Indian Institute of Science

Variational Methods in Mechanics and Design

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NPTEL Online Certification Course

Hello again, so we are talking about the properties of the function space right, so we took the simple example of a taut string.

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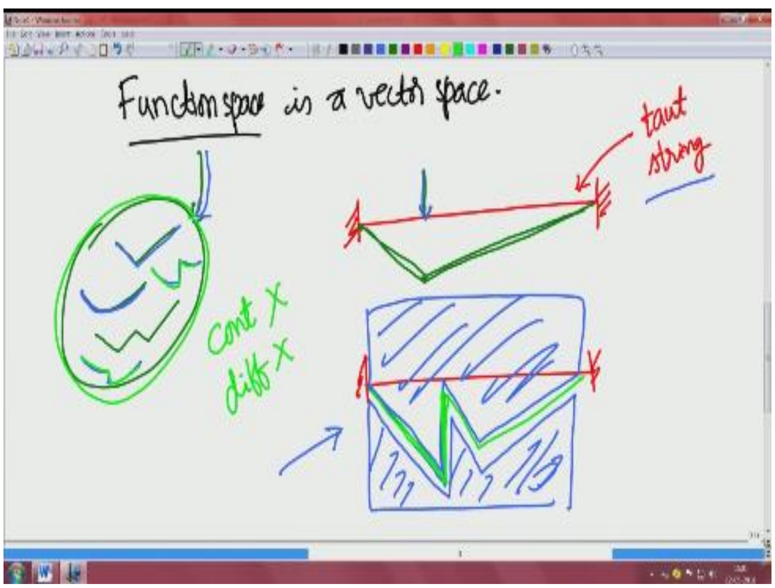

And we wanted to know what kind of shapes this taut string will take when you apply different kinds of forces on it, right. So all of those we have to search in order to find one that minimize the potential energy, so we are talking about that function space here we say what kind of function we said that they did not be continuous like this they can be is they need not be differentiable like this smooth they can be non differentiable like this kinked one which is what happens in this particular case when there is a force like that, right.

But we also said that if you take this particular case there are two wedges that squeeze this wire so that it will have a discontinuity over here, okay let me change color so we have here a discontinuity right the same x value has a range of values, so this type of things are also allowed things that have discontinuity. So now our function space consists of what does contain tomatoes or oranges or apples or what right, then they are not continuous okay they are not differentiable right, so what any, anything can be belong to this functions pace so we can find our answer here that is not true.

So there are some certain properties that you can define so that your function space is properly characterized so you know where you are searching okay, you are searching in your room you have to know what kind of a room that is okay, so for that we need to understand what is called a normed vector space, okay.

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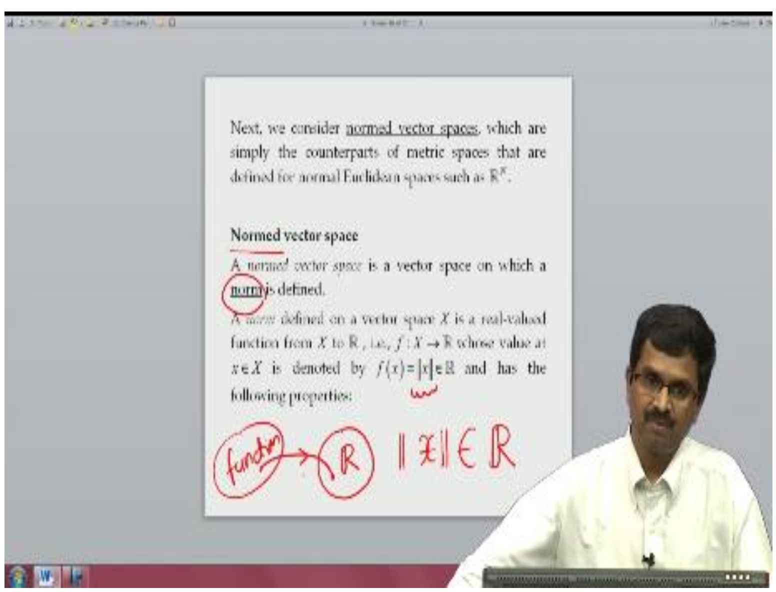

And some properties of that we know what a vector space is now, now we add a qualifier called normed, right what does it mean it simply means that along with a vector space there is something called and norm, also norm is like a metric something that says something about it there are some people let us say there is a norm we say if we can measure the height of every person weight of every person let us say then there is a norm, okay.

You can talk about two different people there and what is the distance between them that is the norm, so norm is maximally defined are denoted let us select this if I take this f(x) here so when we have been you say something is a norm it is it talks about a characteristic of that like height of a person like I said a norm is defined in a vector space is a real valued function so this x that we put like this okay, is a norm will be a real number it should belong to this real number space R.

So you have a function space okay, let us call this a function space okay and then we have the real number space right, say norm takes a function and maps at a real number we can calculate it that is what is the norm so when you have a norm defined for a vector space we get what is called a normed vector space.

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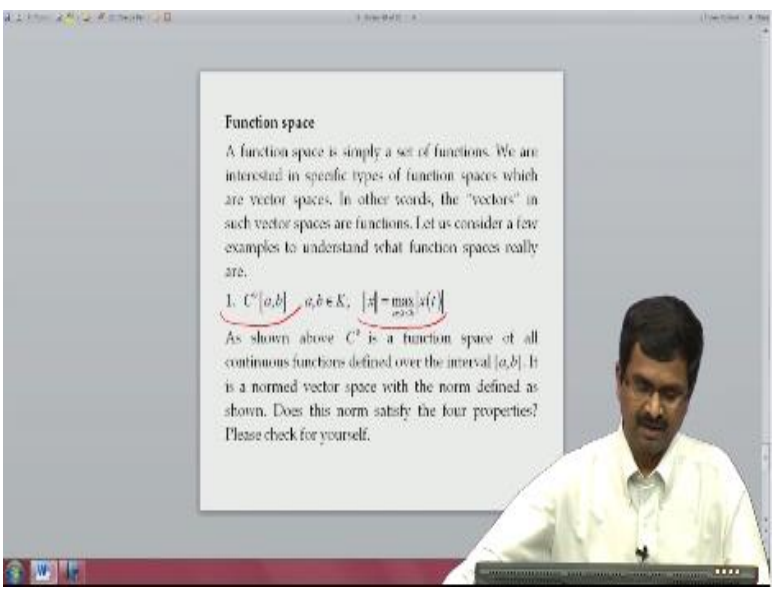

That norm has to satisfy for property that are shown here first is that the norm has to be non-negative just like metric is matrix should be non-negative and so is the norm both are the same okay, and if the norm of an element belong to vector space is zero that means that element should be a null vector the zero function in that, okay. Similarly the norm of α times x is equal to absolute value of α times norm of x.

