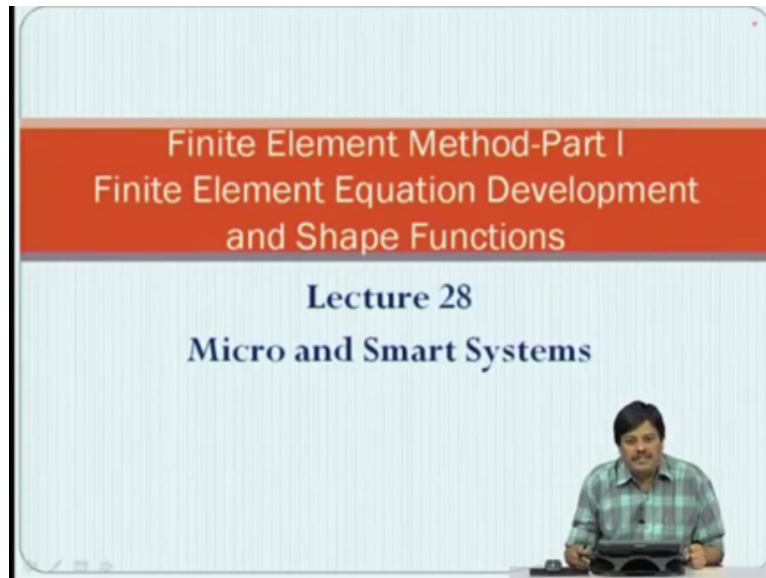


Micro and Smart Systems
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Lecture - 28
Finite Element Equation Development and Shape Functions

So this is lecture number 28 of the micro and smart system course.

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And here we would be talking about the finite element method development as such and as a part of it we will develop the finite element equations and the shape functions that are required for the development of many elements. So just to give a summary of what we talked about in the last class.

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Finite Element procedure

1. It uses Weak form of the Governing Equation with the weight function \mathbf{r} is same as the dependent variable
2. The system under investigation is split into many sub-domains called elements and over each of these sub-domain, the dependent variable variation is assumed and converted in the form

$$\bar{u}(x, y, z, t) = \sum_{n=1}^N a_n(t) \phi_n(x, y, z)$$
A screenshot of a video lecture slide. The slide has a light blue background. It contains the title "Finite Element procedure" and two numbered steps. Below the steps is a mathematical equation for the shape function expansion. In the bottom right corner, there is a small inset video of a man in a green and blue checkered shirt sitting at a desk.

We said that finite element uses the weak form of the governing equation with the weight function v which is same as the dependent variable function, which is given in equation 1 here. So the system and the investigation is split up into many subdomains, which we call it elements and each element has a number of nodes. Over each of these subdomains, the dependent variable is assumed in certain form of a series, which is given by here equation 1 where $a_n(t)$ and ϕ_n has certain meanings.

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FEM procedure

- This equation is the standard form for most of the approximate method that was described previously. However, here in FEM each of these have specific meaning
- $a_n(t)$ represents nodal degrees of freedom
- $\phi_n(x, y, z)$ represents shape function normally denoted as N
- The above variation of dependent variable, when substituted in the weak form of the governing equation and minimized as per PMPE OR HP, we get

1. Set of algebraic Equation for Static problems
2. Coupled set of ordinary differential equation for dynamic problems

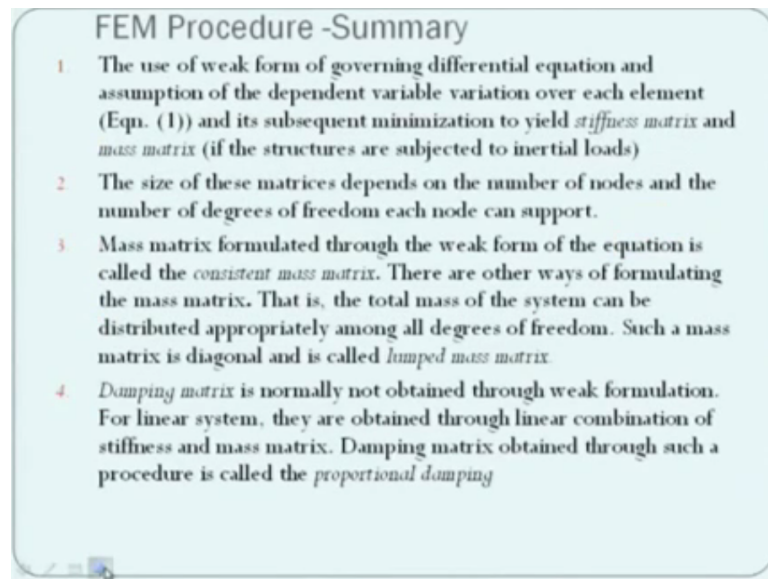
Let us see what these meanings are. So the equation 1 is the standard form for the most of the approximate methods that were described previously; however, here in FEM each as I said earlier has specific meaning. For example, an $a_n(t)$ represents the nodal degrees of freedom that is each element will have a node. For example, if it is a rectangular element, we have 4 nodes and the degree of freedom in each of this node represents this $a_n(t)$.

And the ϕ_n of x, y, z represents the shape function, which is normally denoted by N in finite elements. The above variation of the dependent variable when substituted into the weak form of the governing equation and minimized it we get 2 sets of problems. If the problem is simple static that is the variables where a_n is just a function is a constant and the ϕ is not a function of t .

Then we get a set of algebraic equations, which is normally for static problems. If the problem is dynamic, then we get a couple set of ordinary differential equation that is basically a governing differential equation is converted into algebraic equation for static problems and

a couple set of ordinary differential that is the PDE is transformed into a set of ODEs for dynamic problems.

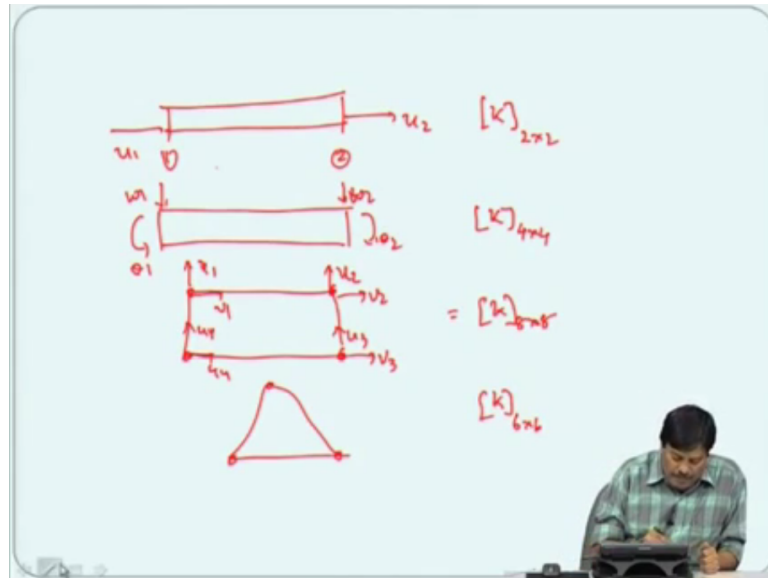
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Let us summarize the procedure now. So basically what it says here is the substitution of the assumed function into the weak form of the governing equation and the subsequent minimization will give us a set of 2 matrices what is called the stiffness matrix and the mass matrix and the mass matrix will be there only if the structure is dynamic in nature that is it is subjected to inertial loads.

And the size of these matrices depends upon the number of nodes and the number of degrees of freedom the each node can support. For example, in order to explain this in little more detail let us take a simple rod problem okay.

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So the rod problem basically will have 2 degrees of freedom that is the axial displacement u_1 and u_2 and this is node 1 and node 2. So in this case, the stiffness matrix k which we represent will be $2/2$. If the same problem is now a beam problem that is when the loading is in the transverse direction in this direction, then it will take 2 degrees of freedom that is the w_1 and θ_1 the slope and w_2 and θ_2 slope.

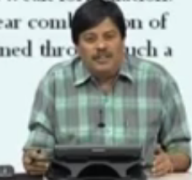
So basically in which case this stiffness matrix will be $4/4$. The other possibility is these are all 1-D elements. Suppose we have a rectangular element, then this has 4 nodes and each node can support u_1, v_1 , then this is u_2, v_2 , this is u_3, v_3 , and u_4, v_4 and stiffness matrix will be $8/8$. Suppose we have a triangular element other possibility then we have 3 nodes and if each node can support 2 degrees of freedom then the matrix k will be $6/6$.

So basically it depends upon how many degrees of freedom it can have. We will come to the definition of degrees of freedom little later.

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FEM Procedure -Summary

- 1 The use of weak form of governing differential equation and assumption of the dependent variable variation over each element (Eqn. (1)) and its subsequent minimization to yield *stiffness matrix* and *mass matrix* (if the structures are subjected to inertial loads)
- 2 The size of these matrices depends on the number of nodes and the number of degrees of freedom each node can support.
- 3 Mass matrix formulated through the weak form of the equation is called the *consistent mass matrix*. There are other ways of formulating the mass matrix. That is, the total mass of the system can be distributed appropriately among all degrees of freedom. Such a mass matrix is diagonal and is called *lumped mass matrix*.
- 4 *Damping matrix* is normally not obtained through weak formulation. For linear system, they are obtained through linear combination of stiffness and mass matrix. Damping matrix obtained through such a procedure is called the *proportional damping*



Now the mass matrix formulated through the weak form as I said what we get there is called a consistent mass matrix. There are other ways of formulating the mass matrix, which we will not go in detail in this course, it is beyond the scope of this course so in which case if that is done, the consistent mass matrix is a completely full matrix depending upon the what structure it is $2/2$ for a rod, $4/4$ for the beam etc.

However, there are alternate ways of formulating this mass matrix. Suppose you take the total mass and length only to the translation degrees of freedom then you will get a lumped mass matrix, which will be completely diagonal. There are various advantages of using the diagonal matrix when you do a dynamic analysis, which we are not going through in this course.

