**Indian Institute of Science**

**Variational Methods in Mechanics and Design**

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**NPTEL Online Certification Course**

So far in this course we have discussed calculus of variations its history and the kind of problems that we encounter in geometry and mechanics and then we took detour for three lectures where we learnt about the theory of finite variable optimization right both necessary and sufficient conditions so we really are now waiting to go to calculus of variations but before that there are some formalities I must say or what we call mathematical preliminaries okay.

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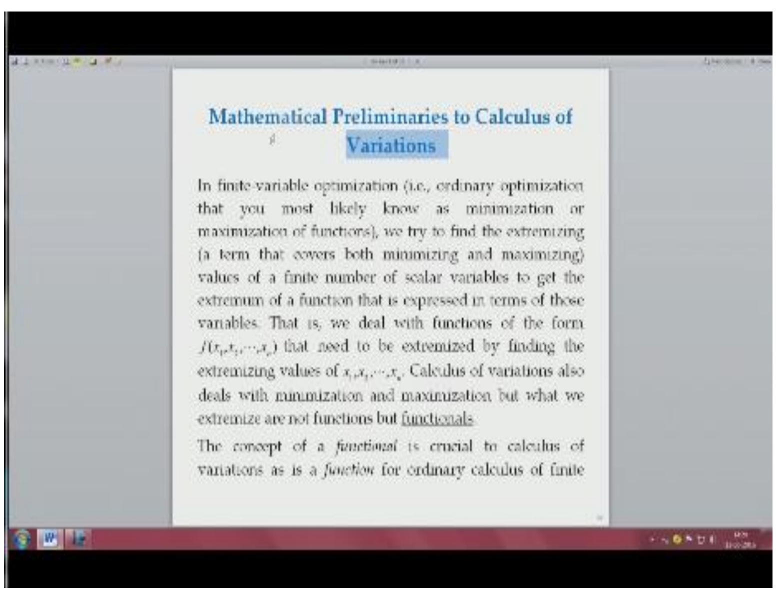

So we need to discuss these mathematical parts primary part of it before we jump into calculus of variations in this the one thing that we need to understand is the notion of what is called a functional normally infinite variable optimization that we have already discussed the function is what will be there as objective function that were called objective function or constraint functions whereas in calculus of variations we have what are called functional we have seen that in the lectures that discussed the kind of problems we encountered in geometry as well and mechanics which are cast in the form of calculus of variations where the unknown is a function itself in a finite where optimization we have unknowns as finite variables their continuous likex1 x2 x3 the continuous over a domain but they are finite in number right in calculus of variations we will have function itself as an unknown such as.

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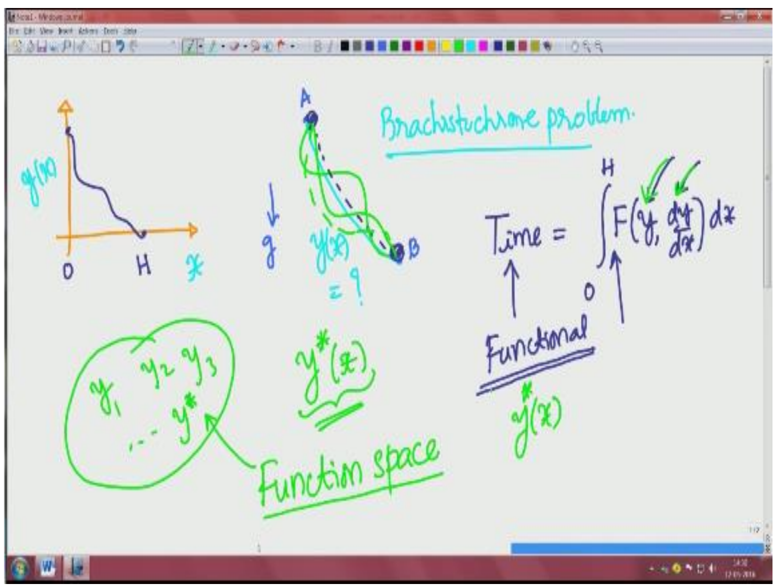


If I just take the drastic own problem that we say there is a point here there is a point here and there is gravity acting we say from to go from point A to point B we want to find the function that passes through these points and takes the minimum time that is what if you remember is what is called the Brachistochrone problem okay so this function if I call it y(x) that is what we do not know we want to find that one which minimize the time so if I say x here and y(x) here so we get some functions there are lot of them let us say a point A see a point B is here a lot of function that are possible we want to know which one of them will be the answer that is the one that minimizes the time.

Now the time that we had if you recall we had like an integral right so this time of this scent if I keep a little bead here for it to fall and come over here there is certain time taken that time turned out to be integral where if I say this is x here right so the 0 to let us say H it was 0 to H and we had something which dependent on this y(x) as well as dy/ dx that is what we had okay so here this is called a functional so in calculus of variations we minimize or maximize what we call a functional as opposed to minimizing or maximizing a function in finite variable optimization.

So there the variable were x1 x2 x3 xn whereas here we will have y(x) we can have more than one also we can have y1(x) y2(x) and many functions as unknowns but the fact is that the function is the unknown and what we minimize is actually called a functional okay if you look at that the integrand of the functional when the function is the form of an integral it need not be of the integral form a functional is more general than being an integral when it is in integral form such as the one that happened for this Brachistochrone problem then the integrand depends on the function and its first derivative again it can be any number of derivative not be only one derivative it can be just a function and the first derivative second derivative and so forth okay.

So such a thing is called functional understanding functional in this manner will only take us that far we need to know a proper definition of what a functional is okay because when we understand the functional and then how that functional depends on the functions so here y(x) is our function right and it is derivatives also are there so among all these functions that exist so let us I want to find a particular function let us call that y*(x) so here that is a minimizing time is obtained by a particular function which we know as a cycloid okay.