And then we have a triangular inequality equivalent here x + y norm is less than equal to norm of x plus norm of y, okay there are four things that the norm has to satisfy if you want to come up with your own norm you should make sure that all these one two three four properties are satisfied for any two elements are any scalar element that α which belong scalar field should satisfy these four properties.

So now we said already that our function space like a vector space so space of all continuous functions is a vector space because it would satisfy those properties and there is a norm for that, okay you can have a normed function space such as this one.

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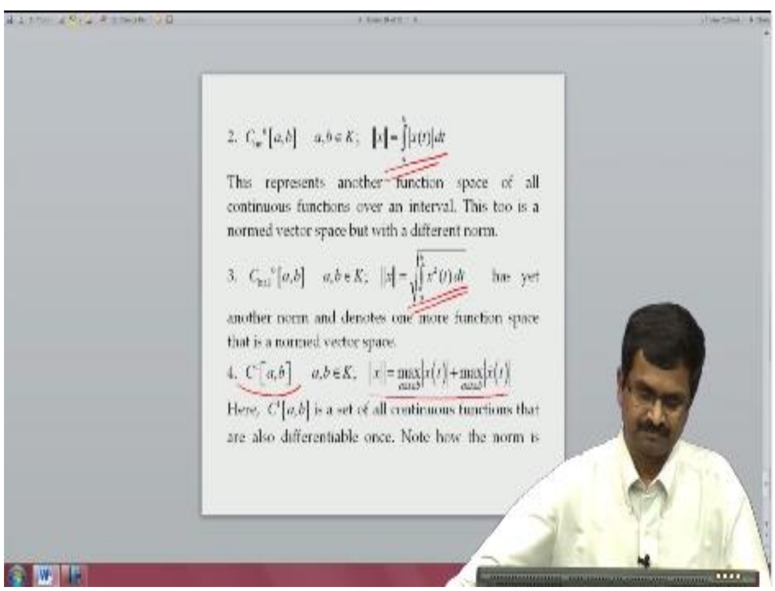

And you can have differentiable functions okay, that we have C1 continuous so we have not only continuous we can also have say that the derivative is also continuous what you call differentiable and for each of them you can define different norms okay, there are lot of different norms that you can define for the same space.

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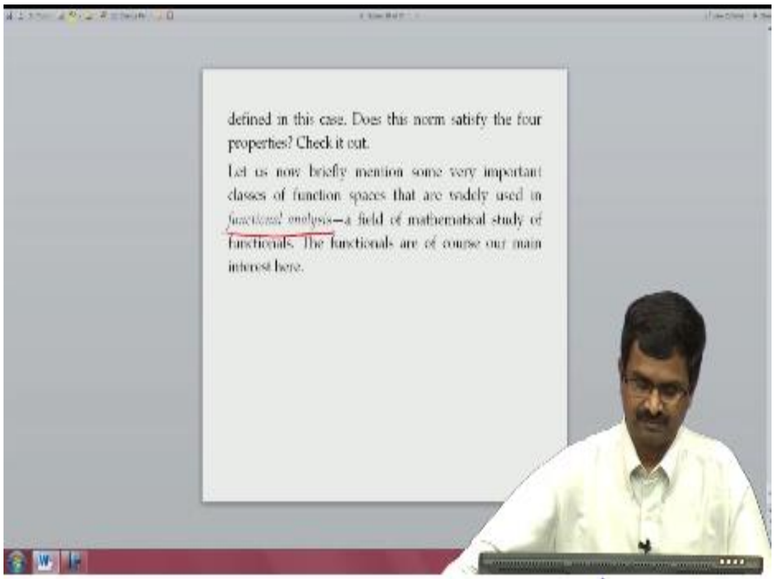

All right, so like I said function analysis is the one that covers all these things in detail what you are talking about is only a basic understanding of various terminology in the context of our optimization or calculus of variations right, calculus variation is optimization over the function spaces we need to understand what role these different terminology are going to have, okay.

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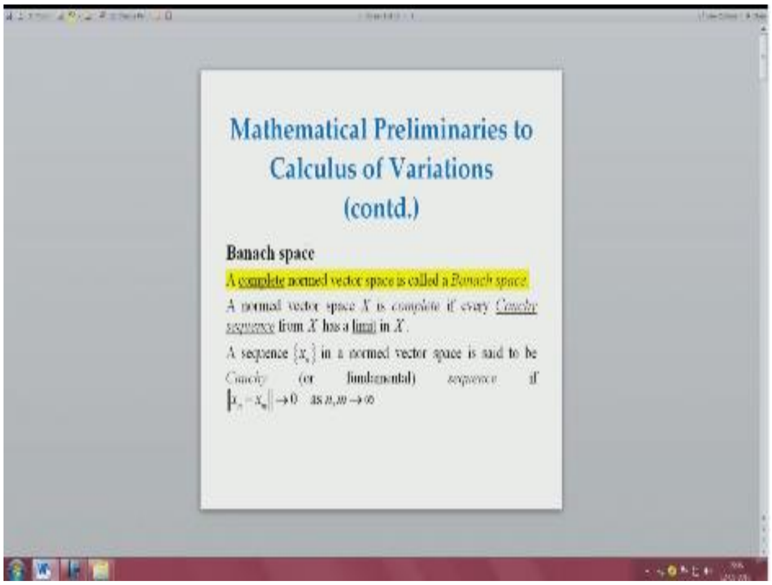

So one of these spaces which is a normed vector space we said vector space at a function speed that has a norm defined on it, so something you can say this element that is a characteristic that is the norm. Now let us talk about something else called Banach space right, so again the motivation for this comes from our notion of.

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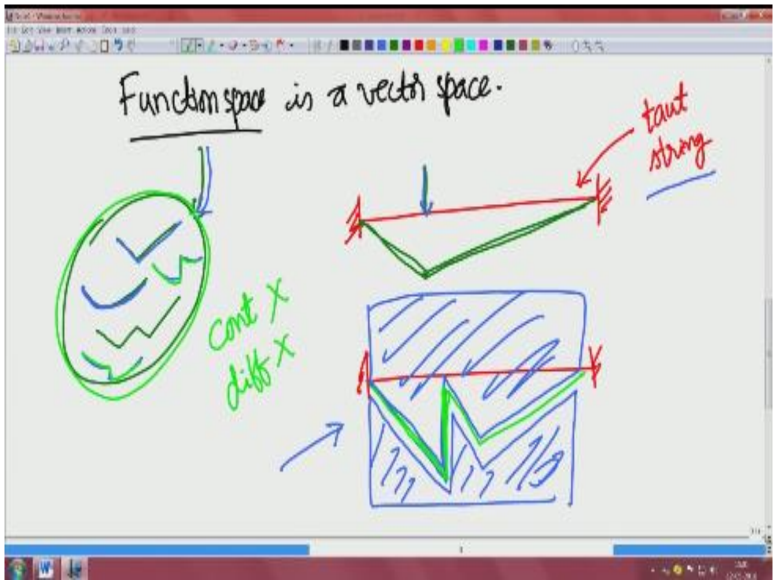

So we said that in this particular case what should be our function space what kind of taut continuous taut differentiable so what right, so for that we want to define what is called this Banach space.