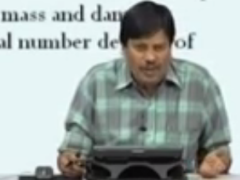
The damping matrix another matrix that is coming from the weak form is many times not used in weak forms for many reasons because damping is a very complex phenomenon, which is not well understood even today, but that has to be there. The damping comes because whenever there is time dependent force then this force will not could be sustaining the same amplitude it will die down after some time because of the various reasons such as the environment, such as the material in which it is, the responses traversing etc.

But however to represent it they formulate the stiffness and mass matrix and take the damping matrix as a combination of the stiffness and mass matrix and such a matrix is called the proportional damping matrix.

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FEM-Summary (Cont)

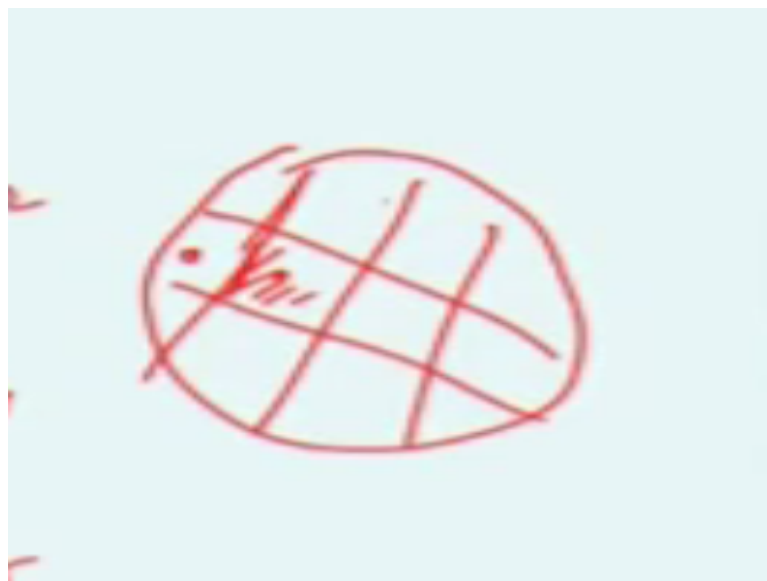
- 5 FEM comes under the category of stiffness method, where the dependent variable (say displacements in the case of structural systems) are the basic unknowns, the satisfaction of compatibility of displacements across the element boundaries is automatic as we begin the analysis with displacement assumption
- 6 Equilibrium of the forces are ensured only within the element. Global Equilibrium is not ensured. It is accomplished by assembly of the elemental matrices that are sharing the common interfaces
- 7 Similarly, the force vector acting on each node, are assembled to obtain global force vector. If the load is distributed on a segment of the complex domain, then using equivalent energy concept, it is split into concentrated loads acting on the respective nodes that make up the segment. The size of assembled stiffness, mass and dam matrices is equal to $n \times n$, where n is the total number degrees of freedom in the discretized domain.



There are two categories of numerical method of solving under FEM, one is based where the forces are basic unknowns these are called the force method which is also a kind of FEM method, but it is not very popular, but the conventional FEM method which is extensively used is called the stiffness matrix or the stiffness method where the dependent variable say the displacement in the case of structures or it is current and magnetic field in terms for the Maxwell's equations are the electromagnetic problem are the basic unknowns.

And the satisfaction of the compatibility of the displacement across the element boundaries is automatic as we begin here.

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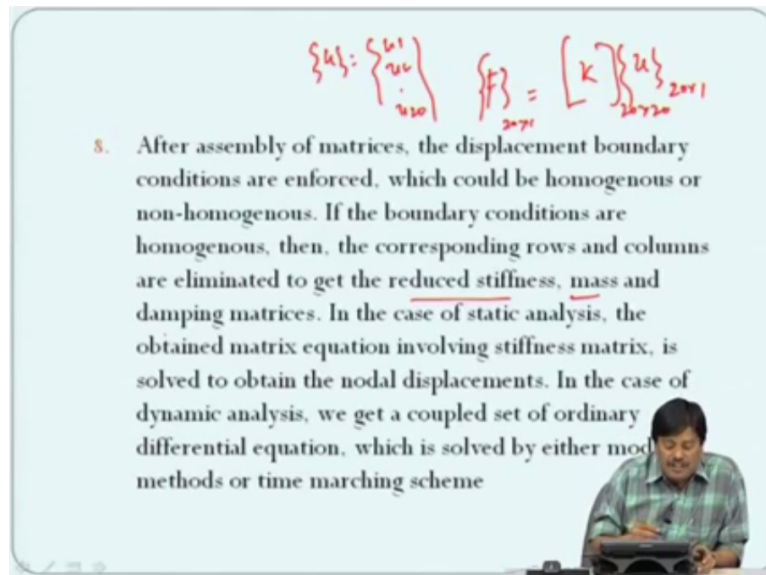
So in order to make this very clear let us again take a domain and this domain is split up into say number of element as shown here. Let us isolate here so in this case the displacement

coming from this element and the displacement coming from this element are compatible at this edges because we start with the compatibility. However, because of the applied load somewhere else there will be forces generated and the forces across this inter element boundary will not be compatible, will not be in equilibrium.

So this equilibrium has to be in force. How do we do that? So we generate the matrices here for each of the stiffness matrix when we assemble it, the assembly process ensures that the forces here are in equilibrium. So basically there are two aspects, which we said in the elasticity, one is the compatibility and other is the equilibrium. Here the compatibility is ensured whereas the equilibrium has to be enforced.

The assembly of matrices will basically ensure that the equilibrium is satisfied. So when we assemble the whole matrices so in the case of a rod we have a 2/2 when we assemble all the matrices say it has 10 degrees of freedom so the total assemble matrix will be 20/20. So it will be n/n. So if there are n degrees of freedoms, the assemble stiffness matrix will be of the order n/n.

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Handwritten equations on the slide:

$$\{u\} = \begin{Bmatrix} u_1 \\ \vdots \\ u_{20} \end{Bmatrix} \quad \{F\} = \begin{Bmatrix} F_1 \\ \vdots \\ F_{20} \end{Bmatrix} \quad [K] = \begin{Bmatrix} k_{11} & \dots & k_{1,20} \\ \vdots & \ddots & \vdots \\ k_{20,1} & \dots & k_{20,20} \end{Bmatrix}$$

8. After assembly of matrices, the displacement boundary conditions are enforced, which could be homogenous or non-homogenous. If the boundary conditions are homogenous, then, the corresponding rows and columns are eliminated to get the reduced stiffness, mass and damping matrices. In the case of static analysis, the obtained matrix equation involving stiffness matrix, is solved to obtain the nodal displacements. In the case of dynamic analysis, we get a coupled set of ordinary differential equation, which is solved by either modal methods or time marching scheme

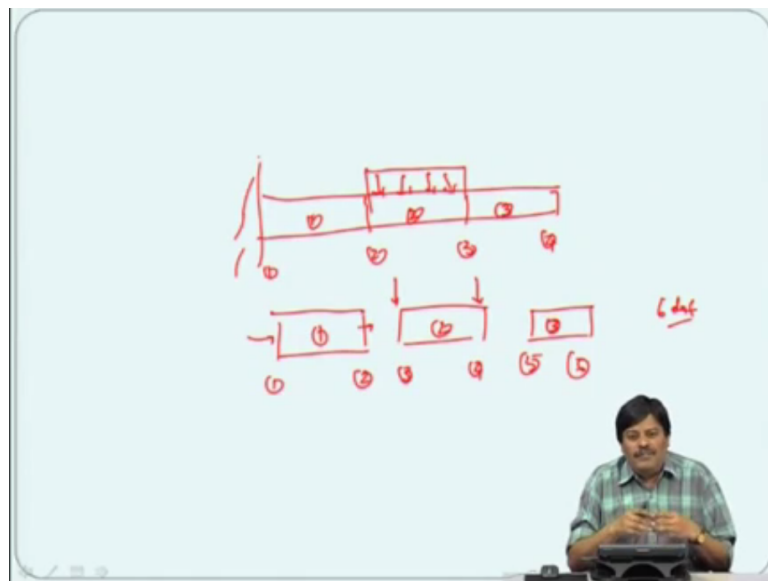
The slide also features a video inset in the bottom right corner showing a person sitting at a desk, looking at a laptop screen.

Then what we do? After the assembly process, then we get a stiffness matrix say your k will be say 20/20 for a 10 element rod matrix let us say and this 20/20 will corresponds to the 20 degrees of freedom in the rod. Each element will have 2 degrees of freedom; there are 10 elements so there are 20 degrees of freedom. So in that 20 degrees of freedom, you cannot solve it without applying the boundary condition.

Because this matrix will be singular so basically you will have your force vector F which is $20/1$ will be equal to $k \cdot u$ which is $20/1$ where u will have basically u_1, u_2, \dots, u_{20} . So out of which we need to say which are the boundary conditions are known, suppose u_1 is 0 so that has to be eliminated. So when we eliminate it, we get a reduced stiffness matrix and similarly we can get a reduced mass and damping matrices if you are doing a dynamic problem okay.

Then basically the reduced stiffness matrix is solved for the applied loads. So many times the load would be basically distributed.

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For example, if you have a structure say we have a rod where some portion is having some distributed loading. So in the finite elements, we can only handle concentrated loads. So basically we split this element into sub elements so this will be element 1, so this will be element 2 and this will be element 3. So I will name it as 1, 2, and 3. So each will have say 2 degrees of freedom.

So this will have node 1, 2, 3, 4 so you will have 1, 2, 3, 4, 5, 6, so 5 and 6 has a common node 3 in the global direction. So now we have 6 degrees of freedom okay. So now what we do basically is this distributed load has to be converted into concentrated load acting in these 2 element edges and when we assemble it corresponding to the degree of freedom 2 and 3, we have the nodal vector going into the matrices.

