Indian Institute of Science

Variational Methods in Mechanics and Design

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Hello so far at least in the last lecture we discussed in mathematical flummeries for calculus of variations today we jump into calculus of variations that is to say that we will establish necessary condition or there are the groundwork for developing the necessary conditions for the minimum of a functional okay in normal calculus you have functions and you have gradients or derivatives if f of X is a function DF / DX is its derivative or if F (X) a function is dependent on X and Y then you have partial derivatives that is ∂ f / ∂ X ∂ f / ∂ Y so ∂ f / ∂ X if you take X component vector ∂ f by ∂ take the Y component vector you get a vector in three dimensions.

Let us say a function is dependent on X Y and said then you will have ∂ f / ∂ X ∂ f / ∂ Y ∂ f / ∂ Z those three if you take as three components XYZ week I am using right hand so that is correct XYZ if you have then you have a vector that becomes the gradient in two dimensions we have only x and y so between those two you will have a vector based on some magnitude right so that is your vector in 2d do one dimension just have derivative.

That is what we do with functions but what if you have functional which we discussed in the last few lectures a functional is basically a mapping from a function space to a real number a scalar right functions we have a function where commit real number now a functional that is a real number if you say that is a minimum then we have to do some perturbation we have to see to first order what is a change for that we need a concept that is equivalent of taking a derivative of the gradient and that is called variation so variation is the derivative counterpart for calculus of function for calculus of functions normal calculus.

You have derivative for calculus of variations which is functional calculus we need equivalent definition and that is the variation in particular that variation has a name associated with per centum Gateaux and that is what we will discuss Gateaux variation or a first variation of a functional so let us look at the notes that are also available along with this video lectures if you look at the nodes and I will highlight.

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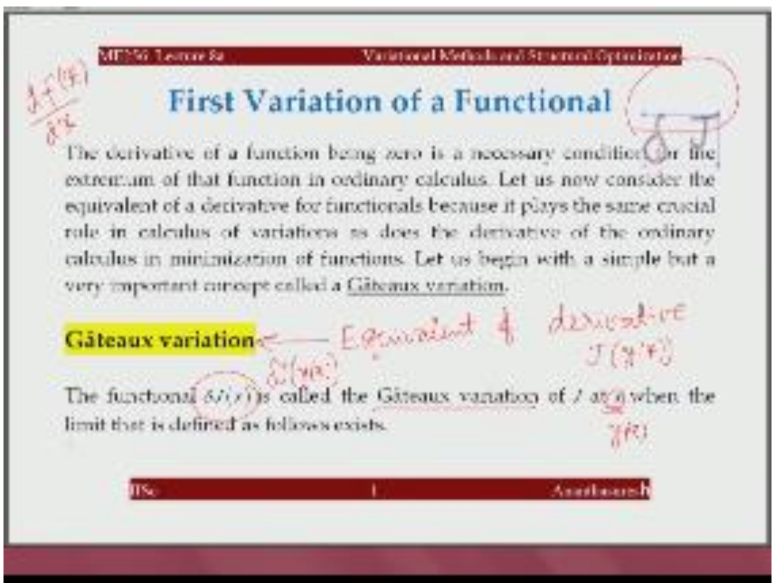

A few points as we go along today this concept of the two variation okay that is equivalent of derivative this is the equivalent of derivative that is there in the normal function calculus okay so what is this Gateaux variation first of all it is denoted like this if j is a functional if j is a functional we put λ there that indicates that this thing is the variation of j just like we have a function f of X to write derivative we say DF / DX right in a similar way we write λ J okay that is our Gateaux variation okay now what does it say go to variation here of j at x when I say X it is actually a function ideally speaking I should have put J as a function of Y of X in which case it will be at Y of X.

So if I say my functional j is a function of Y of X and its derivatives right then ∇ J of Y of X will be variation of j at y of x okay let us not get confused here in the nodes x is denoted as a function or a vector space that we talked about here if I switch back to y of x then ∇ l J Y of X right ∇ J Y of X variation of j where y is the variable with respect to that function at a particular function is define in terms of a limit that we will see next okay.

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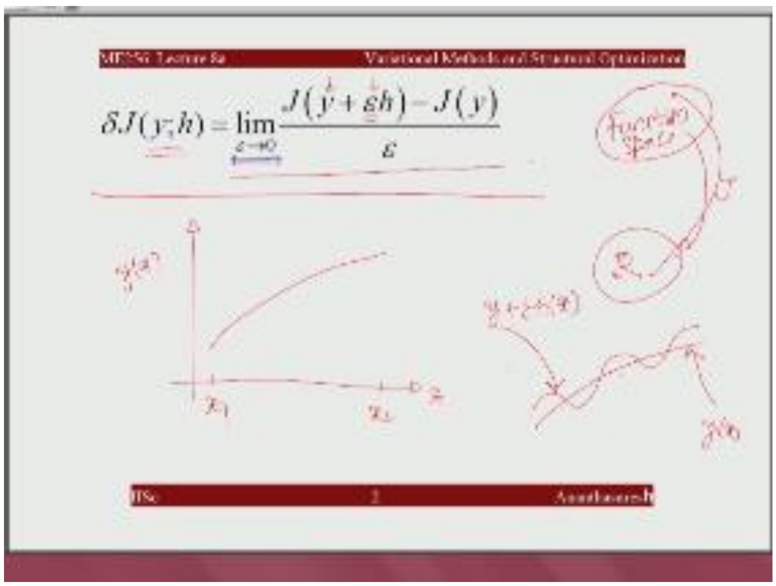

This is what it is now I have changed it back to Y of X not like that so we have this functional we have the function space here this is the function space many there are lot of functions in it that is going to be mapped by this functional J that we have okay, to a real number space here are so if I take any function here there will be a mapping of a real number here that is what J- right now at a particular function let us say this is my Y of X right at that particular function if you want to compute the equivalent of the gradient or what we call the variation that is defined like this okay.

So how is it defined it is limit some ϵ tending to zero that ϵ appears here so we have taken y of x if I were to plot it and say this is X and I am plotting x of x here from some limits x1to x2 if I have a function like that what I am doing here that I have y of x but I am adding something adding h of X let us say I define h of x in some particular manner okay, so this H of X that we have okay if I add over it okay, so if I let us I have a function like this my age will be like a perturbation over it right that is h but then I am putting a small value ϵ divided by it.

And making an ϵ 10 20 so if this is my Y of X the squiggly line over it is y plus ϵ times H of X where H of X is an arbitrary thing you can make it whatever x a small number because that is tending to 0 ϵ tends to 0right that is how got to variation is defined mathematically okay.