Let us cycloid is given denoted by this y * (x) we need to find it by searching through a number of functions so we do not know there are a lot of different curves that are possible that go through these right so among all of those functions let us think of a set of all these functions let us say there is a y1 there is a y2 and there is a y3 and so forth many functions but there is also lucking there y * which is our answer okay we are to search in this space okay which we actually call a function space just like our variable x belongs to a real number space okay x1 x2 x3 each of the bailout real number space similarly here we are deal with what is called a function space.

A space that has a lot of functions now which function what kind of function properties that is something that we want to discuss today which is what we call mathematical preliminaries of calculus of variations the first one you understand is functional okay before we discuss what a functional is.

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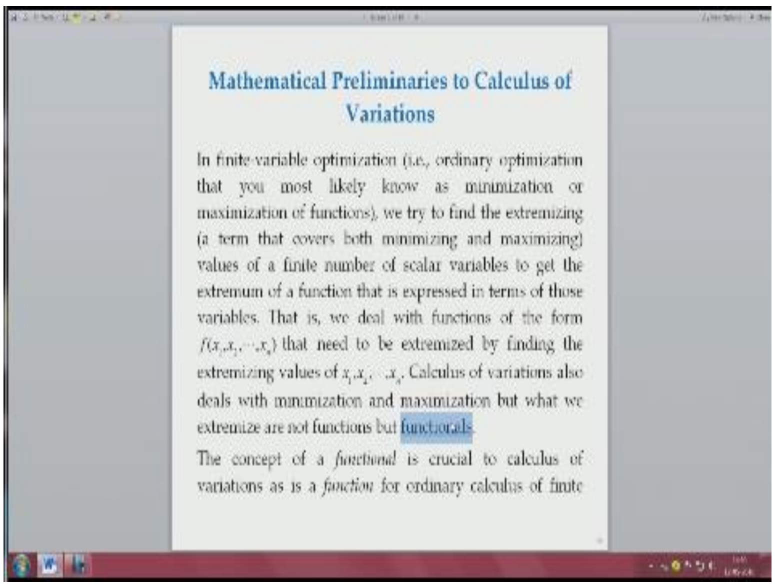


We can actually talk about or remind ourselves what a function is what you want talk about functional but let us talk about what functional is our aim well.

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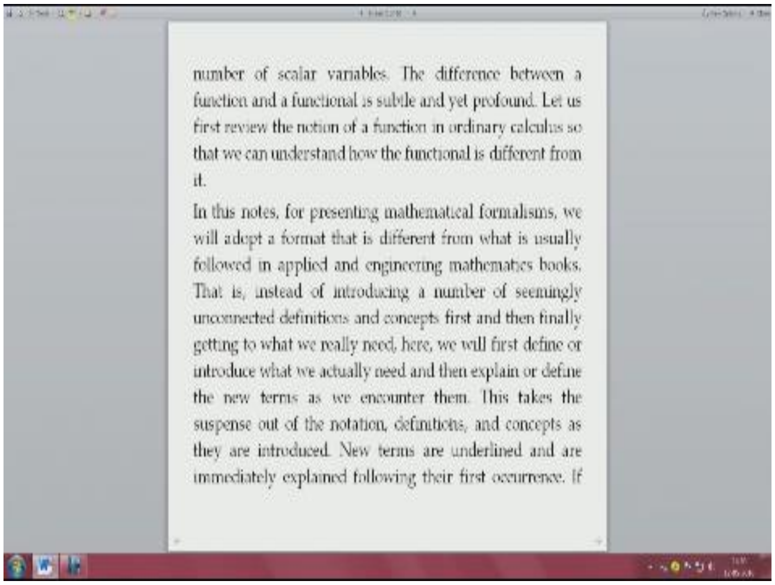


Let us see what a function is first there is lot of text here that you can read leisurely let us discuss this text is written so you can understand offline as well but let us jump in and then see.

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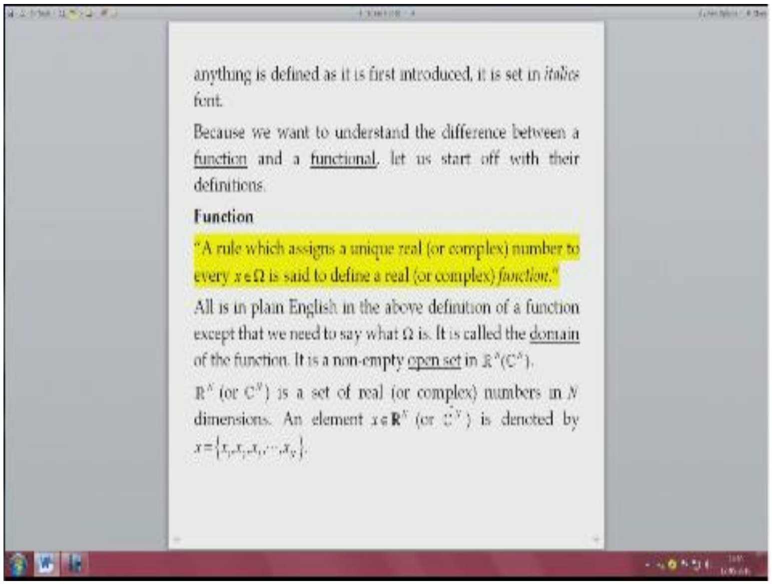


What a function is okay a function which is highlighted here you read it says a rule which assigns a unique real or in brackets will put complex let us leave out the parenthetical partner a rule which assigns a unique real number to every x belonging to Ω some domain or which this function is defined he said to define a real function so that means that for every x there is a unique f (x) or y(x) which our symbol you are using it or the function that is what a function is a rule it says that in the domain let us say x1 to x2 is your domain in that real line if I pick any point I would like to know a unique number that is that function okay that is if you have if you draw let us switch here okay.