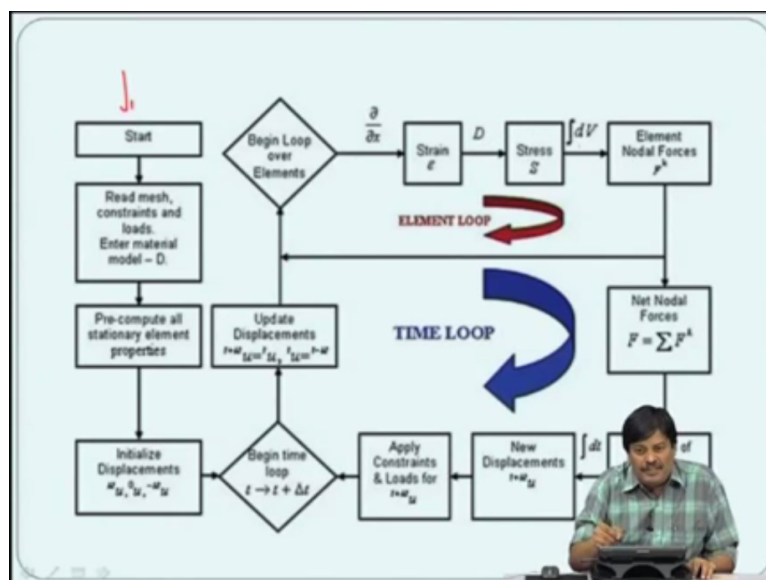
So essentially what we are saying is the distributed load has to be transformed into equivalent concentrated load when we do a finite element procedure. So the resulting equation will then

be solved using the standard algebraic solution methods and we solve for the displacement. Once we get the displacement, we now use the elemental equation. For each element $F=ku$ then get the forces and hence the stresses and whatever quantities you want beyond displacement can be got by post processing the results.

So as I said earlier in my earlier lectures, the finite element procedure has the preprocessing that is meshing the system then the solution of the equation and the post processing. So the 3 steps have to be followed in the finite element procedure and in the case of dynamic analysis we have a PDE that is partial differential equation is converted into an ordinary equation, which are coupled together to many differential equations.

There are various methods that we use to actually solve these equations such methods are called the modal methods where we use the Eigen values, we convert the dynamic problem into an Eigen value problem and find the Eigen values and use these Eigen values to find the dynamic response or we can use finite difference scheme, which we talked about little later by actually using the finite difference scheme for the reduced ordinary differential equation, which we call it as time marching scheme. We are not going into details of these right now.

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So to understand this we can take this big flow chart. So this is the preprocessing part the left hand part, so this is the preprocessing part where we start, we read the mesh, constraints where the loads are, what are the types of loads, what are the types of materials okay. Then we pre-compute all the elemental properties and then we initiate the displacement. So if you are doing a dynamic analysis, we can ignore this.

So we begin to do loop over the element, we find the strain displacement matrix, we find a material matrix and if it is a nonlinear problem, we have to update the stress, we do not care to do about it. Then we solve for the displacement, compute the element forces and we do this for all the elements. So before this we generate the stiffness matrix, assemble it. After the we generate the strain displacement matrix we generate the stiffness matrix, assemble it, solve for the displacement, find the elemental forces.

If you are doing a dynamic problem, we have to set the initial displacement to 0 then we need to begin a time marching scheme, then we go about doing it, apply the constraint loads, find the new displacement and this has to be done for each of the time step.

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Finite Element Equations

$$\delta \int_{t_1}^{t_2} (T - U + W_{nc}) dt = 0 \quad (8.69)$$

where the kinetic energy T is given by

$$T = \frac{1}{2} \int_V \rho (\dot{u}^2 + \dot{v}^2 + \dot{w}^2) dV$$

Taking the first variation and integrating, we get

$$\int_{t_1}^{t_2} \delta T dt = \int_{t_1}^{t_2} \int_V \rho \left(\frac{du}{dt} \frac{d(\delta u)}{dt} + \frac{dv}{dt} \frac{d(\delta v)}{dt} + \frac{dw}{dt} \frac{d(\delta w)}{dt} \right) dV dt$$

Integrating by parts, the above equation and noting that the first variation vanishes at times t_1 and t_2

$$\int_{t_1}^{t_2} \delta T dt = - \int_{t_1}^{t_2} \int_V \rho (\ddot{u} \delta u + \ddot{v} \delta v + \ddot{w} \delta w) dV dt = - \int_{t_1}^{t_2} \int_V \rho \{\delta \ddot{u}\}^T \{\ddot{u}\} dV dt \quad (8.70)$$

where, $\{\ddot{u}\} = \{\ddot{u} \ \ddot{v} \ \ddot{w}\}^T$ represent the acceleration vector and $\{\delta \ddot{u}\} = \{\delta \ddot{u} \ \delta \ddot{v} \ \delta \ddot{w}\}^T$ represent the vector containing the first variation of displacements.

So now let us begin the finite element equations. How do we get the finite element equations? So we again talk about we go back and revisit our energy theorems and one of the energy theorems we derived is the Hamilton's principle because we are now trying to derive the equations for the general dynamic equations and then remove the dynamic part of it and solve only for the static part.

So if you go back to our Hamilton's principle, the Hamilton's principle states that the minimization of the total energy between the time t_1 and t_2 that is T is the kinetic energy, U is the potential energy and W_{nc} is the work done by the non-conservative forces and if you integrate this with respect to t_1 and t_2 equal to 0 will give us the governing equation. So now let us go and find out each one of them.

Let us take the kinetic energy for a three dimensional system. Kinetic energy is nothing but integral over the volume of the mass times velocity and the velocity has 3 components that is u, v, w in the 3 coordinate directions so we have u dot, v dot and w dot here. So now we take this and you know that the variational operator operates like a differential operator so we take the variation in the 3 respective directions and then integrate by parts.

So now when we take the variation of T, now we have rho*du/dt*d/dt of delta u, dv/dt*d/dt delta u etc. Now you have 2 time dependent functions, we integrate by parts and then make sure that the first variation vanishes at time t1 and t2 we have done this in the last class so I am not going to repeat it here and when we do this ultimately we get this expression where u double dot is basically the acceleration.

So this can be written in the matrix form as variation of the transpose vector into d dot*v where d double dot is nothing but a vector of acceleration and delta d is nothing but du, dv and dw. So we have converted this equation into matrix form with a variation on the displacement vector multiplied by the acceleration vector.

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The strain energy for a 3-D body in terms of stresses and strains is given by

$$U = \frac{1}{2} \int_V (\sigma_{xx} \epsilon_{xx} + \sigma_{yy} \epsilon_{yy} + \sigma_{zz} \epsilon_{zz} + \tau_{xy} \gamma_{xy} + \tau_{yz} \gamma_{yz} + \tau_{zx} \gamma_{zx}) dV = \frac{1}{2} \int_V \{\epsilon\}^T \{\sigma\} dV \quad (8.71)$$

For linear elastic case, the constitutive law by Equation (???) can be written as $\{\sigma\} = [C]\{\epsilon\}$. Hence the strain energy becomes


$$U = \frac{1}{2} \int_V \{\epsilon\}^T [C] \{\epsilon\} dV$$

$$\int_{t_1}^{t_2} \delta U dt = \int_{t_1}^{t_2} \int_V \{\delta \epsilon\}^T [C] \{\epsilon\} dV dt \quad (8.72)$$

The work done by the body forces, surface forces, damping elements and the concentrated forces are clubbed under W_{ac} . That is $W_{ac} = W_B + W_S + W_D$. Work done by the body forces is given by

$$W_B = \int_V (B_x u + B_y v + B_z w) dV = \int_V \{d\}^T \{B\} dV \quad (8.73)$$

The first variation of the body force work is given by

$$\int_{t_1}^{t_2} \delta W_B dt = \int_{t_1}^{t_2} \int_V (B_x \delta u + B_y \delta v + B_z \delta w) dV dt = \int_{t_1}^{t_2} \int_V \{\delta d\}^T \{B\} dV dt$$


Next, we take the strain energy part. We know the strain energy, we derive the strain energy for a 3-D state of stress which is nothing but is given by this expression where it is a product of stress and strain in each of these planes, which can be written in matrix form in this form and when we take a variation we have to take a variation on displacement or the quantities

dependent on the displacement and the quantity dependent on displacement here is the strain because we know the strain displacement relationship so we write this δU in this form.

Now we take the work done by the non-conservative forces, which can be split up into 3 different forces, one due to body force, one due to surface force acting on the surface and one due to damping force. So now work done by the body force is $B_x \cdot u$, $B_y \cdot v$ and $B_z \cdot w$ and it is a volume integral because B_x , B_y are all force per unit volume. So this can be written in this form in the matrix form and taking a variation of that we get this in this form.

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The first variation of this work is given by

$$\int_{t_1}^{t_2} \delta W_S dt = \int_{t_1}^{t_2} \int_{\partial V} \{\delta d\}^T \{t_s\} dS dt \quad (8.75)$$

Similarly, the first variation of the work done by the damping force is given by

$$\int_{t_1}^{t_2} \delta W_D dt = - \int_{t_1}^{t_2} \int_V \{\delta d\}^T \{F_D\} dV dt \quad (8.76)$$

If the damping is viscous type, then the damping force is proportional to the velocity and is given by $\{F_D\} = \eta \{\dot{d}\}$, where η is the damping coefficient and $\{\dot{d}\} = \{\dot{u} \ \dot{v} \ \dot{w}\}^T$ is the velocity vector in the three coordinate directions. Now using Equations (8.70), (8.72), (8.73), (8.75) and (8.76) in the Hamilton's Principle (Equation (8.69)), we get

$$\int_{t_1}^{t_2} \int_V \{\delta d\}^T \rho \{\ddot{d}\} dV dt - \int_{t_1}^{t_2} \int_V \{\delta \varepsilon\}^T [C] \{\varepsilon\} dV dt + \int_{t_1}^{t_2} \int_V \{\delta d\}^T \{F\} dV dt - \int_{t_1}^{t_2} \int_{\partial V} \{\delta d\}^T \{t_s\} dS dt - \int_{t_1}^{t_2} \int_V \{\delta d\}^T \{F_D\} dV dt = 0 \quad (8.77)$$


Similarly we can write the surface forces in this form where t_s is the surface vector acting in the 3 coordinate direction and similarly we can write damping in this form. While treating damping, there are different ways of damping, the damping could be frictional, damping could be based on material property, damping could be viscous. The most common type which is mathematically easily tractable is basically the viscous damping.

