prime so we can write it as like this.

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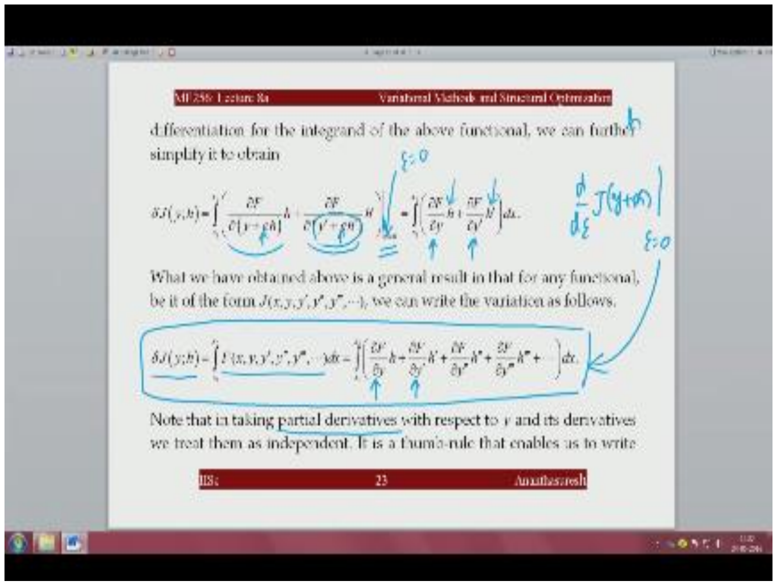

So here we have ∂ f by ∂ y plus epsilon h times h plus ∂ f by ∂ y prime plus epsilon H prime H prime okay so what is means when you substitute epsilon equal to zero so note that here we have subs epsilon equal to 0 okay then epsilon equal to 0 here and here it becomes simply ∂ f by ∂ Y times H ∂ f by ∂ Y prime time's H prime so we are simply doing chain rule again if you go back here so our integrand here when we are integrand is d by d epsilon of f where y is replaced with y plus epsilon X Y parameter plus y prime plus epsilon H prime we are taking derivative of that we are not worried about integration of we are focusing only on this term that is d by d epsilon of f which is now a function of we are not adding x because x does not depend on epsilon as yet epsilon h.

And then we have y prime plus epsilon h prime when you have something like this you want to take derivative you use chain rule so we have this function depending on this first so it is a ∂ f by ∂ of that times that function right and then this is derivative respect to that and then derivative of that if you do this what we get here we will do f by ∂ Y H ∂ f by ∂ Y prime times H prime okay we are taking derivative with respect to epsilon you remember when I take derivative of this with respect to this first we get ∂ f by ∂ y plus epsilon h into d by d epsilon of y plus epsilon h and then we substitute epsilon equal to 0 right.

When you do that what we get is will what will this be with epsilon that is simply will be H right that is why here we have h likewise this would be derivative of this F with respect to this and the derivative of this quantity that will give us this epsilon H prime so in other words if you just want to blindly do it the gutter variation you can simply be written if you have a integrand that depends on YY prime x w prime by triple prime and so forth you have to do like a partial derivative so here when I take ∂ f by ∂ y I should forget that F contains y prime y ∂ ble Prime and so forth I should focus only on why treating that as the variable everything else being constant.

Similarly when I do ∂ f by ∂ y prime I had to forget the fact that why also has why y ∂ able prime y ∂ able Prime and so forth and each of them you do it and then write in this fashion then this partial derivatives if you use that then you can easily write this variation it is exactly what we said is d by d epsilon of j y plus epsilon H where epsilon tends to 0 and the definition of the cultivation terms the limit or they are the same but this is easy to remember okay.