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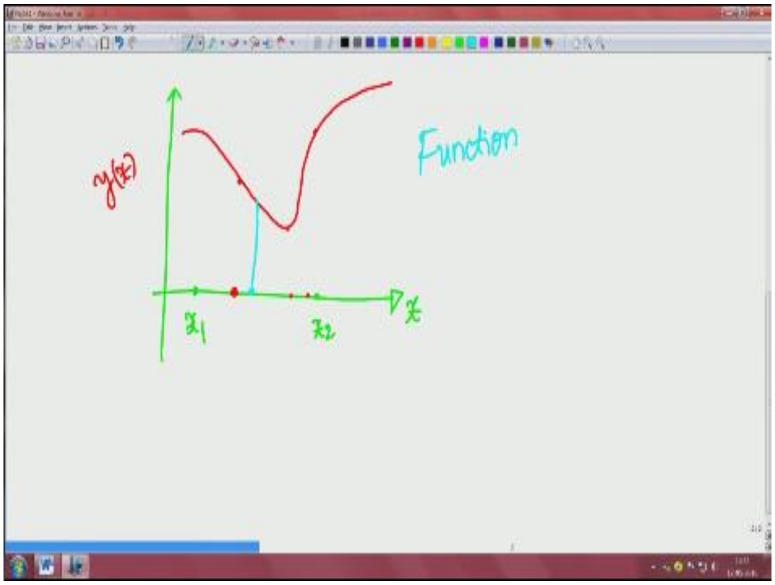


So let us say I take x axis it is our x-axis I define my domain let us say somewhere x1 and x2 okay let us say there is an origin here all right now I want to define a function y(x) or f(x) which ever now any value I take in the domain x1 x2 there must be a unique value y(x) if I take another point another one if I take another point the another one so if I do that I should have a function something like that so for any value that I take over here there should be unique value for y(x) that is what is a function right we all know intuitively what a function is let us understand what a functional is in the same intuitive way okay.

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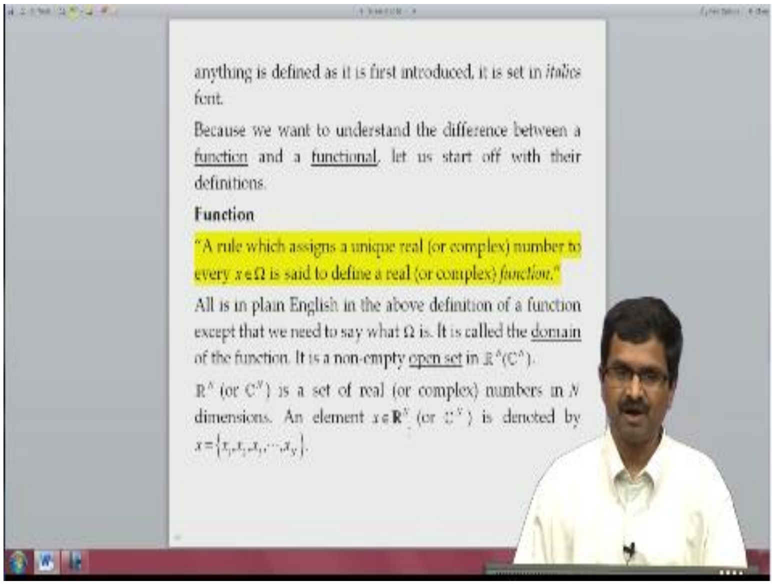


So you understand what a function is now the domain of the function depends belongs to your real number space what is shown as RN over here okay our complex number if you take see n where n depends on dimension let us say you have a function that depends on x1 and x2 then domain of that function is a two-dimensional space because you have to identify two real numbers to calculate that function similarly you can have in 3d x1 x2 x3 that we denote as R3 if it is a complex number will be c1 c2 c3 and so forth okay.

And the domain has to be non-empty open set though all the mathematical rigor with which people define because if you do not have a domain then you cannot define it a function that is why it should be non-empty and you should be open set meaning that it will lack the boundary there are also reasons why people define it in that manner okay.

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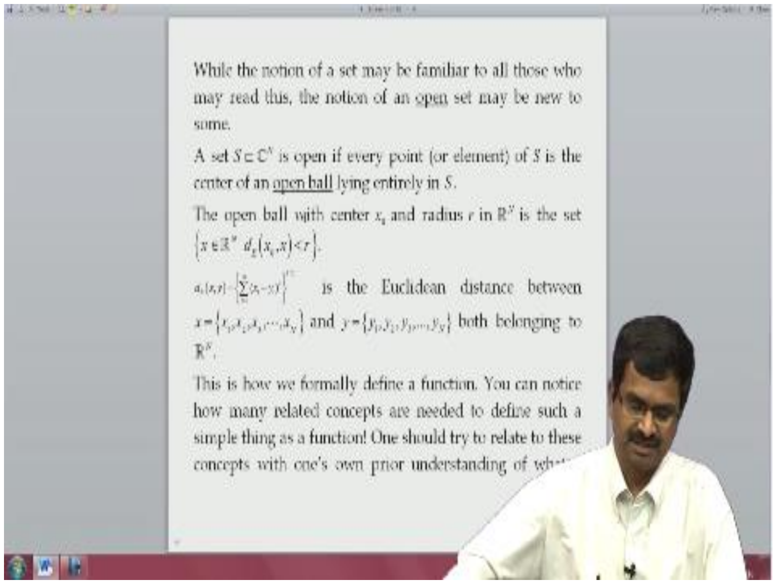


This open set right that we need to understand so why do we need this open set or open ball from the optimization viewpoint if you think so far whenever you define what is a minimum we always talked about that minimum characterized as a local minimum that is only the vicinity of that point it is a minimum or if you say something local maximum in the vicinity it is a maximum not everywhere not like a global look global minima or a global maximum right so in order to have that vicinity around every point you should have like an open ball that is you should be able to define a point a set of points around that point in three-dimensional be like a ball into dementia will be a disk one day we shall be small segment around a point will be plus and minus side either side you will have a few points okay.