So we assume that the viscous damping force is given by this expression, which is directly proportional to some constant, which is called the damping constant multiplied by the \dot{d} , \dot{d} is essentially the velocity vector. So now we put all these things together we get this whole thing in the Hamilton's theorem. Now we simplify each one of them. Let us do that.

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$$\begin{aligned}
 \{d(x, y, z, t)\} &= [N(x, y, z)]\{u_e(t)\} \\
 \{\dot{d}\} &= [N]\{\dot{u}_e\}, \quad \{\ddot{d}\} = [N]\{\ddot{u}_e\} \text{ and } \{\delta d\} = [N]\{\delta u_e\}
 \end{aligned}$$

$$\begin{aligned}
 \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} &= \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \\
 \{e\} &= [B]\{d\} \\
 \{\delta e\} &= [B]\{\delta d\}
 \end{aligned}$$

$$\begin{aligned}
 \int_V \rho \{\delta d\}^T \{\ddot{d}\} dV &= \int_V \rho \{\delta u_e\}^T [N]^T [N] \{\ddot{u}_e\} dV \\
 &= \{\delta u_e\}^T \left[\int_V \rho [N]^T [N] dV \right] \{\ddot{u}_e\} = \{\delta u_e\}^T [M] \{\ddot{u}_e\} \quad (8.83)
 \end{aligned}$$

$$\begin{aligned}
 \int_V \{\delta e\}^T [C] [e] dV &= \int_V \{\delta u_e\}^T [B]^T [C] [B] \{u_e\} dV = \{\delta u_e\}^T \left[\int_V [B]^T [C] [B] dV \right] \{u_e\} \\
 &= \{\delta u_e\}^T [K] \{u_e\}
 \end{aligned}$$

So before we do that now we said that in finite element we assume the variation over each element. That is the variation can be expressed in terms of the shape functions, which I talked about in the first slide multiplied by the an of t, the an of t is the elemental displacement vector of the element, which I call it as u_e of t. For the sake of convenience, u_e means e is the elemental vector whereas u is the global vector.

So now I can get because here \dot{d} is $N \cdot u_e$, N is only a spatial dependent. So the time dependent comes from the displacement and \ddot{d} is essentially $N \cdot \ddot{u}_e$ and $\delta d = N \cdot \delta u_e$. So basically now we put that into the first expression, which is the inertial expression coming from the kinetic energy. So we can write this and δd transposes nothing but δu_e transpose into N transpose multiplied by \ddot{d} is nothing but $N \cdot \ddot{u}_e$.

So when I put this I get this and this quantity is called the mass matrix and the way we can do that is the basically is the consistent mass matrix. Now let us take the strain energy portion which has the strain displacement matrix. So strain displacement we know $\epsilon_x = du/dx$ etc. In matrix form, it is given by here. Now $\epsilon = B \cdot d$. So now I know u, v, w are basically given by the shape function relation, we substitute here u, v, w in this form.

Then we can write $\epsilon = B \cdot d$ and $\delta \epsilon$ is $B \cdot \delta d$. We substitute it into the strain energy expression here that is basically coming from here. Then when we do that we get this form and this form $B^T C B$ is basically the stiffness matrix where C is the material

matrix. We derive this material matrix for isotropic, orthotropic or whatever the material conditions are in the structures.

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$$\int_V \{\delta u\}^T \{B\} dV = \int_V \{\delta u_e\}^T [N]^T \{B\} dV$$

$$= \{\delta u_e\}^T \left[\int_V [N]^T \{B\} dV \right] = \{\delta u_e\}^T \{f_B\}$$

$$\int_S \{\delta u\}^T \{t_s\} dS = \{\delta u_e\}^T \{f_s\}, \{f_s\} = \int_S [N]^T \{t_s\} dS$$

$$\int_{t_1}^{t_2} \{\delta u_e\}^T \left[[M]\{\ddot{u}_e\} + [D]\{\dot{u}_e\} + [K]\{u_e\} - \{f_B\} - \{f_s\} \right] dt = 0$$

$$- \int_V \{\delta u\}^T \eta \{d\} dV = -\{\delta u_e\}^T \left[\int_V \eta [N]^T [N] dV \right] \{\dot{u}_e\} = -\{\delta u_e\}^T [D]\{\dot{u}_e\}$$

$$\boxed{[M]\{\ddot{u}_e\} + [D]\{\dot{u}_e\} + [K]\{u_e\} = \{R\}}$$

So this can be written in this form mathematically. Similarly we can write the body force delta d using this expression here we can write a delta d transpose is delta ue transpose into N transpose and multiplied by the body force vector and this is the discretization of the body force. So this basically converts the body force, which is distributed into the concentrated force acting on the nodes.

Similarly, we can do the surface force and when we put all these things together we get this equation and here delta ue is the incremental displacement, which cannot go to 0 in this expression so the only thing that can go to 0 is the 1 within the bracket and that is what is the governing equation, discretize form of the governing equation in the FEM. So the mass matrix is completely got from the discretized portion that is N transpose N.

The damping is got here N transpose N*eta and k is B transpose CB integral. So we have everything in the discretized form which we need to solve. If we are solving a static problem, we ignore this portion, we ignore this portion, we solve only Ku=R.

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Finite Element Terminologies

- **Degree of freedom:** It is the number of independent motions a structure can support. Example, Rod can support only axial motions and hence each node can have only one degree of freedom, namely axial motion.

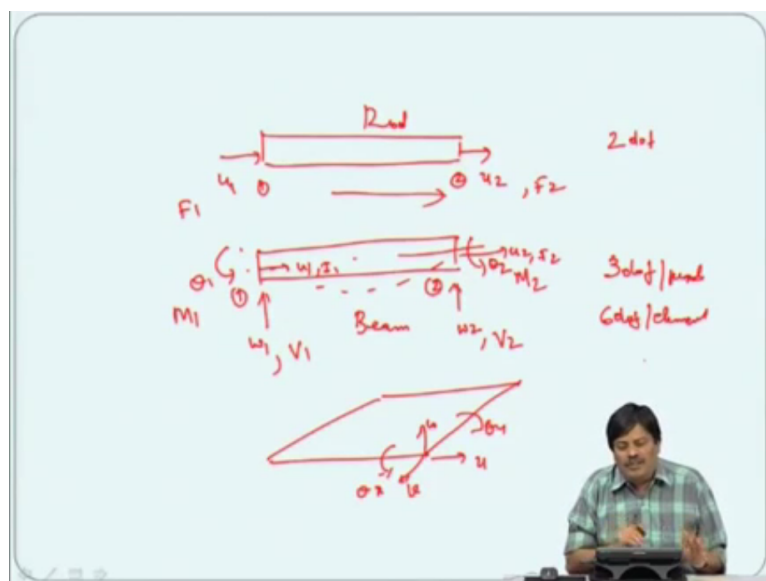
A beam can support both transverse motion and rotation, that is each node in a beam will have 2 degrees of freedom.

A composite plate can support, 3 translational motion in 3 coordinate direction and 2 rotations in the in-plane directions. Hence a plate node can have 5 degrees of freedom. If twist is present, one can add a sixth degree of freedom by adding a rotation about the axial direction

So now let us introduce certain terminologies. Degree of freedom, what is degree of freedom? It is the number of independent motion a structure can support. Example, a rod can support only axial motion in the 2 nodes and hence it has 2 degrees of freedom. It is a number of independent motions. For example, a beam can support both transverse motion and rotation so that is each node in a beam will have 2 degrees of freedom.

A composite plate can support 3 translational motion that is in 3 coordinate direction and 2 rotation in plane direction. Hence, the plate node will have 5 degrees of freedom. If we need to add a twist about the axis then we can add the sixth degree of freedom.

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To explain this a little bit, so let us take a rod. So rod can move only in this direction, it can move only in this direction so basically that means you have 2, this is node 1, this is node 2.

So this is u_1 and u_2 , correspondingly you have F_1 and F_2 so it can support only 2 degrees of freedom. So this is a 2 DOF model. As I said earlier, this is a rod, so this is a beam, beam can bend so it can bend, it can undergo this bending and this bending is caused by the rotation that is the moment that is applied at the end.

So it can support the transverse displacement w at node 1, this is node 2, w_1 and w_2 and it can support θ_1 and θ_2 and corresponding to the w_1 because everything will have a displacement dependent variable determination and a force variation and what is cause and effect as we found in the variational principle. The w_1 is caused by the shear force 1, w_2 is caused by shear force 2 and θ_1 is caused by the moment 1 and moment 2 and as I said if it is a composite beam, the mid plane does not coincide with the neutral plane.

That means there will be an additional component u_1 , u_2 , F_1 , F_2 so it is basically a combination of both rod and beam. So each will have 3 degrees of freedom per node and 6 DOF per element. So if it is a plate, we have plate so you would basically have it can have the translational degrees of freedom u , w , v . Then it can have the θ_x , θ_y and you can also have a twist degree of freedom if there is a twisting angle to it.


So these are called the independent degrees of freedom that you should understand what is an independent degrees of freedom? So every beam will have a degree of freedom that is defined by the motion that is the structure is undergoing.

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Finite Element Terminologies (cont)

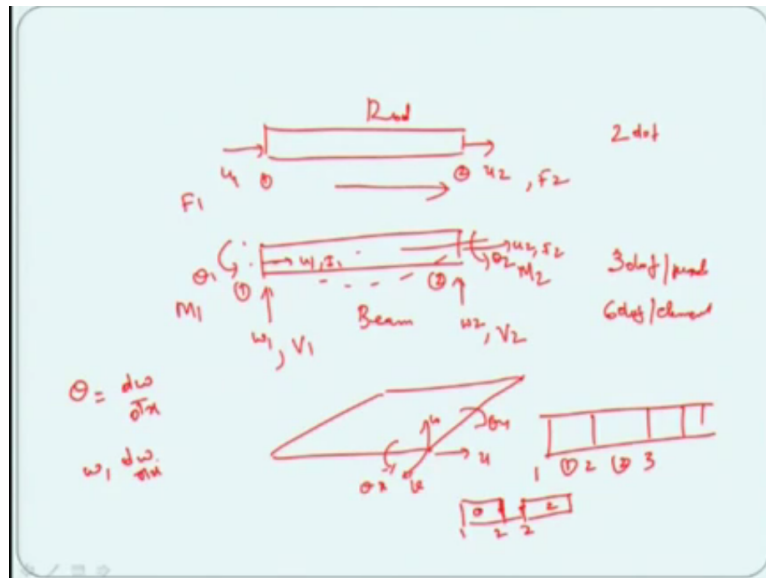
- **Continuity:** Across the elemental boundary, all the displacement needs to be continuous. For example, in the case of rods or plane stress elements, it is necessary that only the displacements be continuous. Such continuity requirement is called C^0 continuous elements.