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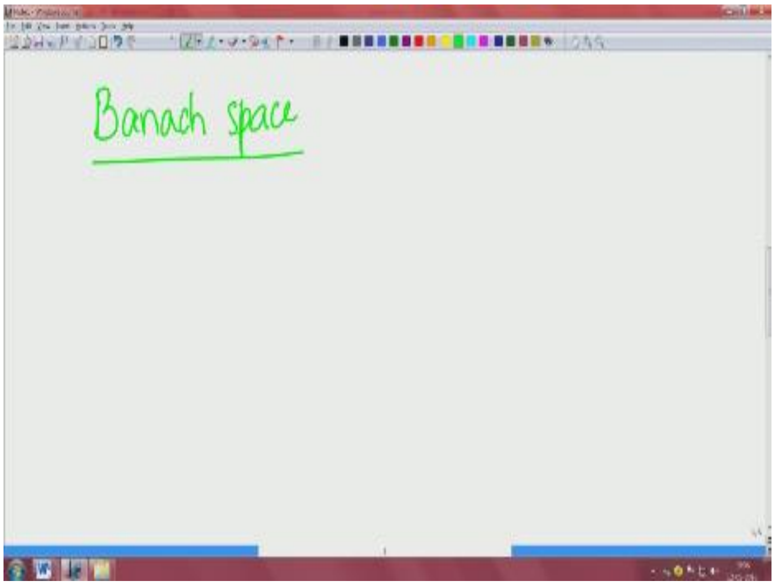

So what is significance of it, this has a lot of significance in calculus of variations okay, let us see the definition first and then come back.

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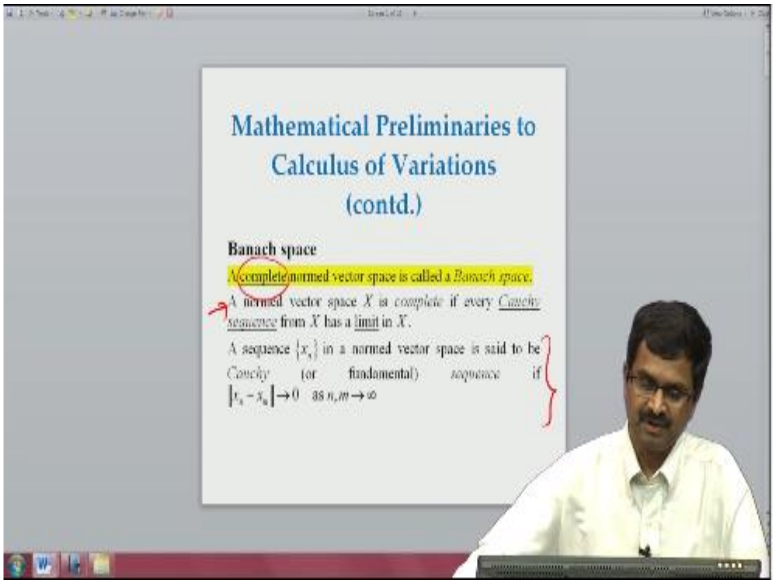

This Banach has space as you see the definition here it is a complete normed vector space okay, so complete have underlined so we have to understand what we mean by complete it is a complete normed vector space that is called a Banach space right, this completeness there is a definition for that also a normed vectors pace X is complete if every Cauchy sequence from X has a limit in X then you have to say what is Cauchy sequence, what is limit and so forth, right.

A Cauchy sequence is basically a converging sequence that is what is the definition here what it says that you can have a sequence meaning that if you start with a function we talk about function spaces they are also vector spaces right, you have function if I start with a function and I have something a ruler operation that gives me a next function, the next function and next function then I get a sequence of things such a sequence converges such that sequence is called a Cauchy sequence.

And a convergent sequence will have a limit it will stop somewhere, right that limit should belong to that space if it does then we call such a thing Banach space okay, that is what all these mathematics in this slide and this slide but you can read leisurely what it says is that you have a vector space or a function space okay, function space is this one kind of a vector space right, you have a function space and that there is a normal so that is you can for every function you can define a characteristic.

Now that function space will be called Banach space if it is complete normed vector space complete means that there should be some convergent sequences which called Cauchy sequences and the limit of the sequences should lie within the space such a thing is called a Banach space, right.

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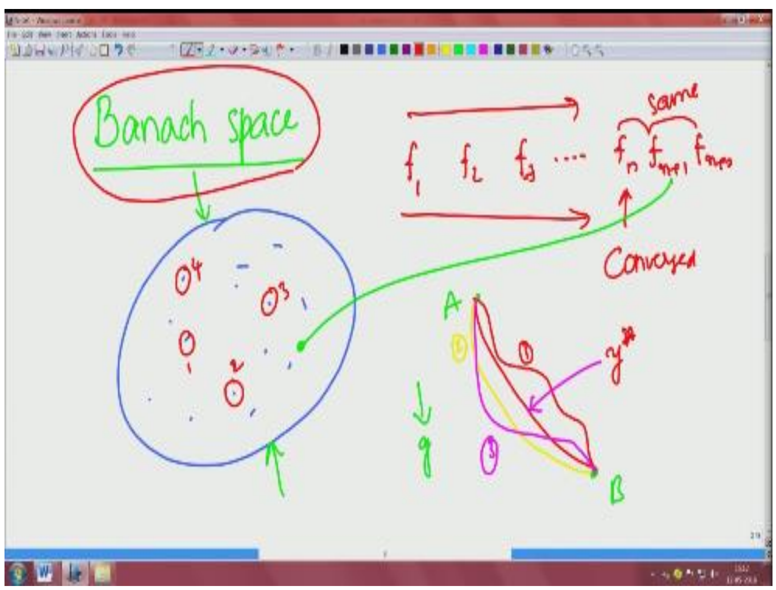

And this is important in calculus of variations because we want to find a function that minimizes your objective functional, right we understand what a functional now is right, we have to minimize that functional way to find such a function so when we searched in a space okay, there are lot of functions now these are not points these are all functions okay, when you have so many functions if I start a sequence I choose this one I call it 1 I choose this 2 I choose this 3 which is 4 and so forth, so I will have f_1 and then f_2 f_3 like that let us say there is f_n.

So let us say after that how much how many more things I take the value of this will there be a limit in the sense that kind of function will not be different from f_n+1, f_n+2 and so for that means that this sequence of functions has converged okay, I have f_n, f_n1 they are all identical f_n+2 they are all the same, let us say almost the same within some tolerance right, such a sequence of functions we call it a converging sequence and the limit of that that this thing what I converge so that also should belong to the space if it does such a thing we call Banach space, okay.