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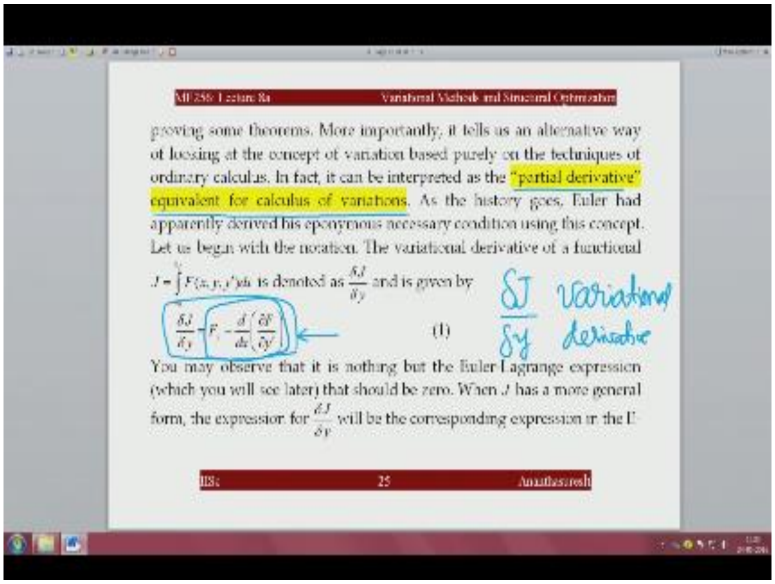

This is what you would need two calculations but you should verify that this is same as that that basically comes from this definition okay now there is one more concept called variational derivative which is also useful to discuss and this one right now you take it for granted this later we will see how this comes about Δ j x Δ y is called the variational derivative it is the equivalent of partial derivative for calculus of variations okay in other words when do we take partial derivative when you have multiple functions right.

Multiple variables you tables with respect to one that becomes a variation derivative which is given by a something that looks strange now that for you this one is going to look strange we will come back to this to see what this variational derivative is a variational derivative okay.

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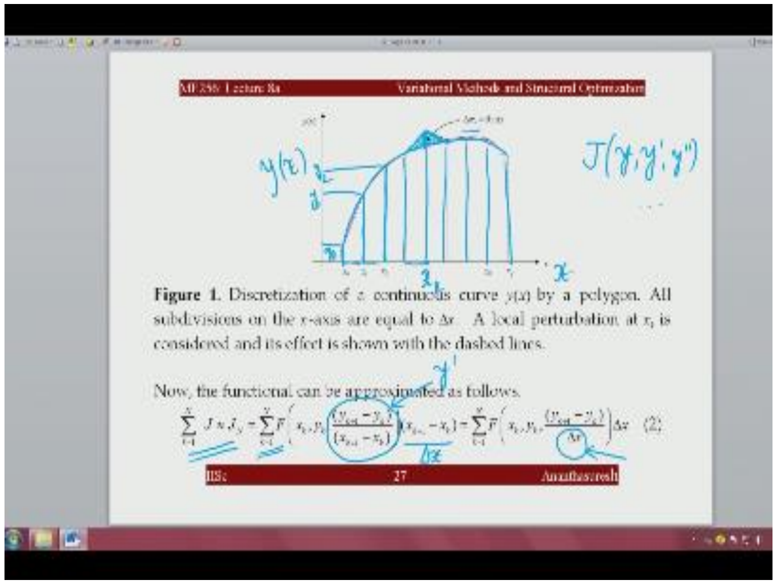

So this requires a little bit of discussion where we want to look at a function okay this variation derivative let us say I have a function y of X right so this is y of X and this is X I have some function what if I perturb this function only at one point let us say this is XK only there I have perturbed right then this function becomes like as if the string we are pulling it up so what was this function suddenly becomes this and this area that is shaded is denoted as Δ sigma k because I have part of the function at XK because of this what is the change in my functional is only a y that we are doing.

But remember that our functional depends on Y and its derivatives right how many are there may be right now if you take that and approximate that now then when you take this j and try to put a bit at let us say X K and you do that perturbation at every point okay you say that you do not know what this height this height this side you are discretizing when you do that and try to consider that the total change there you can write it as summation of this thing okay.

The function x x k plus 1 minus xk this is more like Δ X right that is a small interval that we are taking okay if you notice we have X K Y K instead of Y Prime what we have put here is y k plus 1 minus y k / xk plus 1 minus xk it is basically finite difference approximation of YK if you notice this quantity here it is nothing but Y Prime okay so I am taking this j integration being turned into summation over discretized set of interval instead of taking the continuous 0 2 or X 0 to X final XF we are making it a function at specific points and everywhere we are trying to see what happens if I put up a little bit okay.