In the case of beams and plates, which has slope d.o.f and when the slope is derived from displacements (), in ~~which case~~, both the dependent variable and its first derivative needs to be continuous. Such continuity requirement is called C^1 continuous elements. However, if the slopes are independently interpolated (as in the case of Timoshenko beam or Mindlin plate), the maintaining C^0 continuity is sufficient



The other one is the continuity. So as I said earlier across the element boundary all the displacement need to be continuous that is a derivative should exist. For example in the case of a rod, or a plane stress element, it is necessary to have only displacement to be continuous such a continuity requirement is called C0 continuous element.

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That is for example if you have a series of rod element and we isolate it, this is element 1 and element 2. So the element 1 this will have 1 and 2, 3 so the axial displacement 2 due to element 1 should be equal to the axial displacement at 2 due to element 2. That is not at all an issue because it can be maintained, but if it is a beam normally in beam the slope theta here is derived from dw/dx from the transverse displacement w .


So if it is a beam then we need to see that the slope at 2 due to element 1 should also be continuous due to slope at 2 due to element 2. So in the case of beam, we need both w and dw/dx to be continuous so there are 2 variables. The primary dependent variable and its derivative needs to be continuous, such a continuity requirement is called the C1 continuous element.

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Finite Element Terminologies (cont)

- **Continuity:** Across the elemental boundary, all the displacement needs to be continuous. For example, in the case of rods or plane stress elements, it is necessary that only the displacements be continuous. Such continuity requirement is called C^0 continuous elements.

In the case of beams and plates, which has slope d.o.f and when the slope is derived from displacements (), in ~~which case~~, both the dependent variable and its first derivative needs to be continuous. Such continuity requirement is called C^1 continuous elements. However, if the slopes are independently interpolated (as in the case of Timoshenko beam or Mindlin plate), then maintaining C^0 continuity is sufficient

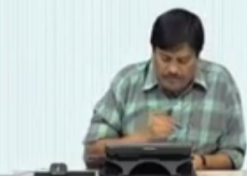


So basically the C^1 continuous element is a problem. It is not easy to satisfy especially when we go from beam to some other higher elements. So that is the problem with the beam so in order to avoid this people introduced what is called shear deformation into the beam formulation and such a beam theory is called the Timoshenko beam theory and if it is introduced in plate is called the Mindlin plate theory.

In these beams, θ is not equal to dw/dx and θ can be independently interpolated so then you need only w and θ to be continuous so that you do not need the first order continuity, which is easier to solve. So we can still build in the C^0 continuity in beam provided we use Timoshenko beam theory.

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Shape Functions



Now let us come to the shear functions.


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Introduction

- Shape functions describe the variation of a dependent variable, say displacement within an element. They are normally expressed in the form

$$\bar{u}(x, y, z, t) = \sum_{n=1}^N a_n(t) \phi_n(x, y, z)$$

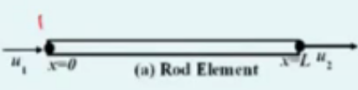
- In the above expression $\phi_n(x, y, z)$ normally referred to as shape function.
- These are constructed using polynomial approximations and the order of which is dictated by the number of degrees of freedom an element can support



So as I said I go back to equation 1 when we said that we approximate the dependent variable or the displacement within the element by a series shown by this equation where an a_n is basically the what is called the degree of freedom or the nodal displacement and ϕ_n is the shape function. These ϕ_n are basically the one which satisfies the governing boundary conditions or they are normally used by approximating this whole thing into a polynomial, which is dictated by the order of degrees of freedom.

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Rod Element



- A rod element can carry only axial motion in the x-direction. Hence, it has 2 axial deformation as degrees of freedom at $x=0$ and $x=L$. If we assume polynomial as assumed variation of deformation, it can only be a linear polynomial. Hence, the assumed variation can be written as

$$u(x, t) = a_0(t) + a_1(t)x \quad (1)$$

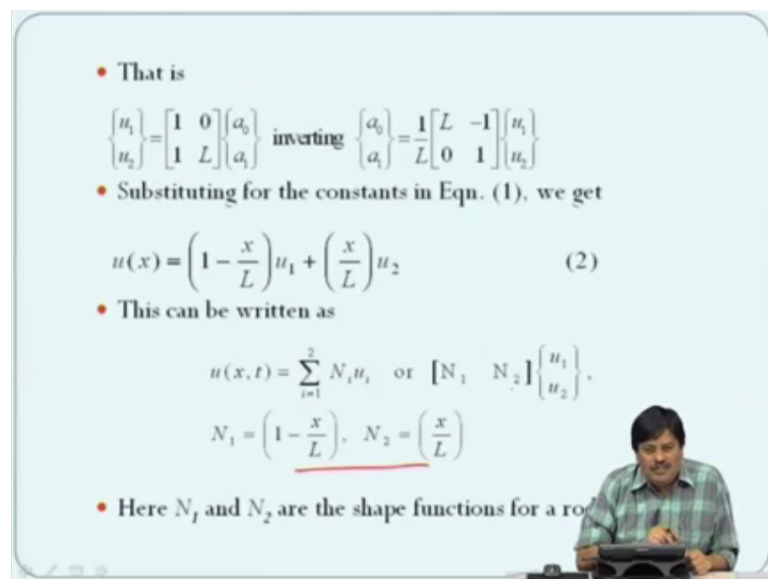
- In the above equation, note that the constants a_0 and a_1 are time dependent if the problem is dynamic in nature.
- We now substitute $u(0, t) = u_1$ and $u(L, t) = u_2$ in the above equation. This will enable us to write the $u(x, t)$ in terms of the nodal displacements u_1 and u_2 .

For example, if it is a rod problem, so for the rod problem we basically have the element has 2 degrees of freedom $x=0$ and $x=L$, so if you want to approximate the variation within the element and use polynomial we need to have 2 unknown constants into the variable that we

are assuming why because it can support only 2 motions that is $x=0$ and $x=L$ that is u_1 and u_2 .

So for these two unknowns we need to have 2 constants so we take this equation u of x $t=a$ naught $t=a_1 t^*x$. So in this above expression, we know that the constants are here a naught and a_1 are time dependent if we are talking about a dynamic problem or it is simply constant in the case of static problem. Now we substitute that at $x=0=u_1$ because we start our axis here this is $x=0$ and u at $x=L=u_1$ the above equation.

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- That is

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} \text{ inverting } \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \frac{1}{L} \begin{bmatrix} L & -1 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$
- Substituting for the constants in Eqn. (1), we get

$$u(x) = \left(1 - \frac{x}{L}\right) u_1 + \left(\frac{x}{L}\right) u_2 \quad (2)$$
- This can be written as

$$u(x, t) = \sum_{i=1}^2 N_i u_i \text{ or } \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix},$$

$$N_1 = \left(1 - \frac{x}{L}\right), \quad N_2 = \left(\frac{x}{L}\right)$$
- Here N_1 and N_2 are the shape functions for a rod element.

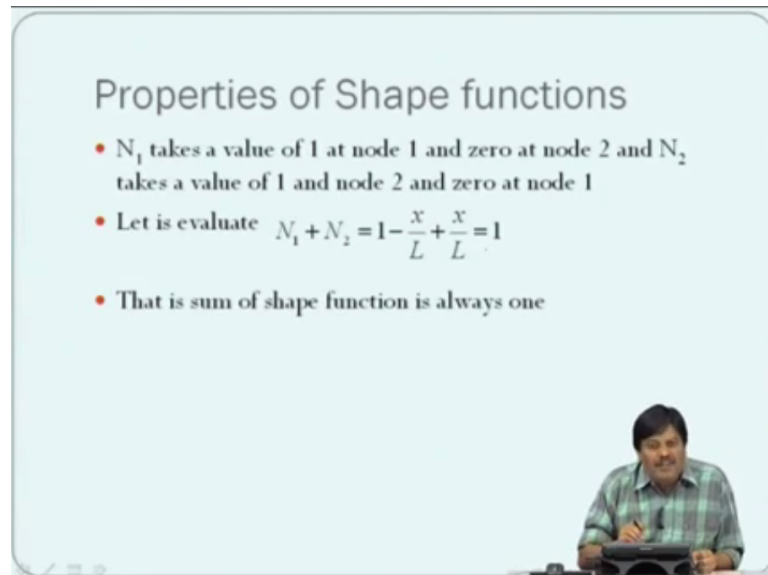
Then we will write the u_1 and u_2 in terms of the unknown constants so that is given by so now I can say $u_1 \ u_2 = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} a$ naught and a_1 and we invert this matrix then we get a naught a_1 equal to $1/L \begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix} u_1 \ u_2$. Now we substitute this back into our equation 1 and we can write and factorize all the u_1 and u_2 separately then we can write u of $x = 1 - x/L * u_1$ and $x/L * u_2$.

So this $1 - x/L$ and x/L are called the shape functions for the rod element so then we can write u of x, t is sum of $i=1$ to $2 \ N_i * u_i$ or you can write this $N_1 * \text{matrix}$ this is N matrix $u_1 \ u_2$ where N_1 are given by this as I said these are the shape functions for a rod element. So what are the properties of the shape functions? If we go back $x=0$ is the left node where displacement is u_1 and $x=L$ is the right node where the displacement is u_2 .