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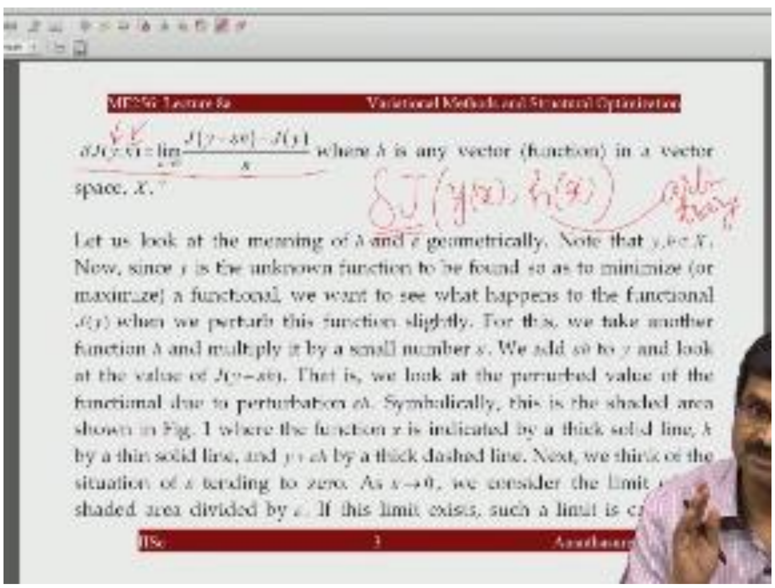

So let us recapitulate once again that the two variation is now defined at a particular function just like derivative of a function will be different at different values of x the two variation is also different functions okay because now we have function space Y of X that is what is variable now so as we vary that for different functions there will be different car two variations different first derivatives okay like if u mass in the brackets token problem between points a and B you join with a curve that curve will have a variation.

If I take another curve that will have a different variation so that you variation depends on the function but it is also defined in terms of a function H so the dirty variation we write λ J at some y of X but also there is H of X that is what you said in all these nodes here right so we have the function y of X we also take a function that is similar to Y of X but a different one in fact this part is arbitrary this is arbitrary that can be anything so Gateaux variation has embedded in it sometimes arbitrary.

That we do not have to worry because there is a way to get out of that arbitrariness okay but definitely got a variation involves this age whenever we write λ J later on we may not write y and H explicitly but implied that it is at a particular fine and it is with an arbitrary function H because that H is their only ϵ is tending to zero nothing has happened to H it is going to stay on okay.

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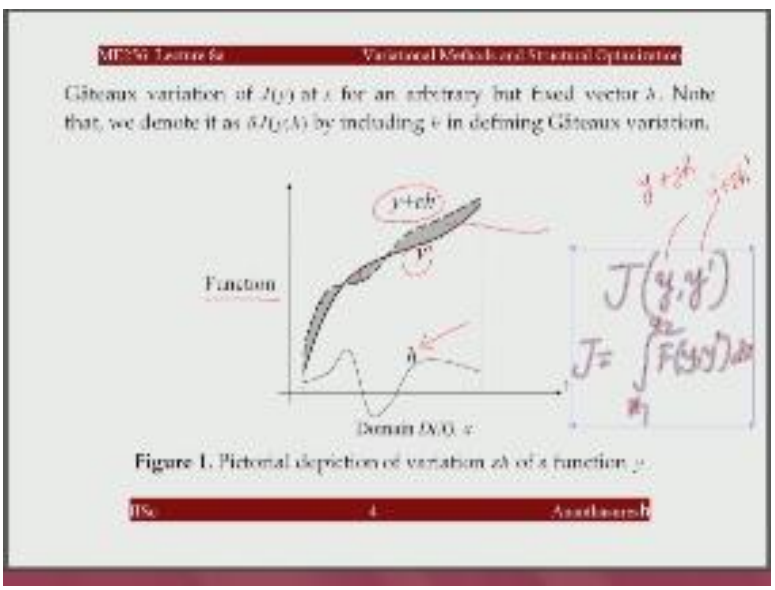

Here is graphically what I had said earlier so we have Y here that is a solid curve and then we have y + ϵ h which is the perturbed one now you can see how H is over here okay h is also a function like why only thing is this H is multiplied by a small number ϵ and then added to while okay that is what is the perturbation between them this shaded region is the kind of variation that at each point so at this value of x this is the variation at this value this is the variation and so forth okay.

So we have this variation ϵ H and ϵ tends to 0 what happens to the functional so this is where plotting only function but then we have J which depends on Y and its derivatives and soon right because J if you take it an integral there is an integrand that y and y prime and other things okay which is from some quantity X 1 to X 2 so that is dependent on why and how does it change if Y is changed to y + ϵ H and ϵ tends to 0 when y tense becomes y + ϵ H Y prime will become y'+ϵ h' right now what happens to this J when we make this ϵ 10 20 is what we want to discuss okay.

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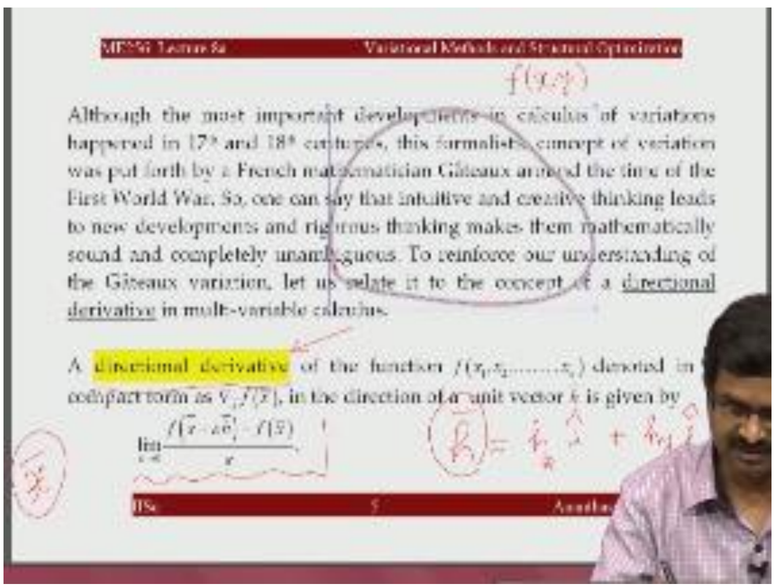

This concept of got a variation which you defined in terms of limit also has an analogy in calculus of functions not our calculus of variations but calculus of functions which is to say the directional derivative let us say we have a two dimensional problem that means we have function of x and y right at a particular point in a 2d domain let us say over this text I am going to draw something right let us say this is our two-dimensional domain okay so these are two dimensional domain inside that okay if I take a particular point there will be a vector gradient ∂ DF ∂ f /∂ X ∂ f / ∂ Y right that will be the gradient.

But does not tell you how the function will vary in a particular direction that you are interested in let us say you take a direction H okay that is now each will have let us say a component HX in I that is X direction hy in the Z direction if you have something like that in that direction you want to know if I have a function at a particular point okay how does that vary that is given by this concept of directional derivative.