So the open set is something that is defined here as set belong s let us say belongs to real or complex domain at every point of s there is center of an open ball that is at every point there is an open ball then that is like a point that will have open in the sense it does not contain the boundary something on the boundary on the other side there is nothing if it is inside there are points all around that is what we mean by this open set so that at every point is open wall so that we can check whether a point is local minimum or maximum so whenever we want to check around that point there should be a few points.

If it is not there will have trouble right so that is why this console open set is important for a function and so is it is for the functional as well.

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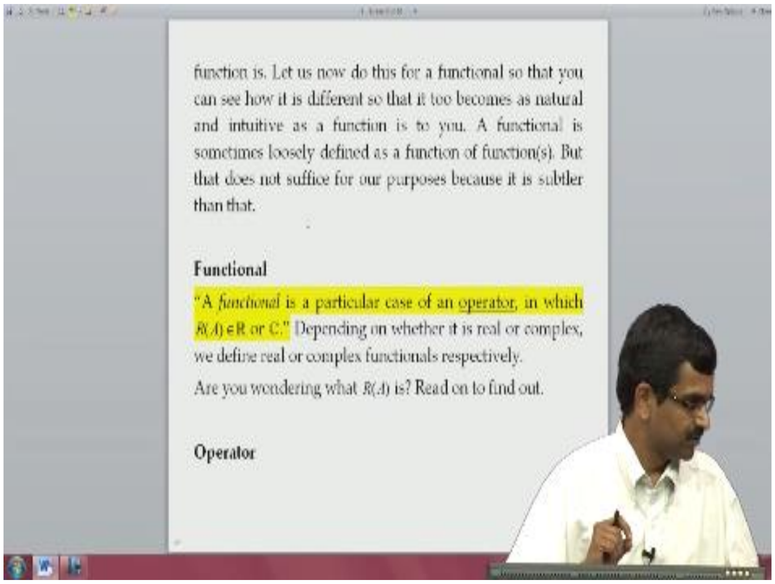


Now we come to the division of a functional so a functional is a particular case of an operator in which RA which is the range of that domain that you have belongs to real or complex number let us not worry do complex numbers now just focus on real numbers what it says is that a functional is simply an operator that assigns a real value for every function that is there so if we go back to our notion of a space okay.

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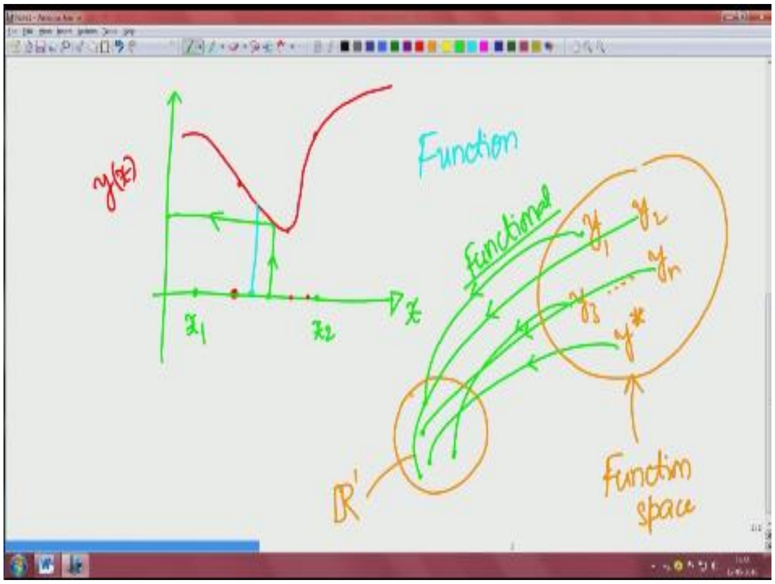


Let us call this for now a function space which contains all functions will be y1 there will be y2 y3 and so forth let us say yn and so forth and then there will be some y * also that we are after we want to find that minimizing function so what this functional does is we have the real number space okay let us say this is R1or just R that it contains all real numbers I could have drawn it like a line also okay real number space I can also put a real line right what this functional does is it takes every function let us I take this y1 it is a point in the function space there will be a corresponding point in the real number space okay that is what it a functional does functions operator that takes a function okay.

Let is a y3 will have a point and y * will have a point and yn will have a point and y3 you will have a point and so forth okay for every one of them there is a corresponding real number that is what is a functional okay just like a function is a mapping or what we said there is a correspondence from real number that is real number here to a point that correspond to this y here okay in a similar way a functional takes a function and finds a corresponding real number okay that is what is a functional is.

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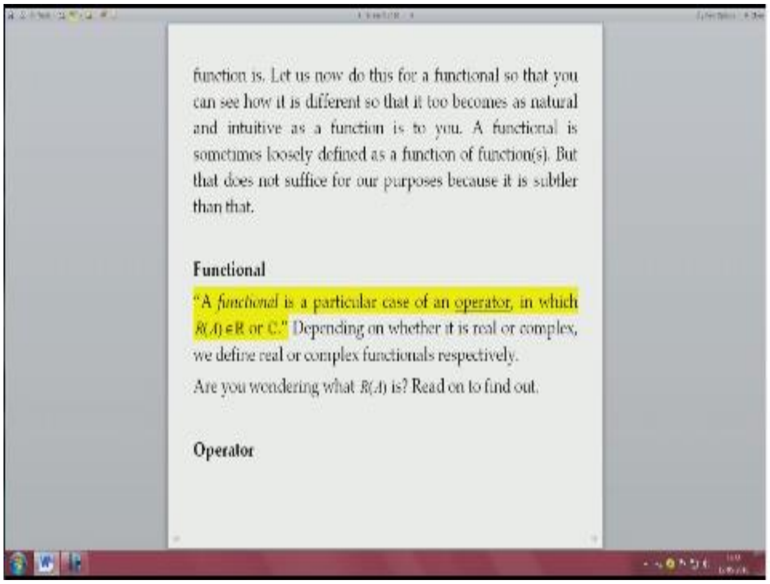


But then we put an operator's where to define what this operates in the definition we said a functional is a particular case of an operator in which the range is real number space.