Suppose we substitute $x=0$ here that is at the left node, $N_1=1$ and N_2 will be 0 so it takes the value of unity at node 1 where the displacement has to be satisfied. So at node 1 we need

displacement to be u_1 so u_2 should go to 0 so that is why $x=0$. At $x=0$ the N_2 goes to 0, N_1 is 1. At $x=L$, N_1 goes to 0 and $N_2=1$ so N_1 takes the value of 1 at node 1 and 0 at node 2 and N_2 takes the value of 1 in node 2 and 0 at node 1 and let us evaluate the sum of the n_1+n_2 which is equal to 1.

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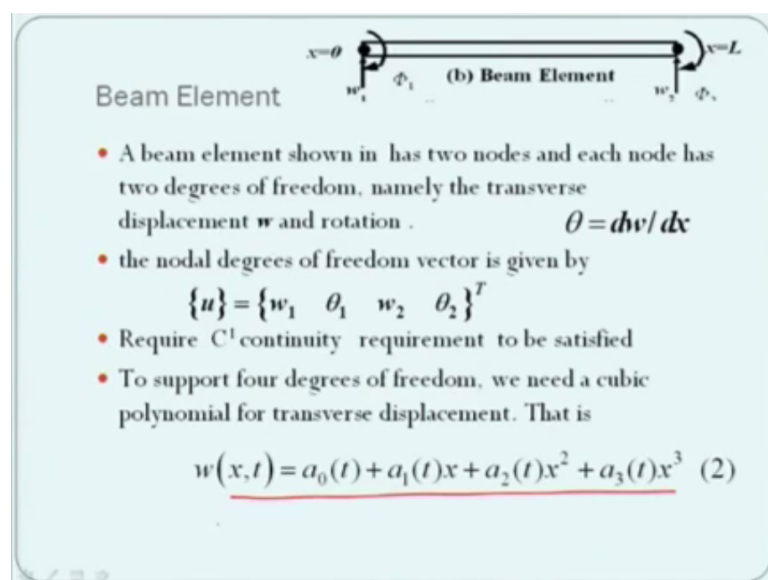


Properties of Shape functions

- N_1 takes a value of 1 at node 1 and zero at node 2 and N_2 takes a value of 1 at node 2 and zero at node 1
- Let us evaluate $N_1 + N_2 = 1 - \frac{x}{L} + \frac{x}{L} = 1$
- That is sum of shape function is always one

So the sum of shear functions are always equal to 1. So these are some of the properties of the shape functions. It takes the value of unity at the node where you are evaluating it the coordinate and all other places it is 0.

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Beam Element

(b) Beam Element

- A beam element shown in has two nodes and each node has two degrees of freedom, namely the transverse displacement w and rotation $\theta = dw/dx$
- the nodal degrees of freedom vector is given by $\{u\} = \{w_1 \ \theta_1 \ w_2 \ \theta_2\}^T$
- Require C^1 continuity requirement to be satisfied
- To support four degrees of freedom, we need a cubic polynomial for transverse displacement. That is $w(x,t) = a_0(t) + a_1(t)x + a_2(t)x^2 + a_3(t)x^3$ (2)

We will do the same this with beam element. As I said the beam element has 2 degrees of freedom for node, which is w is the transverse displacement and θ is the rotation. So the minimum order that is needed here is it has to have 4 degrees of freedom are there so with the

assumed polynomial field if we assume polynomial should have at least 4 constants for determination of this 4 displacement boundary conditions.

So that means minimum we need a cubic so the higher order approximation that is required because of this 4 degrees of freedom will lead to continuity requirement which is C1 mainly because as I said earlier both w and dw/dx have to be continuous at the interelement boundary. So we take a cubic polynomial here. So what we do we substitute at $x=0$, $w=0$ so that is given here.

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- Now, we substitute
 $w(0,t) = w_1(t)$, $\theta(0,t) = dw(0,t)/dx = \theta_1(t)$,
 $w(L,t) = w_2(t)$, and $\theta(L,t) = dw(L,t)/dx = \theta_2(t)$
- Inverting the above matrix, we can write the unknown coefficients as $\{a\} = [G]^{-1} \{u\}$
- Substituting these coefficients in Equation (2) we can write the transverse displacements as

$$w(x,t) = [N_1(x) \ N_2(x) \ N_3(x) \ N_4(x)] \{u(t)\},$$

$$N_1(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3, \quad N_2(x) = x\left(1 - \frac{x}{L}\right)^2$$

$$N_3(x) = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3, \quad N_4(x) = x\left[\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)\right]$$

At $x=0$, you have w_1 , at $x=0$ $dw/dx = \theta_1$ because the slope is derived from the displacement. At $x=L$, the transverse displacement w_2 and at $x=L$ θ_2 dw/dx will be θ_2 . When we substitute this, we can relate u with θ and inverting it we get the unknown constants a_0, a_1, a_2 can be related to the nodal displacement by this and when we substitute this back into our original equation here for a_0, a_1, a_2, a_3 we can write $w = N_1, N_2, N_3, N_4$ of u so $n_i \cdot u_i$.

That is the standard form for all finite element formulation and we get the following shape functions. So a lot of mathematics has to be done just it is a matrix inversion. Once you do that we get this. So let us look at these shape functions, does it satisfies the shape function properties? That is we need that it takes the value of 1 at where it is evaluated. Suppose we substitute at $x=0$, we expect that the N_1 should go to value 1 and all other should go to 0.

So at $x=0$, we clearly see N_1 is 1 and N_2, N_3, N_4 are all 0. If we substitute $x=L$, we see that this is $x=L$ we have $(())$ (38:34) N_2 will be 0, this will be 0, N_3 will be 1 because it is the transverse displacement that is 3-2 and N_4 will be equal to 0. Then what about the theta degrees of freedom. So the N_2 will not go to 0 when $x=0$ because what we are looking at N_2 is the shape function for the dw/dx that is the rotation which is got from w .

And similarly N_4 it is the shape function for theta at node 2 at right node so we will find that the dN_2/dx that is because theta is derived from the w , we need to take dN_2/dx and substitute at $x=0$ you will see that this will be equal to 1 and this will be 0 and when $x=L$ dN_2/dx basically this will be equal to 0 and this will be equal to 1. So that is precisely what I have written here.

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- These shape functions satisfy all the properties of the shape functions
- Note that, N_2 , which corresponds to shape function of the slope at Node 1 does not take the value of unity at $x=0$. However, dN_2/dx is equal to 1 at $x=0$. Similarly dN_4/dx and not N_4 that takes the value of unity at $x=L$

Note that the cubic interpolation functions are derived by interpolating w as well as its derivative dw/dx at the nodes. Such polynomials are known as the Hermite family of interpolation functions, and ϕ_i^w and ϕ_i^{θ} are called the Hermite cubic interpolation (or cubic spline) functions.

The graphs show: $N_1(x)$ starts at 1 at $x=0$ and goes to 0 at $x=L$; $N_2(x)$ starts at 0 at $x=0$ and goes to 0 at $x=L$; $N_3(x)$ starts at 0 at $x=0$ and goes to 1 at $x=L$; $N_4(x)$ starts at 0 at $x=0$ and goes to 0 at $x=L$.

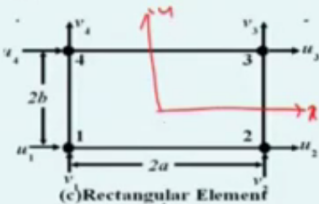
These shape functions essentially satisfies the properties of the function and I have specifically said that N_2 which correspond shape function of this slope and node 1 does not take a value of unity at $x=0$ even though it is evaluated at that node, only $dN_2/dx=1$ and if you plot this you see that the variation, N_1 takes the value of 1 and 0 at 2. The slope dN_2/dx takes the value of 1 at node 1 and 0 elsewhere.

This is N_3 which is again corresponding to the transverse function so this is how the shape functions vary.

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Rectangular Element

- Let us now consider a rectangular finite element of length $2a$ and width $2b$
- This element has four nodes and each node can support two degrees of freedom, (namely) the two displacements, $u(x,y)$, and $v(x,y)$ in the two coordinate directions. Since there are 4 nodes, we can assume the interpolating polynomial as



(c) Rectangular Element

$$u(x,y) = a_0 + a_1x +$$

$$v(x,y) = b_0 + b_1x +$$

Now let us go to a 2 dimensional element and suppose we have a rectangular element of length $2a$ and width $2b$, which has 4 nodes and each node can take 2 degrees of freedom that is u and v in 2 horizontal directions so it will have totally 8 degree of freedom element. So we have $4u$ degrees of freedom and $4v$ degrees of freedom that means the polynomial is not only x it is also y dependency is there.

It will have x and y and it should have 4 constants both in u and v so we take the form $u = a_0 + a_1x + a_2y + a_3xy$. Similarly, we have $v = b_0 + b_1x + b_2y + b_3xy$ so there are 8 constants corresponding to 8 degrees of freedom here. So now we substitute the coordinates of the element. So here the axis is exactly at the middle so we have a middle axis this is x and y .