That is a complete number to complete meaning that limits of all convent sequence belongs to that space itself, why is it important for us is important for us in calculus of variations because if I have brackets token problem point A to B I want to find in the presence of gravity A function that minimize the time these cycloid. But before that let us say randomly I take another function okay, from there let us say I start my initial guess one and that may take me to another function a low and then let us say that is second function and then I take some other function I can call it 3 1 2 3 and many more functions.

Finally a converge to my y* here right, after that there is nothing further to go that is a best, right. So this convergence sequence has a lot of role to play in calculus of variations right, and that final function should belong to the function space that I originally took to search, right we are searching in a space which should contain our solution that means that somebody has to search for something in a room the first thing you should ask is do you know for sure that the what I am looking for in that room, right.

If it is not there is a futile search right, so our optimization or calculus of variation we do to minimize a functional we should search in a room Banach space okay, that is what it means all right. So we need to have the answer in the room that we are searching it okay, that is what it actually means right.

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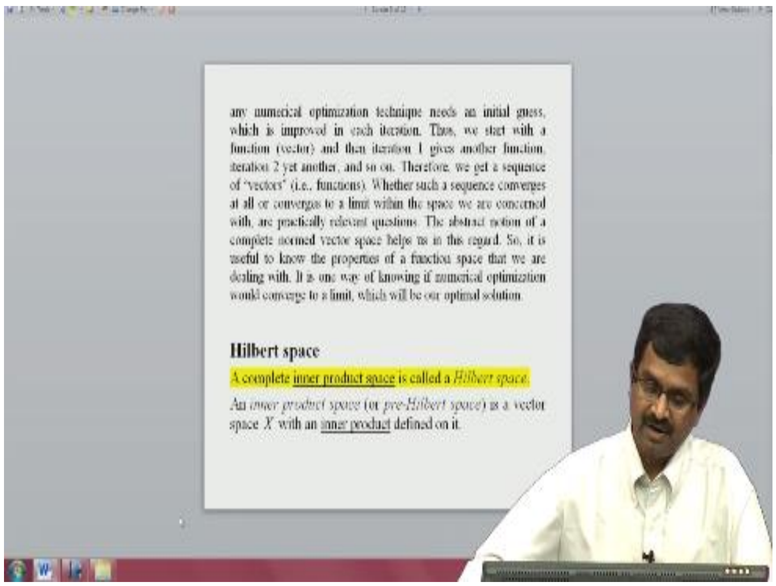

So there is another space called Hilbert space right, the Hilbert space has a different thing what is called here a complete inner product space is called a Hilbert space, what is the Hilbert space it should be complete inner product what is inner product in a product we can define as shown here right.

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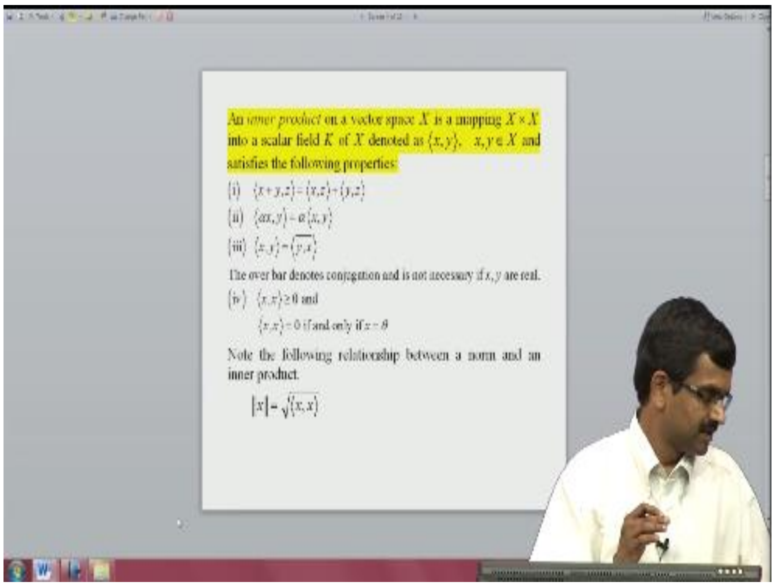

The norm we needed element itself X and we put two bars on it saying the norm, right so if i go back here all right, let us s go to the next page.

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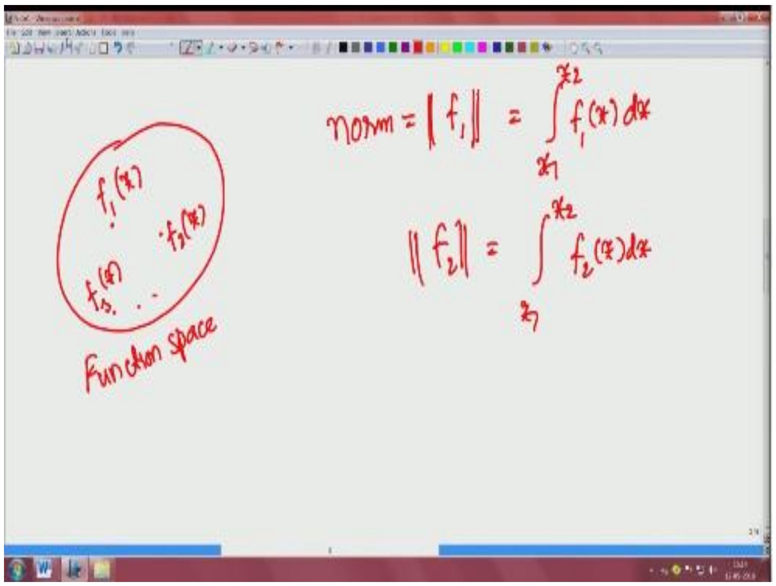

Yeah, so if I have a function space I have different functions have $f_1(x)$ let us say $f_2(x)$ and then $f_3(x)$ and so forth so all of these are points in that function space so for every one of them we can define a norm and norm can be the norm of the function which we say we have this f_1 we have a norm on it right, you know it can be an integral so if the domain is from x_1 to x_2 I can say $f_1(x)dx$ a real number now should be real number I can define.

So similarly I can define for f_2 also f_2 norm is x_1 to x_2 f to dx right like that I can have the norm so as opposed to this which talks about if I take a $f_1$norm of f_1 tells me something about f_1 just like in a population a group of people norm can be height of the person, right for every person there is a height every person there is a wait those are norms so you can take one element of the function space or vector space and talk about norm.