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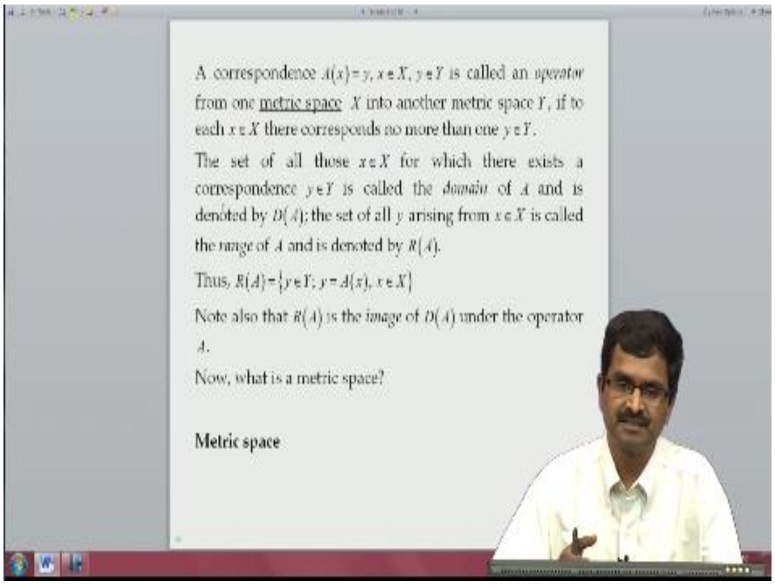


Now what is an operator again is a correspondence okay a(x) = y x is given we will find y by doing operation called a(x) like f(x) okay so this operator is defined from one metric space to another space we are underlined this metric space well define it what is a metric space okay and there will be the domain and range that you are familiar for a function so similarly here also when you say it is an operator there will be a domain for the operator you take an element from that from the domain and find something in the range which is the output okay.

Now we say the metric space we have to now talk about what a metric space is so this is something that you find in rigorous mathematics you have to define everything precisely so a lot of definitions will come and if you really want to know the mathematics of this very well we should take a course on real analysis or function analysis that will be a lot of work for engineering students so what we are going to do in the next this lecture and the next lectures is to provide you the background to study that if you wish to okay so intuitive understanding of various terms.

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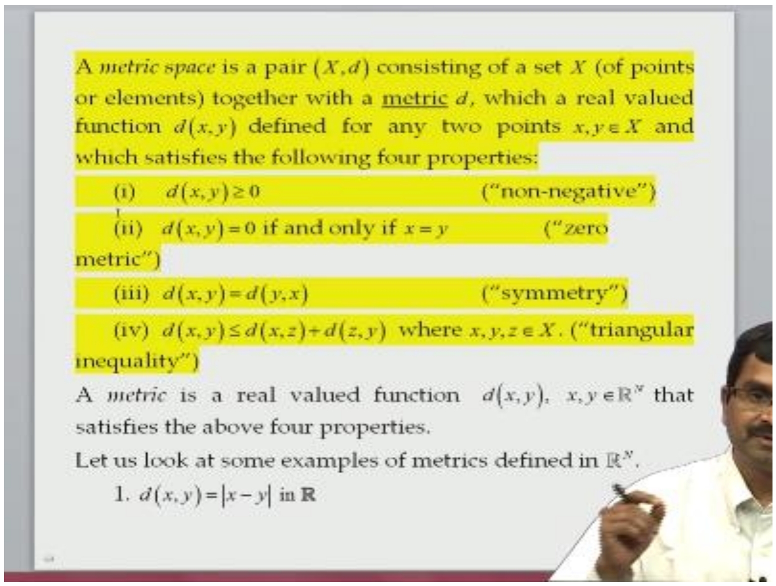


So let us understand what a metric space is, so metric space which is highlighted in yellow here is a pair x and d consisting of a set X together with a metric D okay that is called a metric space in this a metric space that has a metric meaning a measure what does it measure it actually measures the distance between two points with real numbers when you take two points you know how much the distance between them is let us say take a point here another point over there distance between them is as long as the Spanish.

If I take a two dimensional thing if I put two points on a plane I can talk about distance between them but then we are talking about the Euclidean space is that we are familiar with but this spaces can be very general and abstract also so metric space is one where for every point in that space okay x belongs to the points at element of that space there is a distance metric defined okay and this metric has to satisfy for properties which are listed here first of all this distance has to be non-negative this between two point cannot be negative it can be zero okay.

But cannot be negative so it is non-negative and second property says that if the distance is zero that means that two elements of quantity of taken must be one and the same that is what a second property that is called the zero matrix metric is zero between two points X and Y then X should be equal to Y and that there is symmetry also the third property d XY is equal to y X distance from A to B is same as design from BTW if that is satisfied that is another rule that metric space will satisfy the fourth one which is somewhat non-trivial.

But yet very obvious that distance that metric between two points XY is less than or equal to if you take another point Z distance from X to Z and distance from Z to Y or Y to Z because of symmetry okay this is called the triangular inequality it is quite obvious because if I are given let us say three points ABC okay let us say we take a point a point B one point C so this is a B and C okay.