And get this j changed j n r j knew when I do that I get this so we are replacing this y prime with YK plus 1 minus y k /xk plus 1 minus xk instead of taking different intervals xk plus 1 k minus 1and so forth if I make all these intervals equal to one called Δ X I can write it like this equivalent of an loss of generality here okay.

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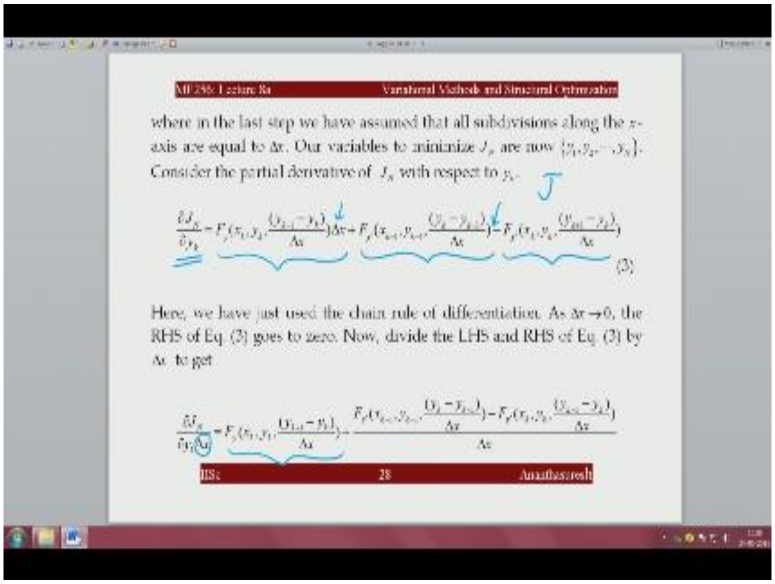

Now this particular thing we have to take partial derivative with respect to this YK so now what happened is that our functional J which was a function of the continuous function y instead of that we have y 1 y 2 YN which are the heights at a put this is y 0 this is y 1 this is y 2and so forth right this is y 0 over here this is y 1 this is y 2 and so forth I can take derivative of this because you want to minimize that functional with respect to that is a particular it is a normal partial derivative okay if I do that in this summation there will be lot of terms as k goes from 1 to m okay.

So if I take their with respect to Y k we will be left with only three terms this is the first term where we have y k and also in the Y prime that is if you look at this we have this approximation is y prime this quantity is y prime we take respect to y q first and then we take this back to y prime in which YK is there and then YK plus 1 also will exist when I take when I take respect to k minus 1 k is equal to k minus 1 so if I take that that will give me two terms this has a minus sign here because now we are taking derivative respect to X K this is K minus 1.

When you take that I had k now when I take at k that I will have this minus y k that gives me the minus sign over here whereas this is plus sign here we are going to take it k minus 1 this would be y k okay so basically we have to look at all the terms or think about all the terms in this summation and try to understand that we get three terms take derivative respect to Y K now this one I am dividing by Δ X okay so I am dividing by Δ X there is this Δ X okay that goes away this term will be without Δ X and these two terms will have Δ X okay.

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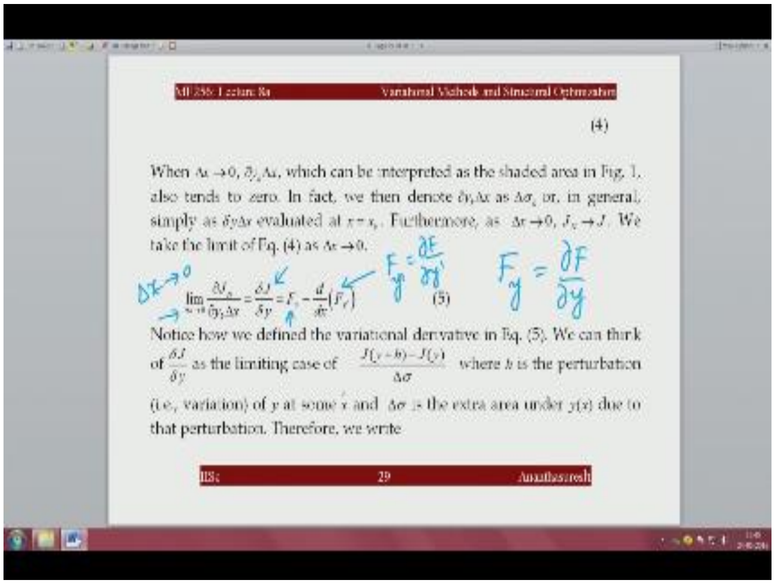