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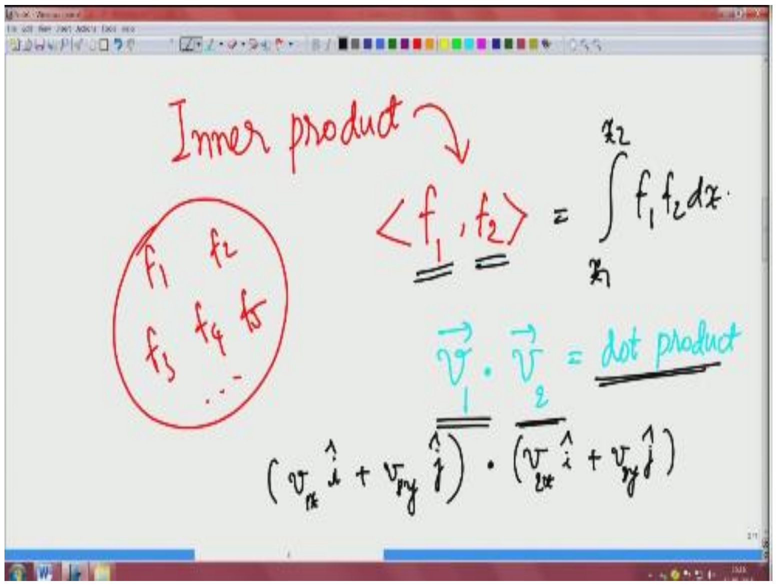

But the one that we now talk about which is called the inner product okay, in a product that requires two elements that is if I have a function space ok let us say there are f_1 f_2 $f_3 f_4$ f_5 and so forth right, in order to define an inner product I need two things i can define inner product between $<f_1$ and $f_2>$ that is enough product ok like if you have two vectors dot product is like a inner product two vectors meaning our mechanics vectors.

If I say there is v1 is a vector normal mechanics vector and there is a v2 I can put dot product right, so that dot product is a inner product, if you want to recall let us say I have v1x+ v1y j i get two dimensional vector when I said dot product v2 x +v2 y if I have you know how to dot product basically v1x times v2x because i dot with i will be one I dotted with j will be 0 and then rod will be 00 will basically v1x+v2y that is a dot product that is the inner product, okay.

It is just that it is defined not for simple vector such as our mechanics vectors that defined for these vectors with our functions for functions also you can define inner product it can be as simple as their let be again for properties for that let us say I can say x1 to x2 I can say f_1 times f_2 dx that is inner product.

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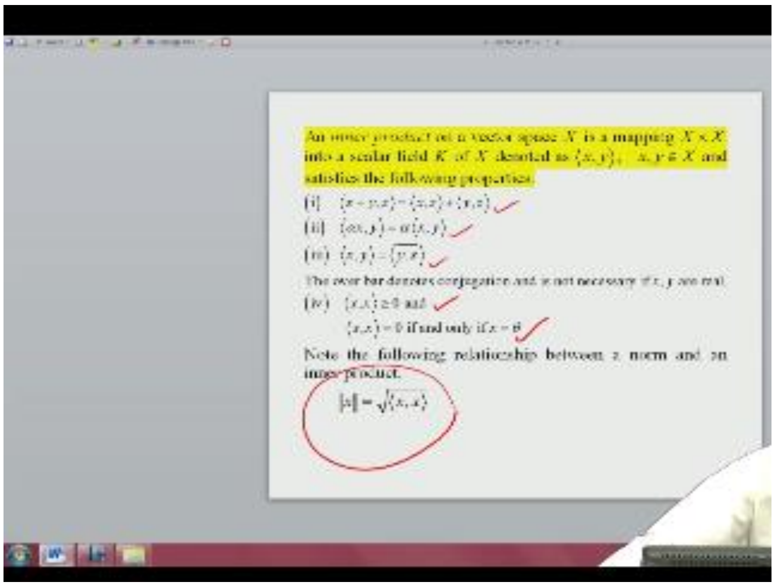

It satisfies certain properties which are listed here when is the inner product x + y z should be equal to x z + YZ and then α x times y α x and y α x, x and y should be equal to α x XY in a product and then x y should be work on we will talk about complex numbers that we YX also that we do not need when you deal with real numbers and then XX is always greater than equal to 0 means nonnegative when X, X is equal to 0 that is only true if X is equal to the X is the null function there okay there is are lay between inner product and the norm such as this you know.

When you say in a product of the element with itself then that will be the norm right what do we need this inner product when you had a norm I said that is a character is like a height of a person or a weight of a person or something like that what do you need between two people when you want to compare things you need to have what can so that you become pair one of the things is the orientation normal mechanics when you say dot product is zero that means that those two vectors are perpendicular to each other right. So it talks about that given angle between two vectors similarly for functions also you can talk about inner product space so that is the Hilbert space.

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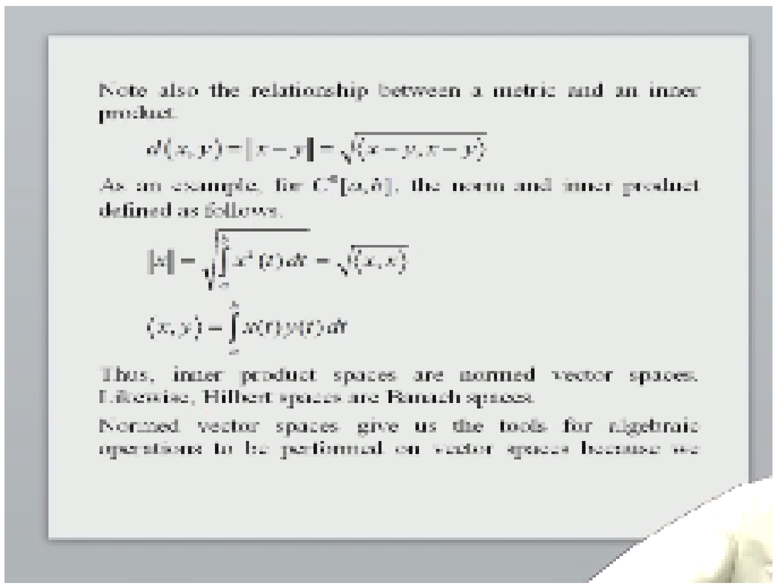

And I this Bona has space and he will Hilbert space also be complete inner product space meaning that if I take a sequence of things that reaches a limit that limit should belong to the Hilbert space right so complete inner product space is a Hilbert space and this Hilbert space in Brahe space are important in optimization as I said that whatever space we search or a room we search should be a bona space so bona spaces are not particular spaces but they are kind of category of spaces like we can say bono spaced our vegetables and Hilbert space or fruit right.

We are not saying which vegetable you are not saying it is Brinjal or lady's finger or some other guard but we are saying all of them are vegetables they are bona have spaces which have a norm and complete normed vector space Hilbert spaces has inner product and they are like fruits we are not saying apple orange a banana but they are all fruits right.