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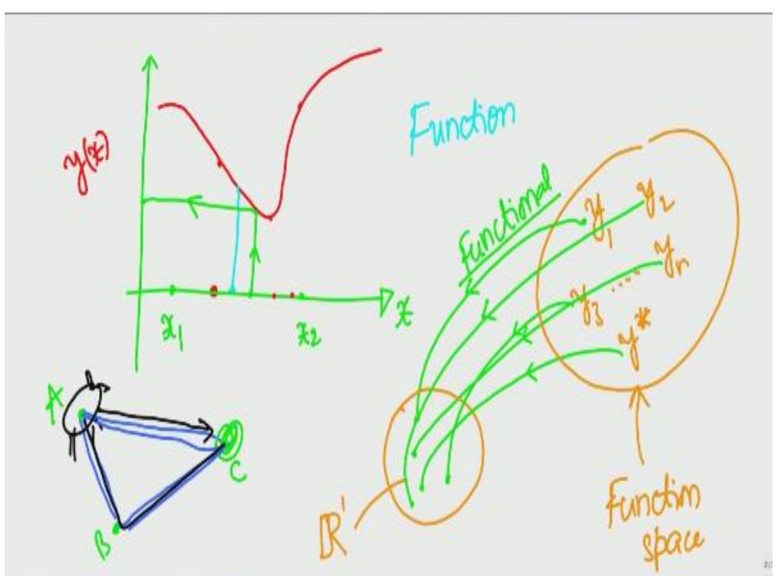

We are talking about two points let us say between a and c  we what it says that if I take a third point the distance from A to B +distance from B to C that will be greater than digital from A to C that is very obvious right but that is what is the property that this metric has to satisfy basically it says that this distance is less than some of these two distances that is very obvious triangular inequality okay my high school teacher used to call this donkeys theorem because if you put some grass over here let us do some green for grass if there is grass here and you have a donkey here that is my rendering of a donkey okay.

If you ask that donkey to goon take the grass the drunk is not going to go this way it is going to go straight like this right so this is a very intuitive obvious thing but the definition is definition the four properties for a metric space we have to understand them intuitively as well as mathematically okay if you define a particular metric distribution two points that satisfy these properties then that will be a metric space okay there are some examples given so if you see at the bottom of this slide this time between two points can be absolute value.

If you take x and y difference of them take absolute value that can be a metric or it could be just 1 or 0.

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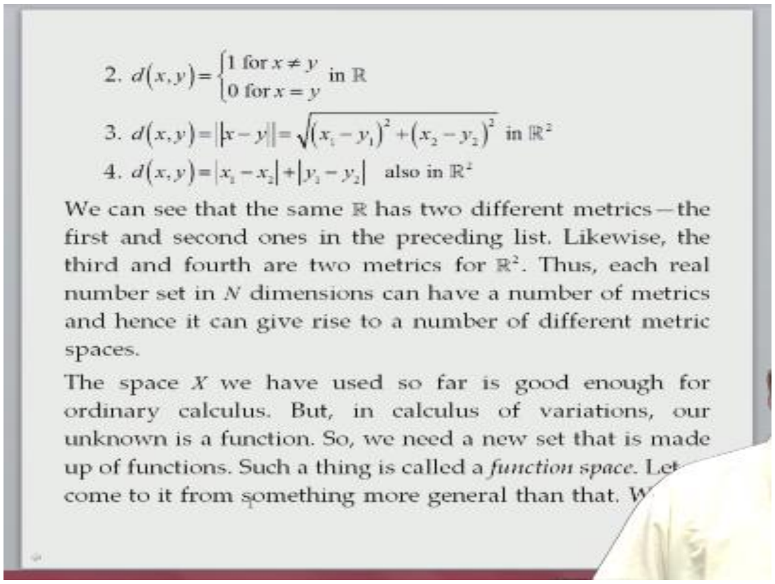

If x and y are different you say distance is one of XY are the same this edition 0 that is also metric you can go back and verify the four properties are satisfied they are indeed satisfied for this then that is a metric or you can have normal distance metric between two points square root of x 1 - y $1^2$ + x 2 - y $2^2$ if you are given two points XY okay or you can have absolute value of x 1 x 2 difference absolutely y 1 y 2 difference add them up that way a metric.

So one can make define a lot of different metric that satisfy those four properties okay so the function space that we talk about is also a space such as this one but we want it to be a little bit more general than what we just defined is metric space so we go to what is called a vector space okay.

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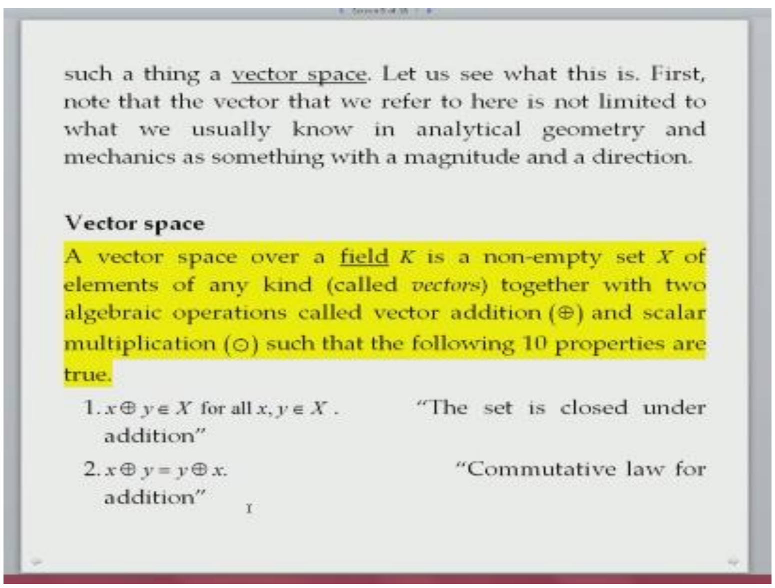


A vector space a little bit more elaborate than a metric space okay and this vector space needs what is called a field which is denoted by the symbol here k that has its own four properties just like metric has but then before they let us think about this vector space vector space is a non-empty set x of elements of any kind that called vectors these vectors are not our normal vector calculus or mechanics vectors that is you know v1 v2 we say not that kind of a vector like a speed acceleration they are all vectors right it is more than that okay.