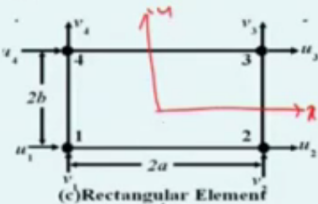
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- In the above equation, we substitute

$$u(-a,b) = u_2, \quad v(-a,b) = v_2, \quad u(-a,-b) = u_1,$$

$$v(-a,-b) = v_1, \quad u(a,-b) = u_4, \quad v(a,-b) = v_4$$

- These help us to relate the nodal displacements to the unknown coefficients as $\{a\} = [C]^{-1} \{u\}$. (Substituting for unknown coefficients, we can write the displacement field and the shape functions as



(c) Rectangular Element

$$u(x,y) = [N] \{u\} = [N_1(x,y) \ N_2(x,y) \ N_3(x,y) \ N_4(x,y)] \{u\}$$

$$v(x,y) = [N] \{v\} = [N_1(x,y) \ N_2(x,y) \ N_3(x,y) \ N_4(x,y)] \{v\}$$

$$\{u\} = \{u_1 \ u_2 \ u_3 \ u_4\}^T, \quad \{v\} = \{v_1 \ v_2 \ v_3 \ v_4\}^T$$

$$N_1(x,y) = \frac{(x-a)(y-b)}{4ab}, \quad N_2(x,y) = \frac{(x-a)(y+b)}{4ab},$$

$$N_3(x,y) = \frac{(x+a)(y+b)}{4ab}, \quad N_4(x,y) = \frac{(x+a)(y-b)}{4ab}$$

These shape functions satisfy all the properties

So this corresponds to $-a$ and this is $+a$ and $-b$ and $+b$ so we substitute this equation into the assumed function here in this function and we evaluate the coefficients we can write a will be equal to this $G^{-1}u$ where my u will be a vector u_1, v_1, u_2, v_2 to u_4, v_4 . So this is the vector of my u displacement. So when I substitute this back into this relation and simplify we can write $u = N_1 N_2 N_3 N_4$ into only u degrees of freedom which will have u_1, u_2, u_3, u_4 this will have only u_1, u_2, u_3, u_4 transpose.

And this will have only the vertical degrees of freedom that is v_1, v_2, v_3, v_4 okay and each one can be given $x-a*y-b/4, x-a*y+b/4, x+a*y+b/4, x+a*y-b/4$. If you look at this carefully let us substitute at $x=-a*b$ suppose we want u_1 which is $-a*-b$ this is $4ab$, everywhere you have a $4ab$ here. So when you substitute $x=-a*y=-b$ here, N_1 it takes a value of unity whereas N_2 will be 0 at $y=-b$, this will be 0.

And similarly when we take the value of say u_2 at $-a$ and b so N_2 takes a value of unity and if you take the sum of $N_1+N_2+N_3+N_4$ it will always be equal to 1 so all these properties are followed and if this is not followed then they are not called shape functions.

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Triangular Element

Deriving the shape functions through conventional means for a triangle is very cumbersome. Here, we will use area coordinates

Consider a triangle having coordinates of the three vertices as $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3)

Consider an arbitrary point (x, y) inside the triangle. This point will split the triangle into three smaller triangles of area A_1, A_2 and A_3 , respectively. Let A be the total area of the triangle, which can be written in terms of nodal coordinates as

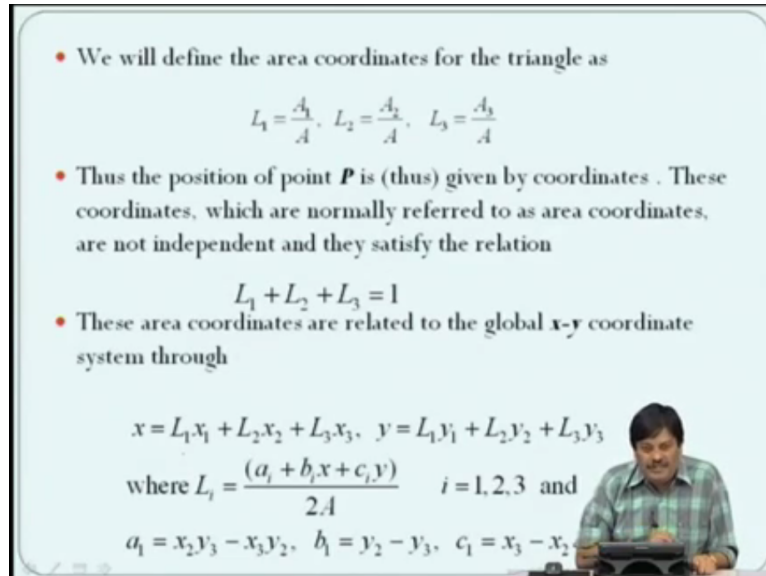
$$A = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

(d) Triangular Element

Now for the triangular element, how do we do it? So now let us look at it is very important element but inconvenient to generate the shape function by the procedure we adopted before even though it can be done. So here we take a slightly different approach where we will not use the conventional coordinates, which is given by you have x_1, y_1, x_2, y_2 and x_3, y_3 . What we will do here is we will take any point for which we require a coordinate.

We split up this into 3 areas, we call them A1, A2, A3 okay and if I want the area, area can be got from the coordinates if A is the total area of the triangle they can be got from the expression given here which is the determinant of this where x1, x2, x3 are the rectangular coordinates of the original element.

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- We will define the area coordinates for the triangle as

$$L_1 = \frac{A_1}{A}, \quad L_2 = \frac{A_2}{A}, \quad L_3 = \frac{A_3}{A}$$
- Thus the position of point **P** is (thus) given by coordinates. These coordinates, which are normally referred to as area coordinates, are not independent and they satisfy the relation

$$L_1 + L_2 + L_3 = 1$$
- These area coordinates are related to the global **x-y** coordinate system through

$$x = L_1 x_1 + L_2 x_2 + L_3 x_3, \quad y = L_1 y_1 + L_2 y_2 + L_3 y_3$$
 where $L_i = \frac{(a_i + b_i x + c_i y)}{2A} \quad i = 1, 2, 3$ and

$$a_1 = x_2 y_3 - x_3 y_2, \quad b_1 = y_2 - y_3, \quad c_1 = x_3 - x_2$$

So now we will define the area coordinate for this problem or the point P it is located by L1, L2, L3 where L1, L2 is the sub area A1/A so if you look at it the sub area A1/A is the L1, L1 is the one where the opposite node is the node 1. Similarly, we take L2 which is the sub area A2/A, L3 is A3/A. The position of the point is thus given by these coordinates, which are normally refer to as area coordinates and or not independent and they satisfy the relation L1 L2+L3.

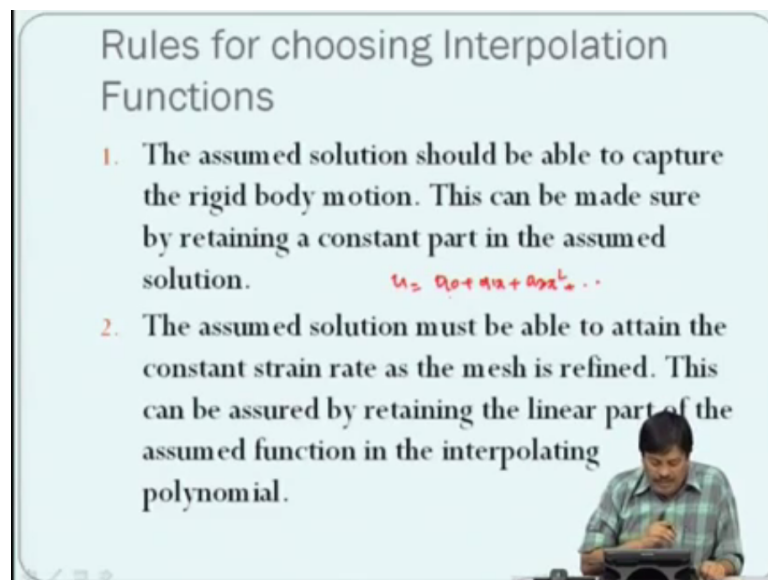
So the moment you solve that it is similar to sum of shape functions satisfying the equation N1+N2+N3=1 where the coordinate x can also be related. I am not going into the derivations of this. Through this relation L1 x1, L2 x2, L3 x3 so from this we can derive the expression between L and the coordinates that is here we say Li will be Ai+Bi x+Ci y/2A where A, B, Cs are given in terms of the coordinates.

So you can get L1 will be A1, B1, C1, L2 will be A2, B2, C2 and these are all cyclic permutations which we can get. So basically the above equation requires to be used we will use this when we want to find the derivatives because in most of our B matrix we need dN/dx with respect to x when we have the area coordinate L we need to use this expression to get the derivative with respect to x.

So that is why we need these relations so this is more convenient to do that. So now we can write these shape functions for the triangle as $u = N_1 u_1 + N_2 u_2 + N_3 u_3$ where N_1 is nothing but your area coordinates L_1, L_2 and L_3 . So here these shape functions also follow the normal rules that is it takes the value of u where $L_1=1$ at the node 1 and 0 elsewhere and L_2 and L_3 is 0 at node 1 so in order to fix this we can go back here.

So at this point $L_1=1$, at here $L_1=0$ and if you come here $L_2=1$ and $L_2=0$, if you come here $L_3=1$ and $L_3=0$. So basically when you say here in this point L_3 is 0, L_2 is 0 but L_1 is 1. If you look at here, L_2 is 1 but L_1 and L_3 are 0. So it satisfies the property of the shape function and at the same time elegantly we can be able to formulate the element in more convenient manner. The conventional method will give us lot of problems.

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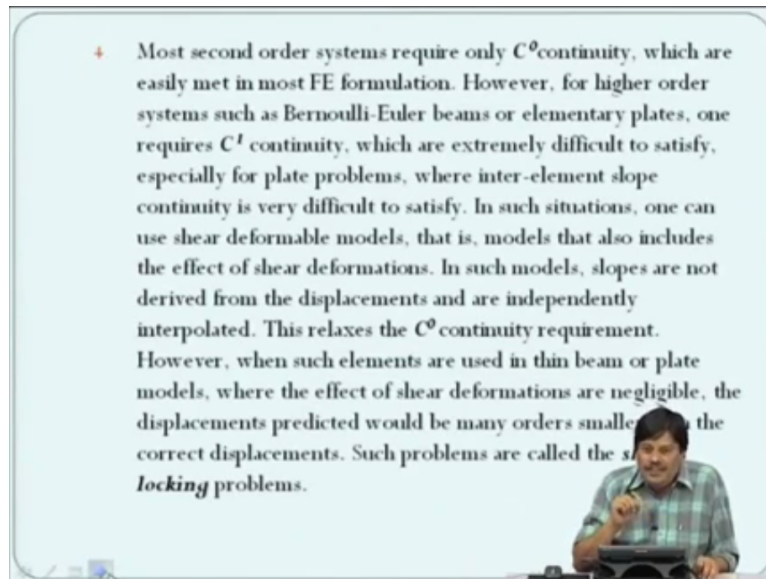
Rules for choosing Interpolation Functions

1. The assumed solution should be able to capture the rigid body motion. This can be made sure by retaining a constant part in the assumed solution. $u = a_0 + a_1x + a_2x^2 \dots$
2. The assumed solution must be able to attain the constant strain rate as the mesh is refined. This can be assured by retaining the linear part of the assumed function in the interpolating polynomial.