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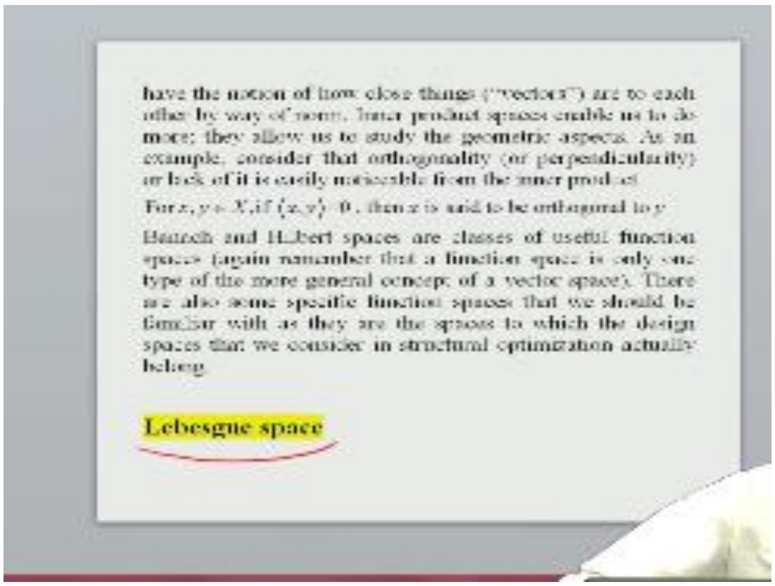

Now we will talk about a few specific spaces as well such as the one which have shown leg space see bana has space and Hilbert spaces are generic spaces categories let us say fruits and vegetable sets all these either a fruit or a vegetable if it is a bana space it is up one kind Hilbert space now another kind which satisfies this property would you find one with the norm other the hill inner product it is a Lebesgue space which is important going back to our the string problem what function should we take right that is where we are getting now this Lebesgue space.

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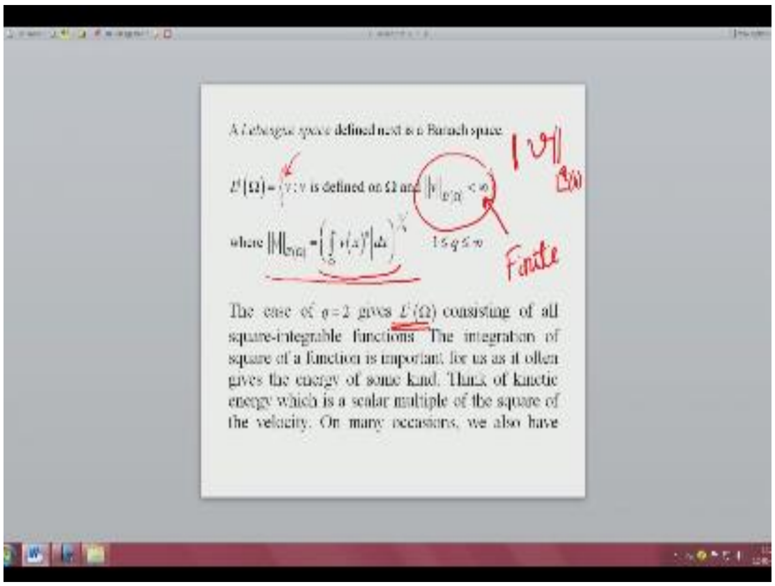

As a particular norm associated with it which is defined here right the Lebesgue space is it contains all these elements V such that a particular norm on it over here right that is norm that we have this V and two bars like this right that norm has a subscript here which is a LQ over some domain right that should be less than infinity we are saying all we are saying is that the Lebesgue norm which is what is defined here the Lebesgue knob should be finite if you go back to a string problem.

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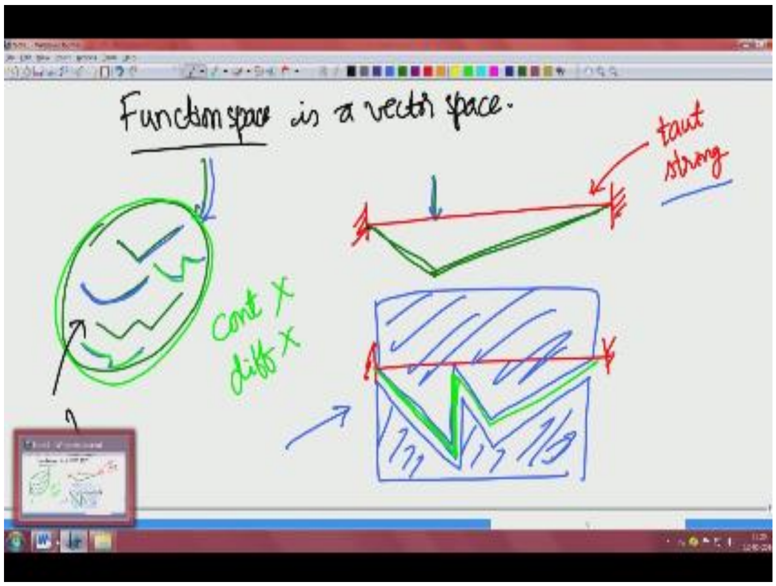

If you recall so we said what kinds of things will be here right not any discontinuous thing but things where this Lebesgue norm is less than infinity meaning finite when you apply these forces on the string we do not want this spring the string to go to infinity right it is some kind of energy in it should be finite you should not be indefinitely having a lot of strain energy or potential energy right it should have a finite one and that is what we have here okay when we say this Lebesgue norm that you should be finite so what it is saying is that the Lebesgue norm is finite it's saying not infinite less than infinity it is finite right what ill buck knob.

Look at this integral so its integral of the absolute value of V raise to q where q is some number right and integrate that integrated absolute value of V raise to q and then take the q through that is a Lebesgue knob right that is kind of some quantity right but that is not enough for mechanics and q can be too that becomes what we call L 2 norm like a distance okay like a function f square DX integral, integral of f square DX is like what is called l2 norm right.