The elements of any kind for example or function space these functions will also be called vectors they have to satisfy certain vector let us understand as a as a term rather than our usual understanding of vector in mechanics okay so this vector space a lot of elements together with two operations one is vector addition you will to add two things and scalar multiplication the scalar multiplication requires this field k basically our normal arithmetic operations okay and these addition and scalar multiplication or not our usual notion of addition scalar multiplication.

That is why they are denoted with a circle around it this +sign circled will be vector addition and the dot which we use for multiplication you put a circle around it becomes scalar multiplication between a vector element that element amount vector space as well as a real number that belongs to the scalar field the first property is that if you take two vectors if you add them up in this fatty particular manner you have to define an operation which is vector addition if you add those two vectors the resultant should also belong to that space that is what it says.

So this set that you are considering the vector space should be closed under addition meaning if you add two things you should get that back okay so in that sense if I call vectors of any kind I said you know if you say a bunch of a basket full of tomatoes does it form a vector space then you should show somebody convincingly that if you take tomato 1 and tomato two if you add them up you should get another tomato if you show that then it is fine but it is not the only property you have lot of other properties right.

If you add you can define that operation of adding to toe matrix if you wish right that is just saying it is like an abstract way but that is what this means that if you define vectors and an addition vector addition in your own way you should show that when you add two of them you should get a vector that belongs to the same space similarly you will have this commutative law if we say x + y should be equal to y +X okay.

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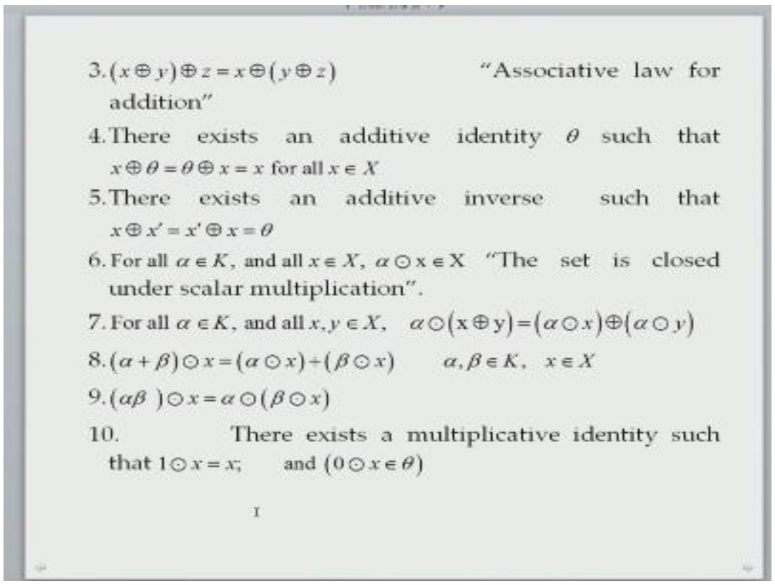


And then there are few more rules like that the associative law x + y in brackets and then Z is same as X +y +Z okay and there should be an additive identity that should be some vector in that space where if you add that you should get back the same add to another vector you should get back that other vector that means that this vector is a null vector that is that denoted wizard θ here so that is a fourth property X +θ should be equal to θ +X should be equal to X okay.

And they should be additive inverse also a negative point if you have to they should be minus 2 similarly if there is X there should be X prime such that when you add them up you should get the null vector okay those five things talk about addition and basically addition right vector addition first by properties next one's talked about scalar multiplication we have a scalar field k which is for now real numbers right and there is our usual arithmetic operations of addition and subtraction multiplication division and so forth the six property says that if α belong to the scalar field X belongs to the vector space capital X then α dotted scalar multiplication with X will belong to X again okay.

If you take a element small X correspond which belongs to the vector space an α bronze scalar field when you multiply them resultant should belong to the same capital X set and likewise associative law α multiplied multiplying X +y should equal α multiplying X +α multiplying why that is vector addition okay similarly α β if you have two scalars multiplied with X will be α x +β times X this all looks simple but just imagine that you want to define a vector space over self-satisfying this ten properties right.

You will find it that you can think of with some effort some ways of defining vector spaces most likely they would have been already defined by somebody else finding a new vector space the shutter is 10 properties is not easy okay so there is a multiplication t like we had additive identity no multiple identity if you multiply scalar get back the scalar one being there okay all these are 10 properties that a vector space must satisfy okay.

And we talked about this scalar field K that K has satisfied for properties which are listed here that is if there are two numbers a B a +B should be equal to a like competitive nature and a +B associate to +C is same as a +B +C and then the ship additive identity a +zero should give rise to a r 0 +I should give rise to a that such as 0 exist in the scalar field additive inverse there is a corresponding minus a so that some of them will be equal to 0 this is a scalar field properties.

So when you have scalar fields are than these four properties and a vector addition and scalar multiplication between a scalar field element and vector space element and this 10 property will discuss now satisfied then you have a vector space okay.