That is defined in this manner that also with anything that is a gradient is always defined in terms of limit here limit ϵ 10 in 20 f of X + ϵ times H here H is not arbitrary although you can ask the rate at which a function changes in any direction in that sense His arbitrary because you can ask in any direction I can ask any direction but His given right in that direction if you want to know how the function changes that is given by limit ϵ 10 20 f of X plus ϵ times h -f of X /ϵ.

Here then input x over bar like we are doing that means that it has x1 x2 x3 many dimensions the problem can have f can depend on n variables right so in that case this is the directional derivative if you look at this and compared with go to variation you will see that it is similar right we have chosen that H of X we have Y of X which was in H of X you have f of X we have taken H here X as a vector then we have h also as a vector if we take Y of X.

Then you have h of x also a function and that is we should look at it that is if we have a functional and you have chosen a function y of X then if you say what is the rate of change of that functional with respect to that Y of X that is a go to variation as it is shown here duration derivative is also similar one direction you choose H what is the change rate of change in that direction it is given by this definition of the directional derivative.

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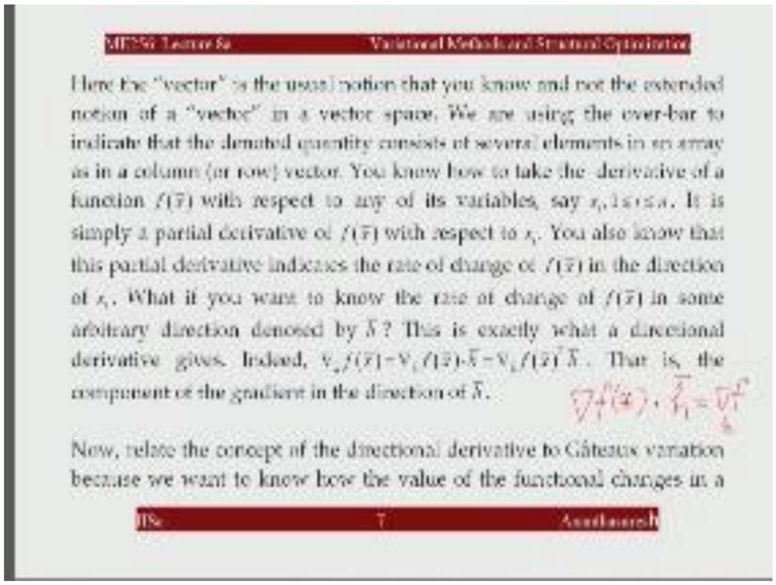

So this directional derivative is a gradient that you take off the function that is we write λ of FX right if I take dot product of that with the direction that you take because gradient is a vector this will be the directional derivative of F in H okay, so you take a function and you take its gradient dot product with H will give you the directional derivative that is similar to go to variation in calculus of variations okay. There is something to remember okay.

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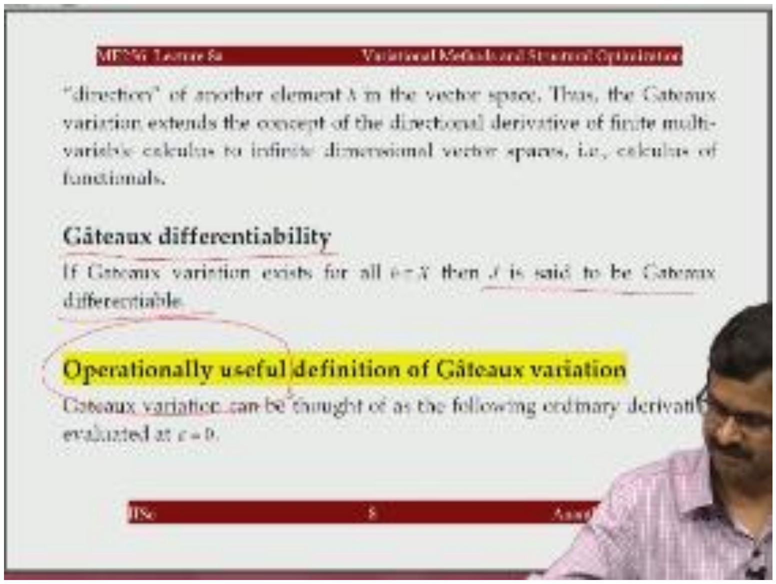

So far we have discussed the equivalent of derivative or gradient in calculus of variation which is greater variation now we introduce a few more terms something called Gateaux differentiability we may not need it for this course but it is good to be familiar with some of these terms if Gateaux variation exists for all H that belong to X then X is our functions vector space now H is arbitrary but if you take any h and take this limit that is the definition of Gateaux variation if it exists.

Then that particular functional J is said to be go to differentiable some definition okay, this limit business is always a tricky thing because every time if you want to find variation you cannot go back to limit and work out all these things that will be inconvenient while the definition cultivation we discussing using limit concept is rigorous and correct we can use a simpler way of defining that which is equally correct and equally rigorous but also operationally useful meaning that we can do operations with it like we do differentiation without even blinking your right.

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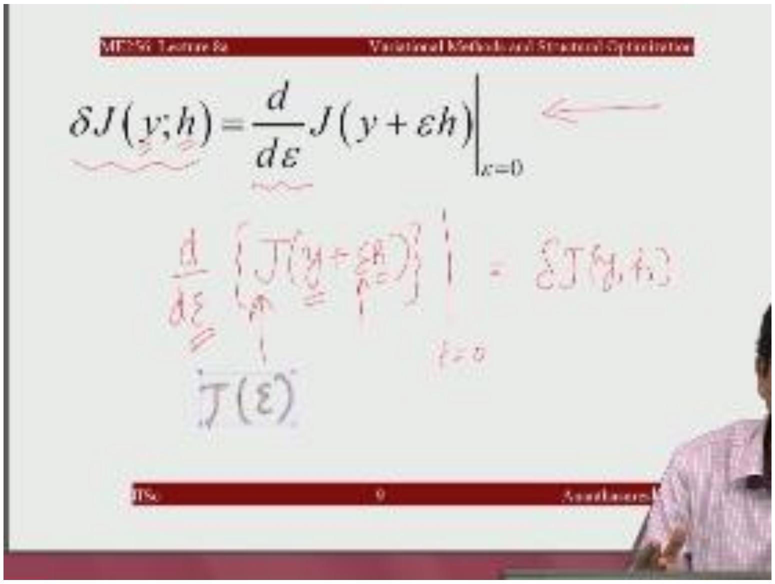

Okay so that one is what is shown here this is the same go to variation which is defined in terms limit now it is brought into the realm of ordinary calculus what do we have here are the redelivery whatever taking derivative of we are taking derivative of j y + ϵ h okay it is a j is a once you are given Y it gives you a real number now we have y +ϵ H meaning wherever why is you replace with y + ϵ H why you are taking their and age it is given to you you have chosen arbitrarily what is the variable ϵ.