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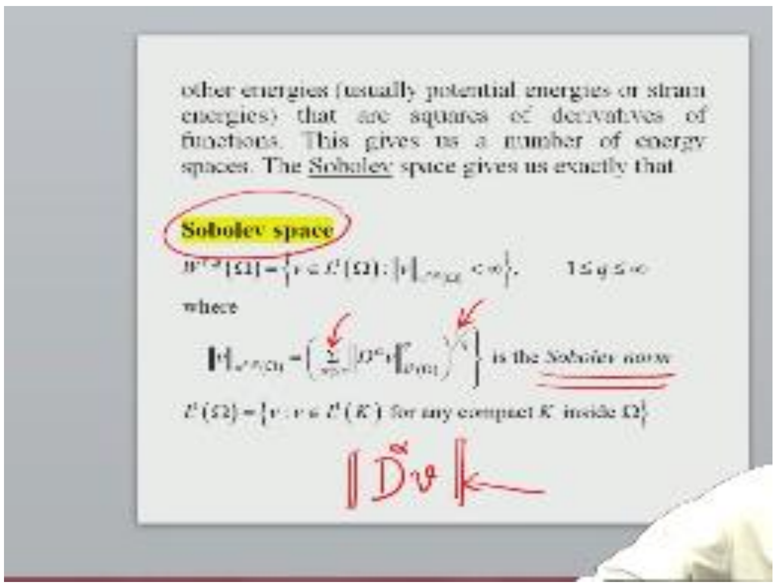

These are all Lebesgue norms and it will become more clear to you when we talk about a more complicated thing called a so below space which has a, a so below norm k sub alone on with a little bit more complicated right it is a there is a summation here over α such that is absolute value is less than R okay and then we have a norm which has this d raised to α v the d is basically differential operator α equal to 1it is a first derivative α equal to2 2nd derivative α equal to 0 it is a function itself that V itself right.

But here we are not saying α should be integer even fractional derivatives are a load which we are not going to get into but know that fractional derivatives also you cannot you do not need to have dy by DX and d square y by DX square you can also have something in between them like derivative 1.3 1.4 or whatever right so it says that all of those if you take and raise it to Q and take the norm and sum them all up and take the cube root of it then you get what is called Sobolev now why is it important for as it looks very abstract it is important for us because our functionals are going to be containing function and their derivatives that is exactly.

What this is saying whatever we have here what is saying is that summation of something right so a function and derivatives summation can be an integral the norm here already though what we have the norm here that can be an integral need not be the only kind it can be other things we have seen already it is something that sums up the whole function okay.

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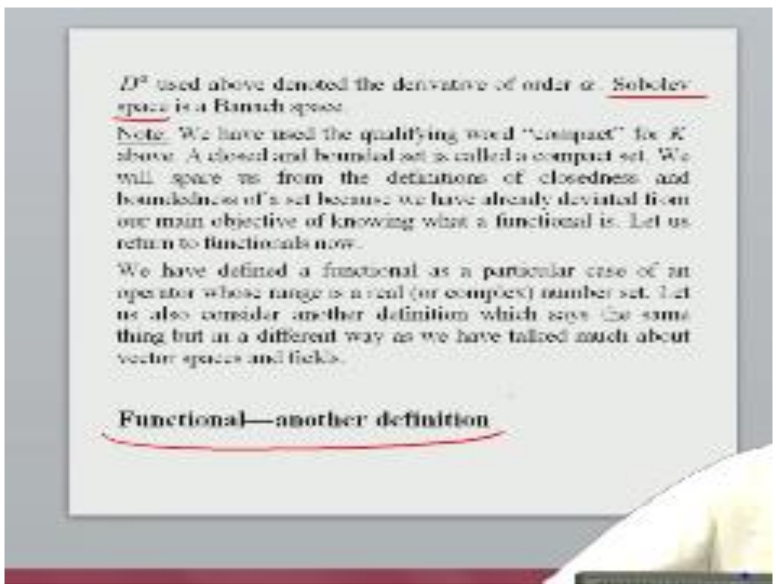

So this so below space that we take we talk about is energy space because most of the strain energy pure energy they all contain the function its first derivative and second derivative and so forth now we need mechanics you do not go beyond second derivative but you can in principle there are energy spaces where those energies are finite because if you see Lebesgue nom we said that should be less than infinity you cannot have infinite strain energy that is why the Sobolev space are important so in calculus of variations when you apply to mechanics in our variational methods we should ensure that our functions belong to Sobolev space.

By the way Sobolev space is a burner has faith at all been proved by Sobolev and others so all of his space is named after a person called Sobolev and so is Lebesgue and mona so for this if we call bona hospice has a space of vegetables Sobolev space is the particular vegetable that is like let us say a Brinjal and Lebesque space is let us say s nah God right these are particular spaces where as one as feels like a vegetable similarly Hilbert space also there will be several specific spaces can be there okay.

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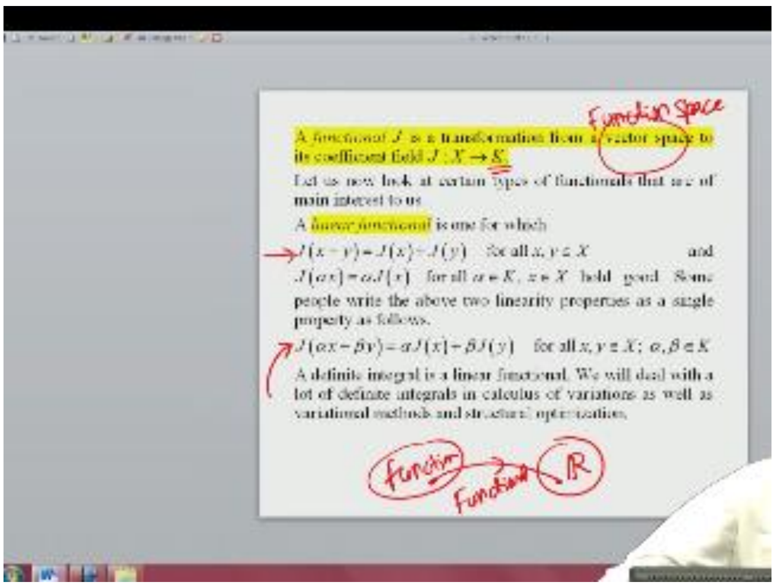

Having understood the spaces now let us take another definition of functional before we leave it we said a functional is simply a mapping from a function space which you understand leave it more than we knew before to a real number space so basically this definitions says that a function is a transformation from a vector space or a function space so when I say vector space here I also mean this to be a function space okay to question field meaning that k that we have right real number okay once you understand that functional in that manner basically it should go from functions that belong to certain space which satisfies all this laburnum or so alone and so forth to areal number space okay.

That is what a, a functional does for everything here it takes you there that is a functional normally people sometimes quickly define a functional as function of functions that will be incorrect because if f of X is a function f square okay which is a function of function that is not a functional right but a functional should take a function to a remembers real number of space that is what is the functional once I got a functional we understand and we can talk about linear function that is like we have been once you understand function.

We have at this linear function non-linear function what is a linear functional something that satisfies this okay j if I say J of X is a functional J of wise another functional if J of X plus J of y is equal to J of X plus y there is a linear functional similarly J of α X is equal to x a of X then that is a linear function both of them can be put together in this fashion here J of α times X plus β times y will be equal to α times j, j of x plus β r MJ of Y then such a functional J we call it a linear functional okay.