Now we go to the next level okay now if you take this quantity and talk about what happens when I make this Δ X go to 0 and we are taking a limit where Δ X is tending to zero okay then what we get is called the variation derivative right so the variation rivet the way we get now we see what we get as FY what I mean by FY is that when I right why our notation is that it is ∂ f by ∂ Y similarly here we have F Y prime while that I mean is ∂ f by ∂ Y Prime how did you get his equation 5 it actually is nothing but what we have over here okay.

If you see this one this is FY meaning ∂ f by ∂ Y and this one this is FY prime and then F Y prime this is that XK minus 1is X K divided by Δ X basically this, this becomes d by DX of F Y Prime in other words this is d by DX off by ∂ Y prime okay so that is what we have here so the directional derivative IS F of F sub y that is ∂ f by ∂ Y minus d by DX of FY prime so this becomes our variational derivative so at every point if you want to know at a particular point if a part of a function what will be the change in the functional is nothing but this variation derivative by the way what we have done in the last couple is how Euler had derived what we now call Euler –Lagrange equations okay.

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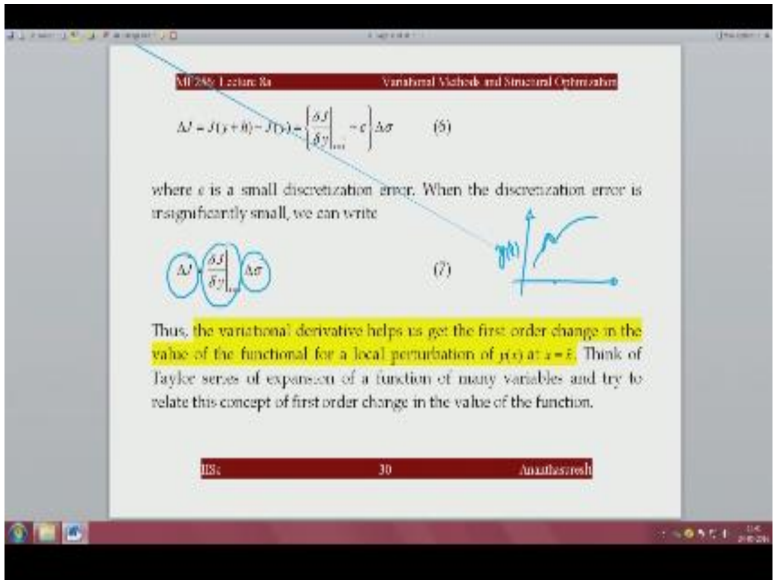

So this variation derivative again Δ sigma if you are call what we said if I have a function okay at a particular point if a perturb it slightly okay then I will have little area I am pinching that right if I take this function and I pinch only one point I will get like this right the new function so new function is going to look something like that we have pinched it there now because of this by the way this is our y function so this is our Y of X this is X right this is our X okay.

Now when I pinched it our particular point that area is Δ sigma that x the variation derivative will give me the change in the functional itself okay so this also helps us derive what we will call oil Lagrange equations but this one is variational derivative okay so what we have discussed in this lecture or several one is go to variation and then fresh a differential and then we had some niggard left fresh a derivative and then we discussed what is called variation derivative all of these are concepts required to establish necessary conditions for calculus of variations.

Where we say that first order term should vanish which is what we will discuss in the next class so that we can do routinely lot of examples so that you can understand how to arrive at the necessary condition and use it to find the function that minimizes thank you.