Now this particular thing we take derivative with respect to ϵ because that is a function of ϵ because in the Gateaux variation why is taken H is taken so what is variable which is only this ϵ right so we take vary with respect to that and then we know in the limit definition we have ϵ tend to 0 substitute equal to 0 and that becomes the Gateaux variation using this age as arbitrary function this is much easier.

This is equivalent to whatever derivative limit that we have defined this is operationally useful definition of Gateaux variation so because it is useful or convenient because it has been brought into the ordinary calculus okay, so you do not have to worry about what the form of this J is you just look at that J as a function of ϵ because Y and HR given to you do not need to worry about them because you are taking derivative only with respect to ϵ so treat this as a function of only this ϵ we can take not normal derivative and then simply substitute $\epsilon = 0$ and you have meta variation okay.

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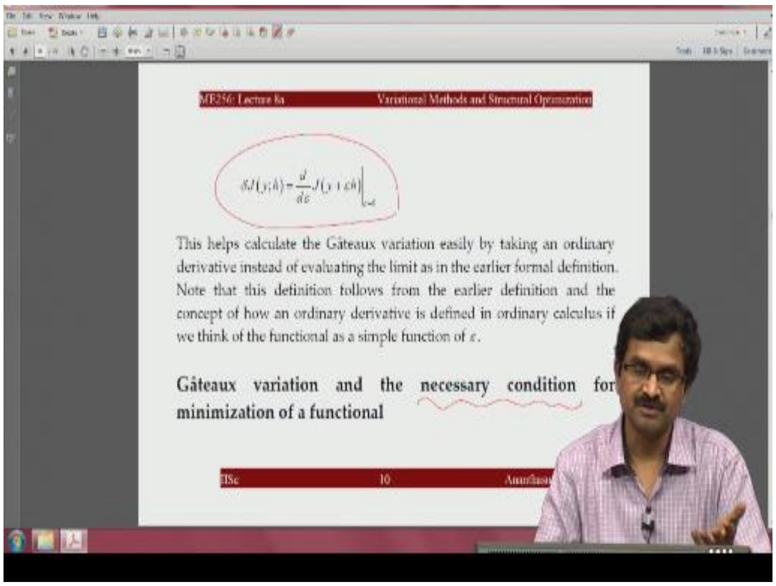

So this is something that we need to understand really clearly this operationally useful definition of go to variation okay, now let us come to this necessary condition that is what we are after one the moment we define derivative we were able to establish the necessary condition for the function that depends on a single variable or multiple variables similarly now for a functional which depends on function now we establish the equivalent of derivative which is the Gateaux variation so we can establish a necessary condition now okay ,for minimizing the functional all right.

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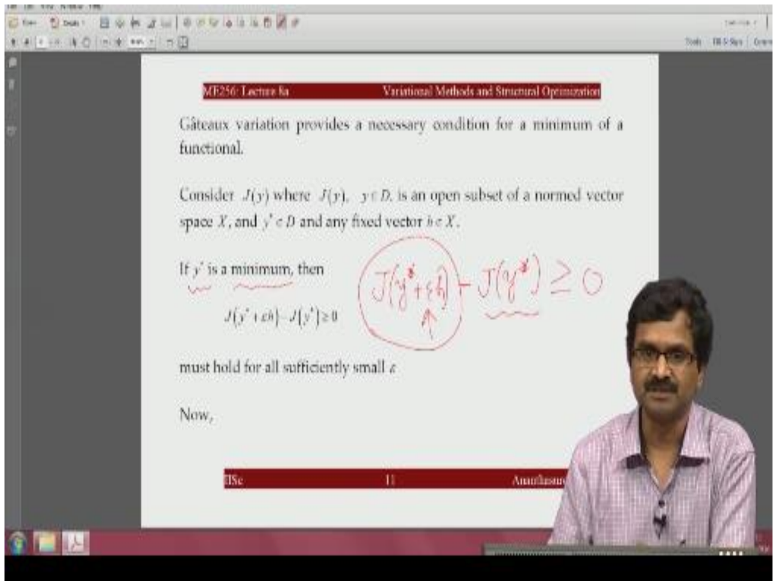

So like we had discussed in the case of functions that somebody tells me a function is F of x and when I want to minimize at a particular x star it is a minimum then what do I do that at x star we portable slightly and then say that to first order there should not be any change because first order change can be positive or negative so the first order term can become passed over negative and we cannot see anything about that function being minimum there so we say first order term should be 0 similarly here we say that first order term should be 0 our starting point is first say if y star that function what to be a minimum then after perturbing.

So we have the functional as a function of y star but then we will have another one after perturbation y star + ε times h the difference of this if really J y star is the minimum value any other perturbation should be larger so we say J y star + ε H- J y star should be greater than equal to 0 that is the definition of a minimum. And the local minimum when we say right then we say for sufficiently small ε what is that change that is how do you expand this right this is we know we can compute it how do you expand this with this thing.

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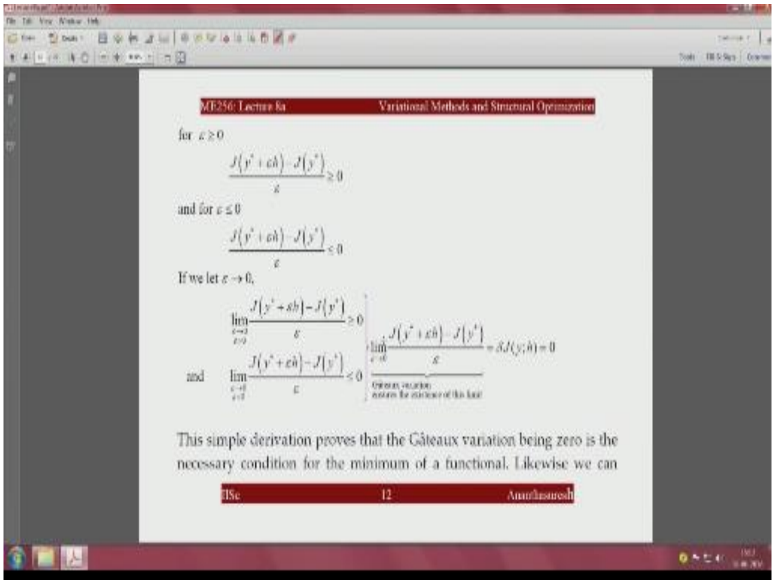

If you think about that then we can write like a first order terms right so what first we are saying is that we take that whatever we want right.