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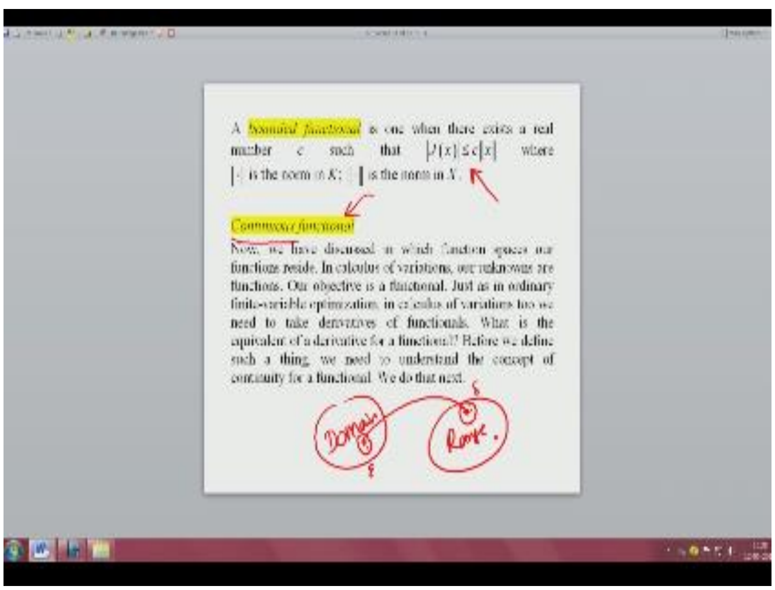

And bounded functional where that is bounded by number C times the norm of that function okay so that is a bond that is also important for us in calc three variations whenever we take a functional which you want to minimize you wanted to be bounded do not want it to go to infinity indefinitely that will cause problems so we need to have bounded functionals and finally we need continuous functionals as we will we know what continuous functions are write a function that has a value if you put up a little bit here there will be so every function will have what we call a domain and a range that these the input that is output if I have some here there is a corresponding anything.

There now if I put up a little bit in the domain the rain should not be going over there it should be somewhere close by if there is epsilon here there should be distance within Δ okay if this is epsilon is around that point If I am there that should be there that is a continuous one right so similarly for function also we can talk about something called a continuous functional be the same a kind of notion that we have right.

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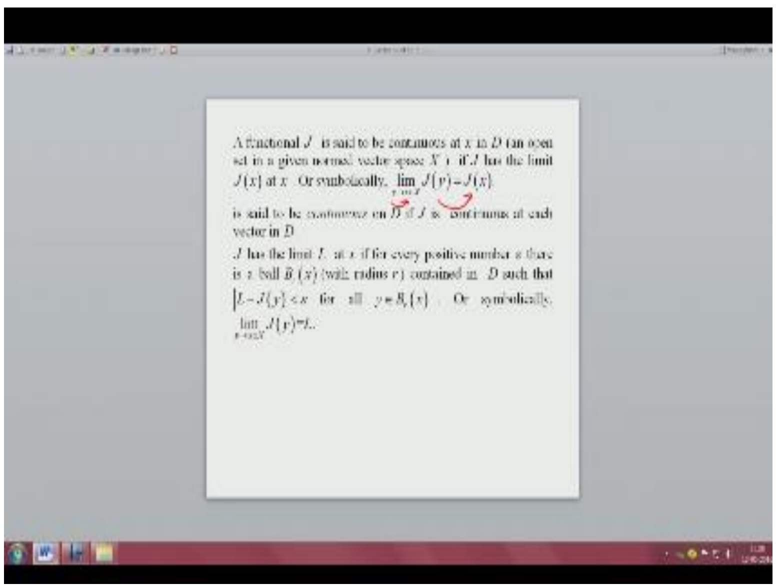

So if I have a functional JC to be continuous at X in D in open C tall those qualifications exist open set and all that right if Jay has a limit J of X at X or symbolically as Y tends to X a little perturbed point in the domain then J of why should tend to J of X so in the domain if they are tending to each other in the range also they should such a thing is called a continuous functional okay so to summarize today let us show here let us just yeah a few things we talked about we talked about what is called the functional.

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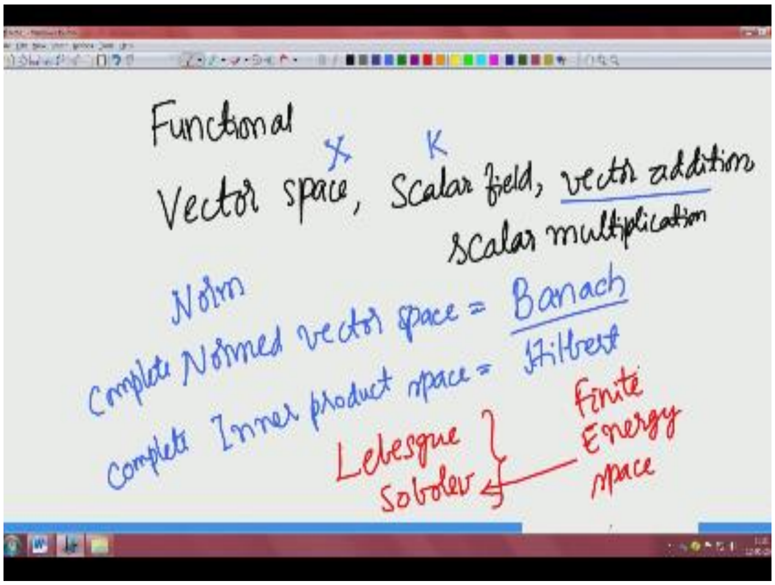

And we defined metric spaces in vector spaces vector space will have a scalar field associated with it and it needs vector addition two of them two vectors elements of that it is able to add following some rules and also scalar multiplication meaning that something that belongs to the scalar field which we called k something that belongs to vector space you know I can just call it capital X was able to add two elements of X that is a vector addition I will multiply a number belonging to k with x right so we discuss this vector space and then we talked about norm.

And then we said normed vector space is called Bona has faced normed vector space right a complete non vector space right so in that sense all the convergent sequences should belong to that that is our banach place we also talked about in product and complete inner product space is called Hilbert space and these are generic categories of spaces okay which is Banas especially important because whatever initial guess we start with a function when you generate a sequence of thing the iterative optimization the limit or converging function should be within that space again remember.

When searching in a room at the end of the search it should still be in the room not outside that is one aha space Hilbert space talks about relations whether things are perpendicular or not and things like that okay that we do not need so much in optimization that we are going to talk about Bernard space is an important thing and then we also talked about Lebesgue space and Sobolev space okay which are important things in terms of mechanics because both of these special below-space they are called energy spaces okay are we can call finite energy spaces right because we saw that should be less than infinity okay like the string problem you cannot have indefinitely infinite energy it needs to have finite energy then all such functions give rise to finite, finite energy are the ones that you should be searching okay with these preliminaries we will jump into calculus of variations in the next lecture thank you.