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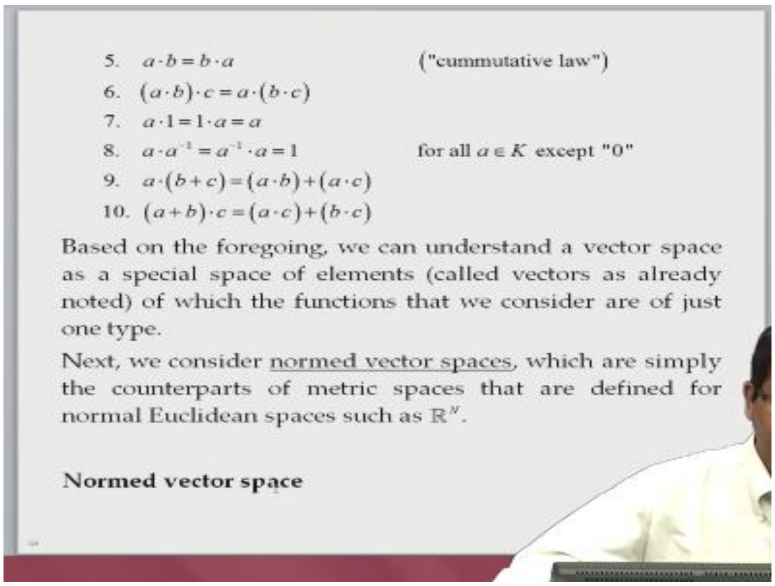


And there are few more actually this is not the four here four square Melvin is that they correspond to the vector space 10 rules a dot B with scalar multiplying B dot a and the rest okay there is a multiplicative inverse A inverse advocate reciprocal the usual arithmetic notion all that will be there okay so this is what is called the vector space this vector space we need to make so much of the discussion here insert new page okay. So why do we need to worry about this vector spaces.

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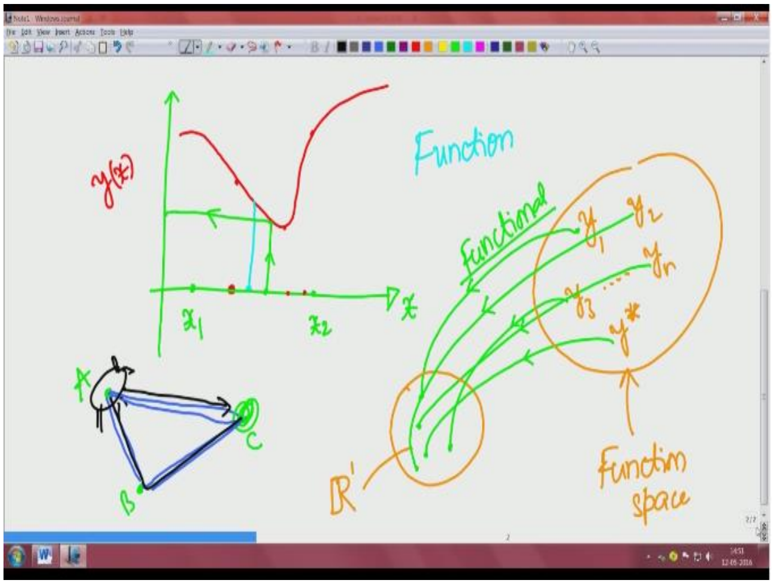


So much because us our function space okay he is actually a vector space our function space is a vector space meaning that a social function space there will be a scalar field there will be vector addition defined there is a scalar multiplication defined meaning that this function space that we consider satisfy the rules of vector space that we just discussed what we need to discuss so much okay.

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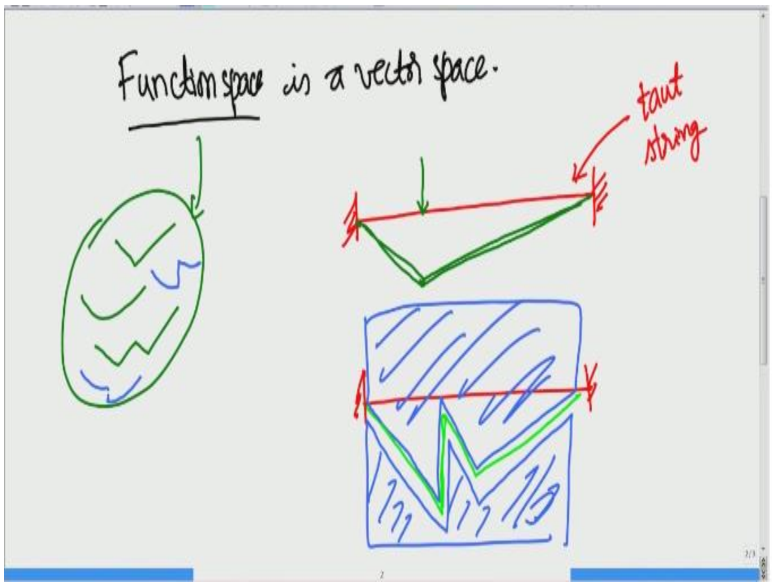


Let us take a simple example from mechanics to understand why do you need to worry about this unction space it is just a group of functions we can just leave it at that but actually it is not as easy as that because if I take let us say a problem where I take a string that is taught that is poled with some tension tart strength okay now when apply some force on it let us apply a force what happens it will deform how does it deform it is going to deform something like this okay.

Now you ask the question when we think about this minimum potential energy principle any elastic system for that matter any system tries to achieve minimum energy minimum potential energy to reach it static stable equilibrium position right similarly the string also when apply thef force it wants to go to a configuration where it will have least potential energy.

So it takes this form it also as if you think of this as a optimization problem for minimizing potential energy that is also a functional as we will see later or already we have seen when we discussed the geometry mechanics in calculus a variation there are lot of possibilities but it takes this particular shape right but you can see this particular shape has a kink in it so we are not saying all differentiable functions are the ones that we have a function space here right.

For this there are lot of shapes here they can be shapes like that their shapes like this the string can take right so now what do we make of this function space what should be there so only continuous once but then even discontinued things are possible so let us just take an example if I take and get something like this here which is really discontinuous over there how do I do that how to take a wedge of a form like this okay imagine that I have a wooden block like this from that side and then another wooden block of matching profile I squeeze the wire between these two.

So this green line that you have shown as no option than having this discontinuity so I need to allow those also here discontinuous functions okay they are there and then sudden discontinuity and then go there because the string needs that also so we need to define what this function spaces which will continue in the next lecture.