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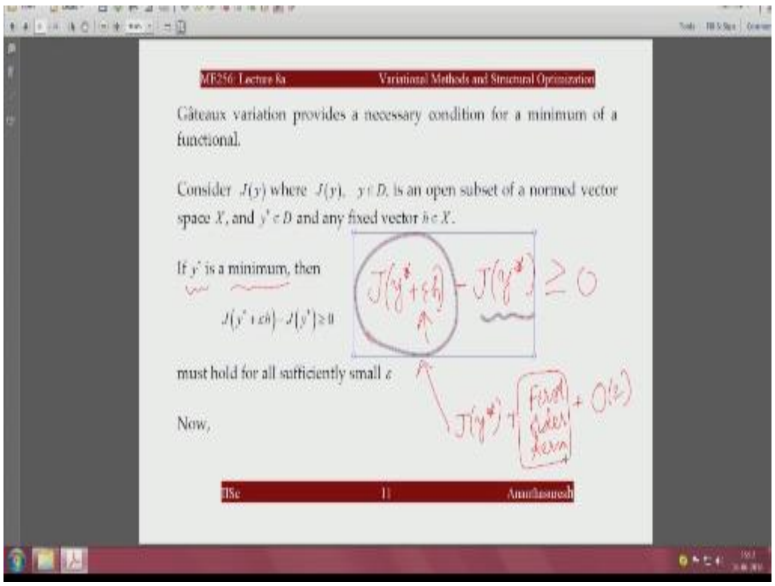

So when we expand it there will be a 0 order term so what we have what is N circle now we have a we divide 0 order term which will be J y star itself right and then we have the first order term first order term and then we have second order and others what we read O 2 write so J y star J y are cancelled so that all depends on this thing right so that is what we will get to and show that this is nothing but got a variation.

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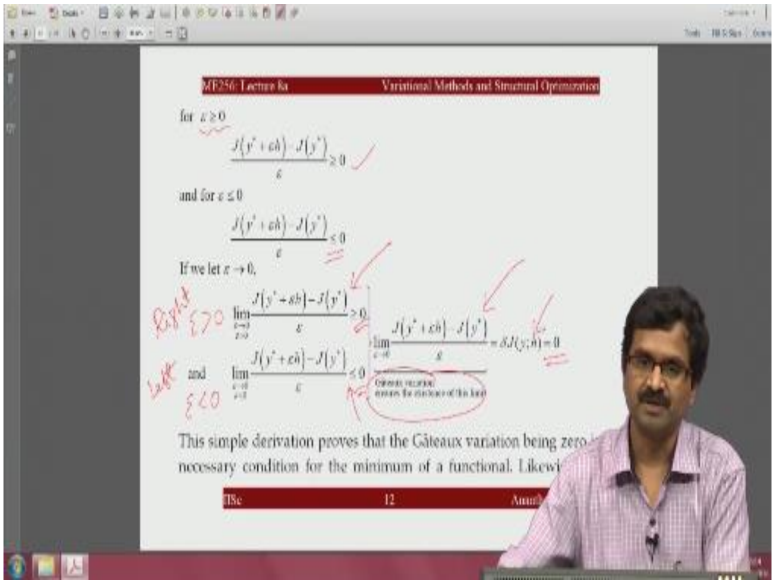

So what we do here is that for ε greater than = 0 we want the numerator part of it to be greater than = 0 if I divided by ε where ε is greater than 0 this is true when ε is less than = 0 this portion will be less than = 0 because the numerator is positive denominator is negative or non positive it will be less than = 0 if we let ε10 to 0 this one and this one here we are saying ε is greater than 0 here ε less than that is we are approaching 0 from the right side.

That is ε greater than 0 and then second one here we are coming to the limit from the right side and here it is the left side we are coming this way this a is greater than = 0 this less than equal to 0 what should it become you should become little equal to 0 right that is nothing but our Gateaux variation that we have that definition of you variation which is to say that first order term is = 0 because derivative is = 0 is the first order term for normal functions for functional the two variation is = 0 that is how we establish our necessary condition you see that we still have that arbitrary edge that will get read of later on .

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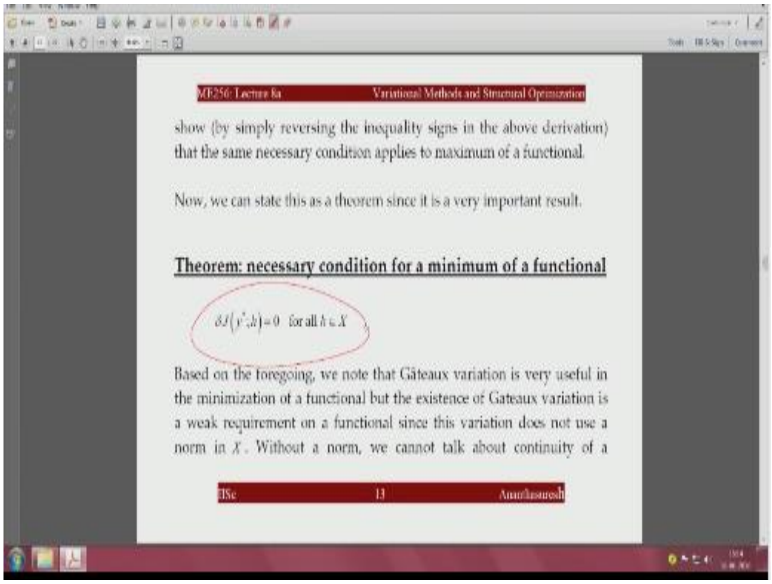

Which will be the next lecture so if you want to state it as a thermo the necessary condition for a minimum of functional is this that is cut variation is = 0 okay.

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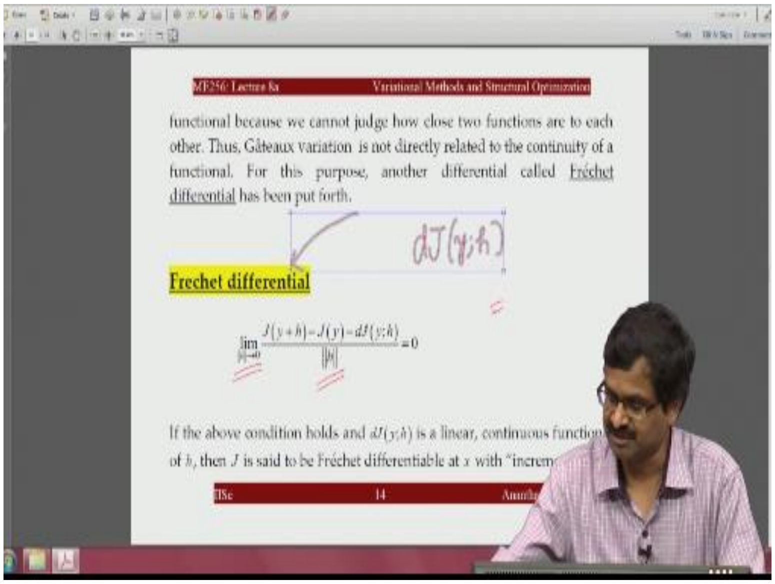

And there is another concept called freshet differential okay, which is a little bit i would say more advanced than got to variation because if you look at this definition okay, in this definition this d J y h if we compare this with go to variation where instead of using d here that put Δ like this right here it is just d so his d J is pressure differential okay, that is this freshet different shell okay that is defined not in terms of ε it still uses h okay, but we are putting the norm of h in the denominator making that norm of h tending to 0 if you do that if this is = 0 that is limit norm of h an arbitrary function h of x there is a norm that we had discussed earlier.