So what are the rules for choosing the interpolating functions? The assumed solution should be able to capture the rigid body motion and this can be made sure by retaining the constant part of the assumed solution that is basically what we are saying here is so if you have a function $u = a_0 + a_1x + a_2x^2$ etc the constant part make sure that the rigid body motion can be present in the assumed function, which is absolutely necessary for convergence for the solution.

The assumed solution must be able to retain the constant state of strain as the mesh is defined and this can be assured by retaining the linear part that is the linear part a_1x in the interpolating function. This is absolutely necessary. These are some of the rules that we need to follow.

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Most second order systems that is the second order system is where the differential equation is order 2 requires only C^0 continuity so you want only displacement to be continuous. We explained this which can be easily met in most finite element formulation. However, for higher order systems such as the Bernoulli-Euler beams where the elementary beam or the plates where the slopes are derived from displacement.

Theta is dw/dx we need to see that both w and dw/dx are continuous across the element these are extremely difficult to satisfy in the interelement continuity. So in such situations what we do is we introduce shear deformation as I said earlier and Bernoulli-Euler beam is converted into Timoshenko beam where the theta is not derived from the slope that is theta is not equal to dw/dx where now we need to have only w and theta continuity that is we go back to C^0 continuity.

Now if you look at it if the beam is very thick shear deformation is very, very predominant we can use Timoshenko theory. Suppose the beam is very thin, there is no shear deformation, you cannot make a C^1 continuous to become C^0 just by relaxing this. So in which case the displacement predicted would be many, many orders smaller than the correct displacements and such a problem is called the shear locking problem.


So we have to be careful well actually migrating from C^0 to C^1 by using different beam theories.

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5. The order of assumed interpolating polynomial is dictated by the highest order of the derivative appearing in the energy functional. That is, the assumed polynomial should be at least one order higher than that is appearing in the energy functional.

$$U = \int_0^L EI \left(\frac{d^2 w}{dx^2} \right)^2 dx$$

- In summary, for all the elements we can express the displacements in terms of shape functions and the nodal displacements as $u = \sum_{n=1}^N N_n u_n$. This spatial discretization will be used in the weak form of the governing equation to obtain the FE governing equation



The order of assumed polynomial is dictated by the highest order of the derivative appearing in the energy functional that is the assumed polynomial should be at least one order higher than what is appearing in the energy function. To understand this now if you take a beam you have u will be equal to 0 to L $EI \frac{d^2 w}{dx^2}$ whole square to dx and suppose you assume a linear polynomial, which you cannot do anyway if you assume it then the energy will go to 0 .

Because $\frac{d^2 w}{dx^2}$ will not exist and suppose you use a quadratic polynomial then $\frac{d^2 w}{dx^2}$ will only be a constant. So we at least make sure that in the case of a beam we need to have something, which is where one more higher order should be there. So what we need to choose is choose a polynomial where $\frac{d^3 w}{dx^3}$ exist. So basically that is what is required here.

So in summary what are we doing here. For all the elements, we can express the displacements in terms of shear functions and nodal displacement, which are given by $u = u^N$ into which is given here $N_i u_i$ and the spatial discretization will be used in the weak form of the governing equation substituted there and then we formulate the stiffness and mass matrices then these matrices will then be used, assembled, solved for forgetting the solution of the structure.

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Finite Element Formulation

• Rod Element Formulation:

A rod can carry one dof/node and 2 dof per element. The elemental variation of deformation in terms of shape function is given by

$$u(x, t) = \sum_{i=1}^2 N_i u_i \quad \text{or} \quad [N_1 \quad N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix},$$

$$N_1 = \left(1 - \frac{x}{L}\right), \quad N_2 = \left(\frac{x}{L}\right)$$

We use the shape function information to derive the stiffness and mass matrix, which is given by

$$[K] = \int_V [B]^T [C] [B] dV, \quad [M] = \int_V \rho [N]^T [N] dV$$

This is in summary about how the finite element process takes place. Now let us look at some of the element formulation, this is a rod element formulation where you have N1 and N2 given by here where $N_1=1-x/L$ and x/L we use this shear functions then we know this is the stiffness matrix, we have derived this so we put this into this system.

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Rod Element (Cont)

- $[B]$ is the strain displacement matrix. Only relevant strain is the axial strain ϵ_{xx} corresponding to axial stress (σ_{xx}) and in $[C]$, only E is relevant

$$\begin{aligned} \text{Hence } \epsilon_{xx} &= \frac{du}{dx} = \frac{d}{dx} [N_1 \quad N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \\ &= \left[\frac{dN_1}{dx} \quad \frac{dN_2}{dx} \right] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \left[-\frac{1}{L} \quad \frac{1}{L} \right] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \end{aligned}$$

Or

$$\epsilon_{xx} = [B] \{u\}$$

$$[K] = \int_V E [B]^T [B] dV = \int_0^L E [B]^T [B] dx \int_A dA$$

$$= \int_0^L EA \begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix} \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} dx = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

So B is the strain displaced only relevant strain is epsilon xx. So only you will have C present here then we put the C back here. So once we use this we can actually get the strain displacement matrices and the C matrix will contain only E the Young's modulus. We substitute all these things into this equation replace C/E, once we do that we get this equation.

And we put this back here and this is the stiffness matrix for a rod we see that it is symmetric and it is a function of the material property E, sectional property A and length.


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Beam Element Formulation

- The Shape function for beam was derived earlier. We use this here. We first derive the strain displacement matrix [B]. The displacement field for the beam is given by

$$u(x, y, z, t) = -z \frac{dw}{dx}, \quad w(x, y, t) = w(x, t)$$
- The relevant strains are

$$\epsilon_{xx} = \frac{du}{dx} = -z \frac{d^2w}{dx^2}, \quad \epsilon_{zz} = \frac{dw}{dz} = 0, \quad \gamma_{xy} = \frac{dw}{dx} + \frac{dy}{dz} = \frac{dw}{dx} - \frac{dw}{dx} = 0$$
- Hence, ϵ_{xx} is the relevant strain and correspondingly σ_{xx} is the only relevant stress and hence the matrix [C] is equal to $E \epsilon_{xx}$



Similarly we can do the beam element formulation where we have the displacement relation between the axial displacement is given by theta times slope. So basically when the beam bends we have so this is the mid plane so we draw this and this is the axial displacement, which is the theta so that is the z times theta is nothing but the axial displacement and this one.

And on using the strain displacement relation we find that this is second order strains and we have the normal strains in the y direction and this is x and this is z so the sigma z in this direction is 0 shear strain is 0 so only relevant material matrix here is again E.

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Beam Element Formulation

- Hence Strain displacement matrix is obtained as follows

$$\epsilon_{xx} = \frac{du}{dx} = -z \frac{d^2}{dx^2} \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix}$$

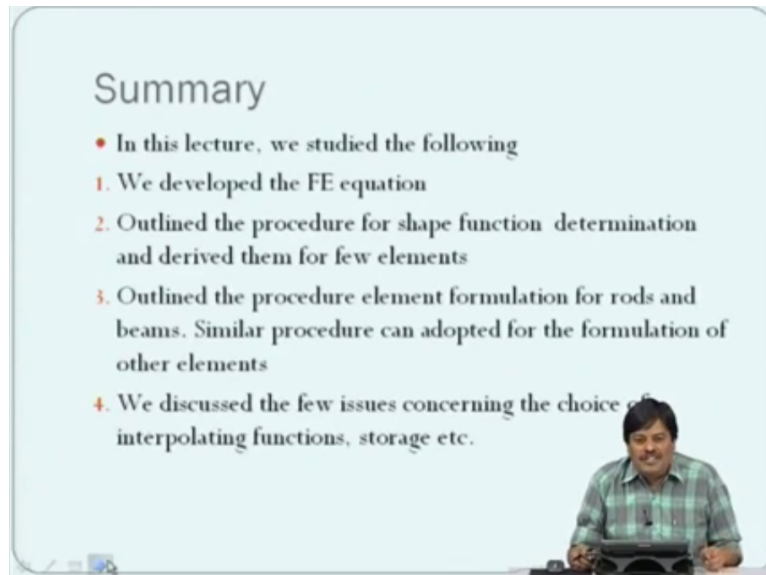
$$= -z \begin{bmatrix} \frac{d^2 N_1}{dx^2} & \frac{d^2 N_2}{dx^2} & \frac{d^2 N_3}{dx^2} & \frac{d^2 N_4}{dx^2} \end{bmatrix} \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = -z [B] \{u\}$$
- The stiffness matrix is given by

$$[K] = \int_V E z^2 [B]^T [B] dV = \int_0^L E [B]^T [B] dx \int_A z^2 dA$$

$$= EI \int_0^L [B]^T [B] dx = \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & 6L \\ -6L & 4L^2 & 6L & -2L^2 \\ -12 & 6L & 12 & -6L \\ -6L & -2L^2 & 6L & 4L^2 \end{bmatrix}$$

Here we plug this back into this equation and derive the strain displacement matrix, which is du/dx which is given by there and when we substitute for u we get this is the second order matrix and we substitute back into the governing differential equation and integrate between 0 to L we get this is the governing equation.

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Summary

- In this lecture, we studied the following
 1. We developed the FE equation
 2. Outlined the procedure for shape function determination and derived them for few elements
 3. Outlined the procedure element formulation for rods and beams. Similar procedure can adopted for the formulation of other elements
 4. We discussed the few issues concerning the choice of interpolating functions, storage etc.

So basically we have done this and now in summary what we have understood here is we have developed the FE equations, outlined the procedures for finding out the shape functions and also derived some of the elements that can be directly go into our analysis like rods and beams and we also discussed few issues concerning the choice of the interpolating functions etc. Thank you.