When that Norm tends to 0 the numerator J Y + h - J y - DJ , Y h go to pressure differential divided by norm of h if that limit is = 0 then this DJ is called the freshet differential or it is a little bit more rigorous because we are involving the norm of h and not just taking ε variation okay.

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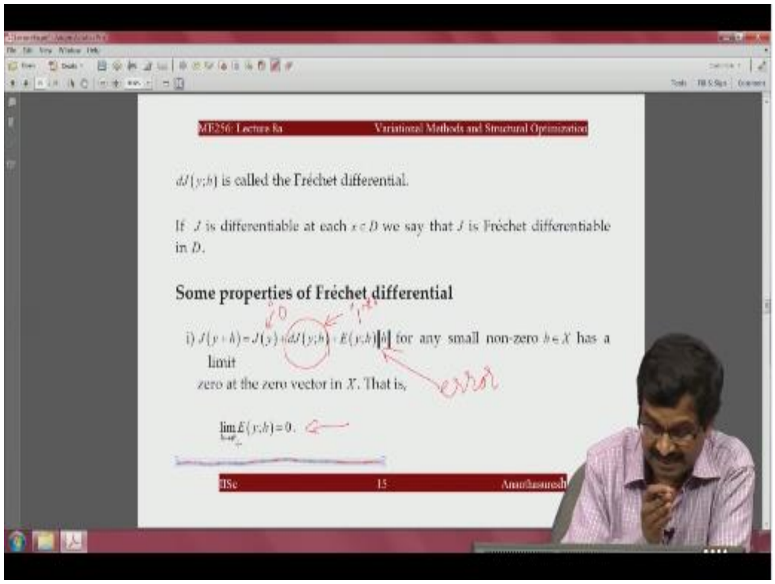

So this has some interesting properties basically to say that if you look at this these are 0 order term J y + h that is I have a why I am adding another function h then there is a perturbation the functional value J that is a 0 order term and then we have this which is the first order term is the first order term right and then their higher order or the error term.

So the third one that we have okay this third one is error always be error like a Taylor series expansion equivalent here right that is freshet differential okay, and this error tends to 0 as h tends to theta θ here is the null function every function space will have a null function like 0 function when this age that is arbitrary tends to the null function this error tends to 0 that is what this says those of interesting properties.

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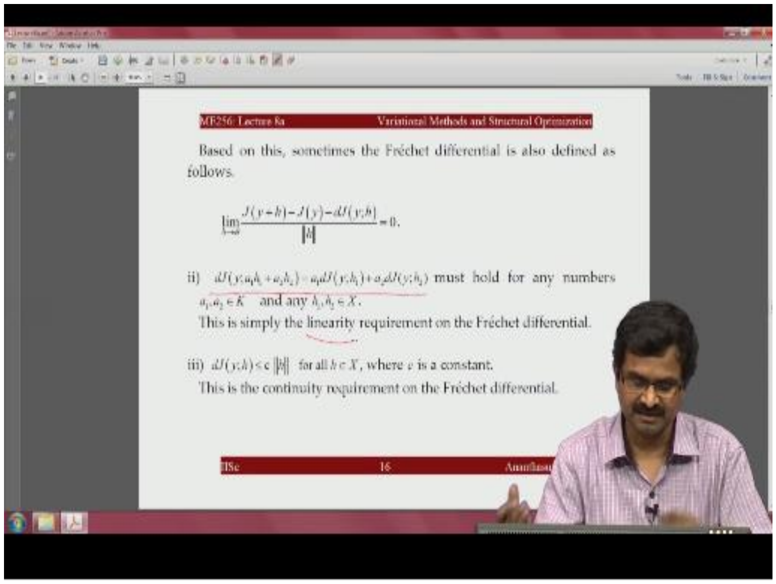

Which you can go through in the note that is going to accompany this video lecture for example this pressure differential is actually linear meaning that you can do operations like that so basically linearity it is a first order term it is linear understand that much okay.

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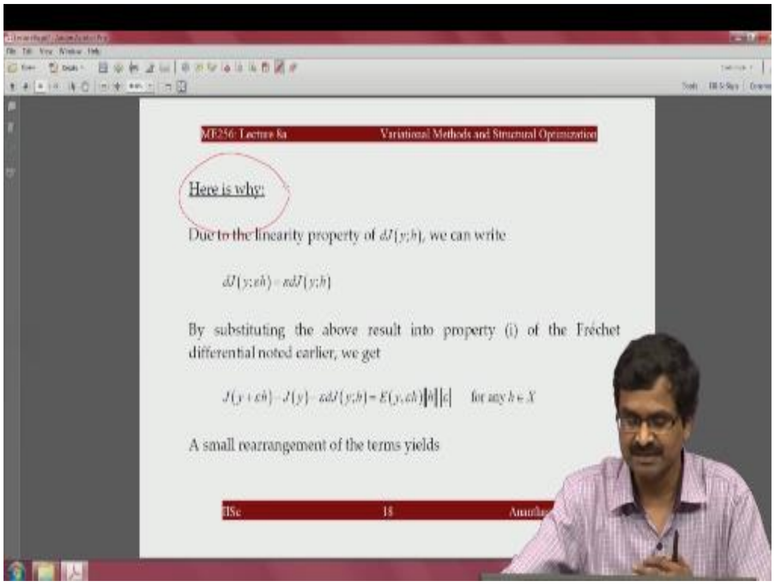

And there is a relationship between Gateaux variation pressure differential relation piece that when pressure differential exists that is + Gateaux variation okay, that that is what it says and there is a way to prove that and you can go through this in the nodes to see why that is the case just a systematic argument to say when pressure differential exists that is equal to Gateaux variation but what we need for our purpose is only Gateaux variation but we just mentioning some of these so that you are familiar with them okay.

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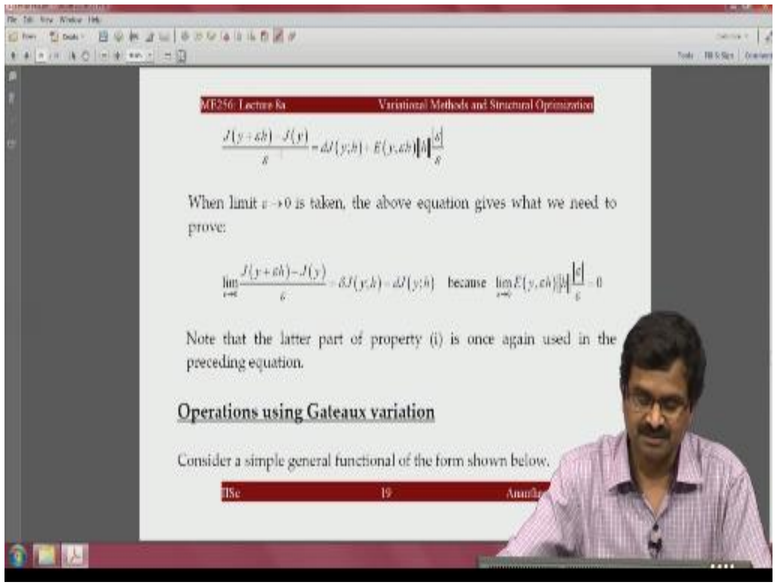

Some operations that you can do with Gateaux variation.

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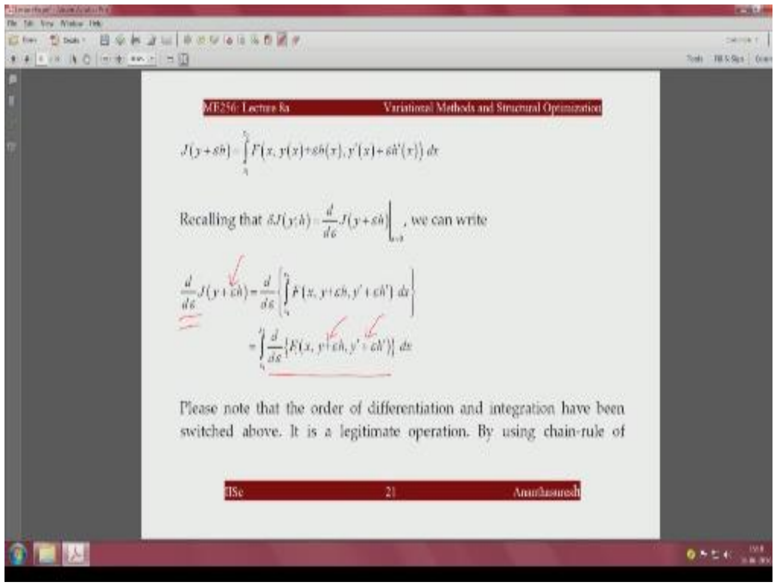

Which you are going to discuss later so again you have A functional whose integrand depends on y of x and also y prime of x we can actually do this you know d by d ε of J Y+ ε h if you do and treat ε a variable that is a variable then you are doing this right d by d ε F of where y is you have substituted Y+ ε h where y prime is you are substituting that with y prime + ε h prime you take derivative okay derivative.

You use normal calculus rules for different creation so here F is a function of Y+ εh so you have to do the chain rule you take derivative respect to Y+ ε h and then take derivative of y plus εh with respect to ε okay that is what we are doing this operation.

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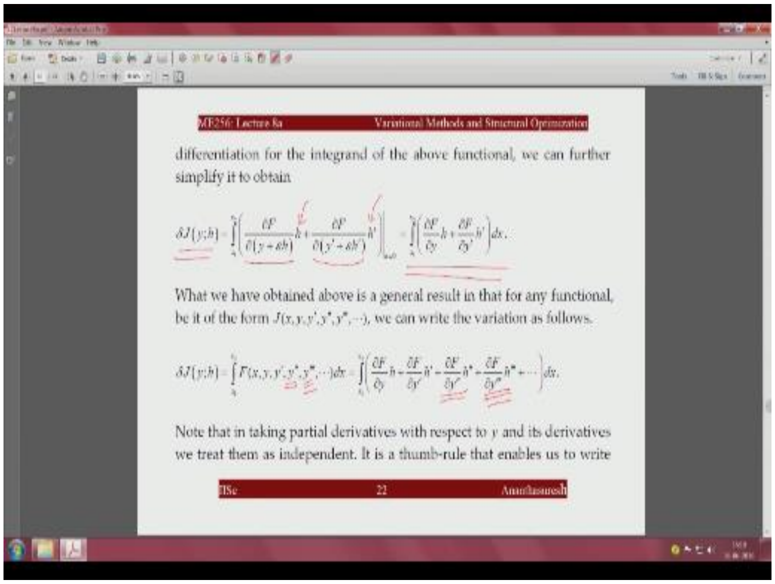

So I am taking derivative with respect to y + ε h and then this is derivative of y + ε h with respect to ε that is a chain rule okay similarly taking derivative respect to y prime + ε h prime taking derivative of y prime + ε its prime with respect to ε that gives you h prime okay so the variation if you want to just write it like that you can do it some people call it Δ rule or Δ property or Δ differentiation that is just a way of remembering but this lecture you should be attention.

Because it comes from a concept of go to variation otherwise if you just know how to write this like that that is good enough you are taking ∂ F by ∂ y throw F by ∂ y prime h Prime similarly if you have y double prime in the integrand then there will be do F by ∂ x ‘’ h’’ /’’’ if you have this will be ∂ F by ∂ y triple prime time’s h triple prime. But it is following from the Gateaux especially the operationally useful definition that we discussed so these operations you should be comfortable with okay.

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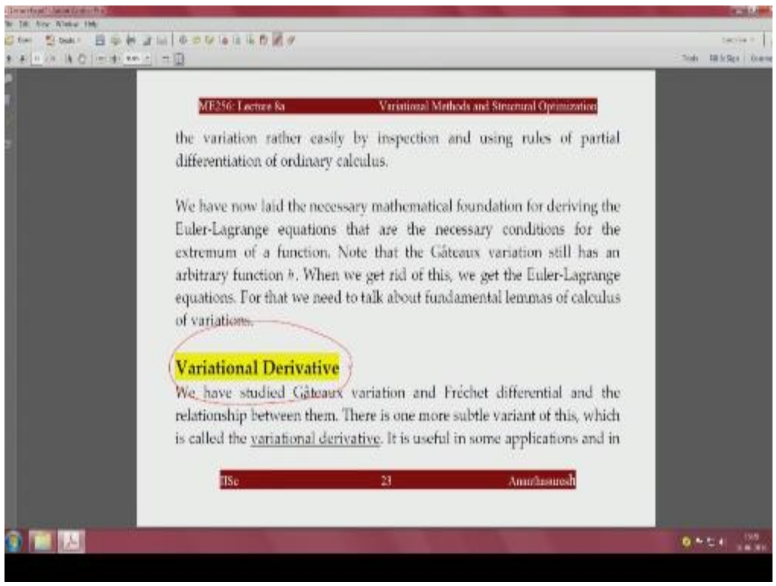

There is another very interesting concept of variation derivative we discussed the two variation and fresh a differential now the third one variation derivative okay, which is very important and in fact this is how the history says coiler had coiler had derived his equations for necessary condition okay what he did again you should read the nodes to understand the details.

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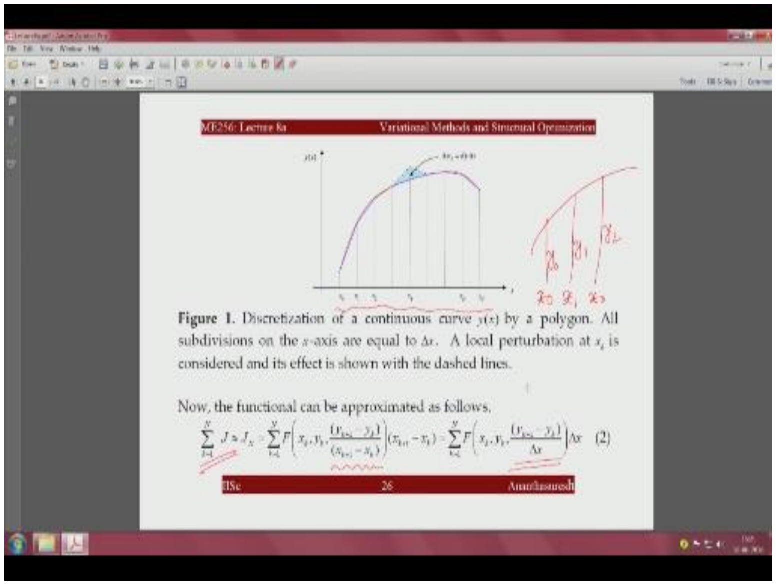

But he had simply used the notion of partial derivative okay, how did he do that what he did was if we look at this graph we have an unknown function okay you pretend that it is not unknown function you are discrediting like here X0 to X n and you say the hides are not known if I say x0 I do not know what that height is x^1 I do not know what that height there I can call it y0we got y1 those heights are not known if I take another point x2 I am not know why too and so forth he basically discrete 'as when it is criticized the integration.

That you have in the functional becomes summation right and then derivatives are represent in terms of this discrete values as a finite difference one so everything is now in terms of y0 y1 y2 which becomes finite variable optimization that is the kind of thing here taken so you have the derivative expressed in terms of finite difference values in terms of the unknown wise y0 y1 y2 and so forth he takes this partial.

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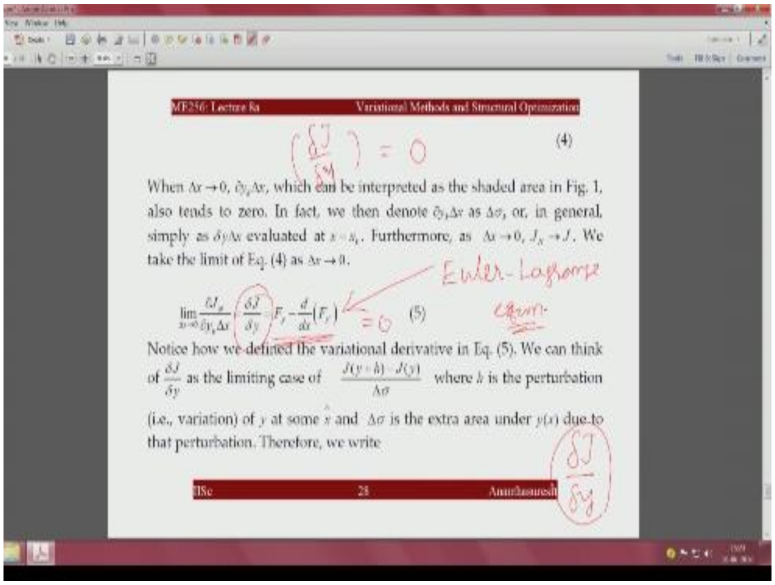

Derivatives makes it equal to 0 okay it is a very elegant derivation so he takes it all with partial differentiation alone and establishes a condition okay that we today call Euler -Lagrange equation okay which we will do in the next class using the Gateaux variation concept and all of that but Euler had done this very effectively using only partial derivative concept by discrediting and making that limit 10 20 then you will get what is called but this particular quantity today we call it a variation derivative which is nothing but a Lagrangian equation I say that should be they should be cool and something should be equal to zero.

We say this is equal to the necessary condition ok but his quantity that we write this Δ j Δ y see today we discussed Δ and J which is got a variation DJ which is pressure differential now here third one variation derivative which we derived for the case of an integral depends on y and y prime but later the next class you will see whatever we get this Euler-Lagrange equation the left side of it equation will have a right hand left side in the case I Lagrange equal right side is 0 whatever is the left side is nothing but this variation derivative that is something.

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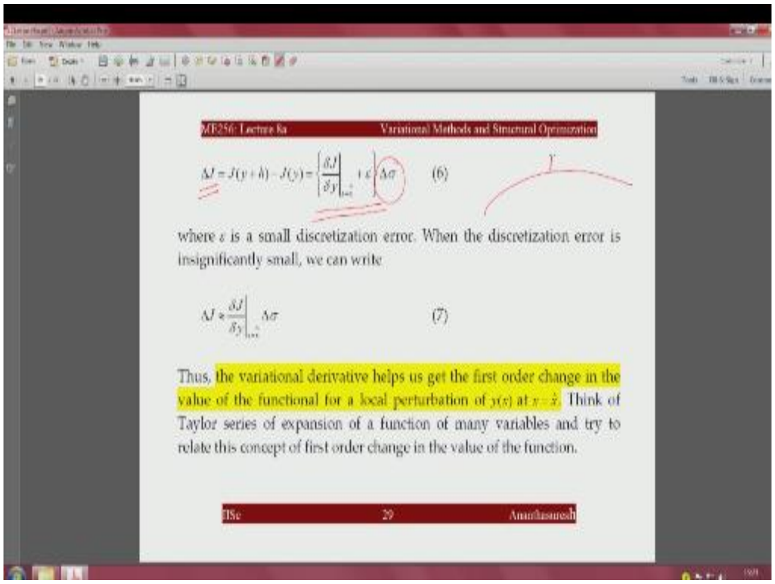

That we should remember because this variation derivative again if I look at the first order change that is again given by this Δ σ here is the perturbation let us i have a function at one particular value if I raise it that function like that this thing okay is Δ σ which will be needing when we discuss constraints in the calculus of variations frame work okay so to sum up today we discuss three things go to variation and then pressure differential and variation derivative 30variation is needed for us in all the things that we do in this course pressure differential is just a concept ray of familiar if you do not understand it is okay.

But if you take function analysis class you will be leading that third is the variation derivative which we need when we discuss how to incorporate constraints in calculus of variations okay next class we will actually derive these Lagrange equations which become the foundation stone for everything else that we will do the rest of the course. Thank you.