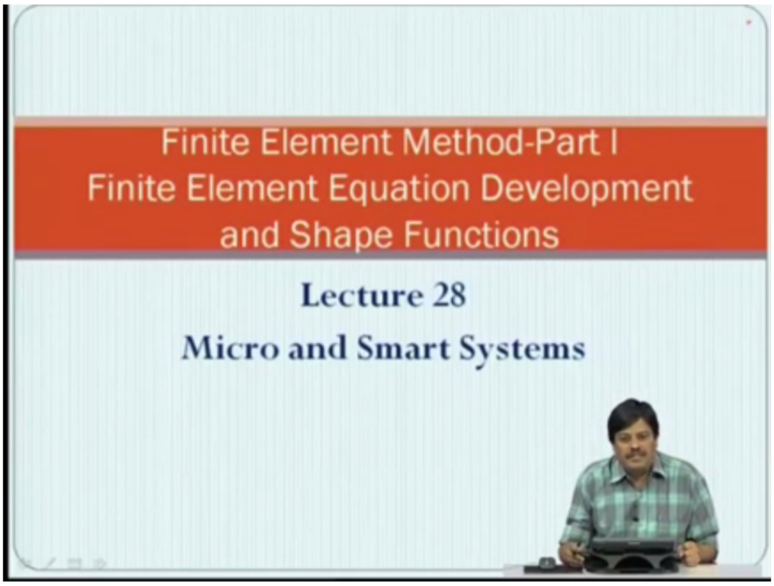
Micro and Smart Systems
Prof. S. Gopalakrishnan
Department of Aerospace Engineering
Indian Institute of Science - Bangalore

Lecture - 28
Finite Element Equation Development and Shape Functions

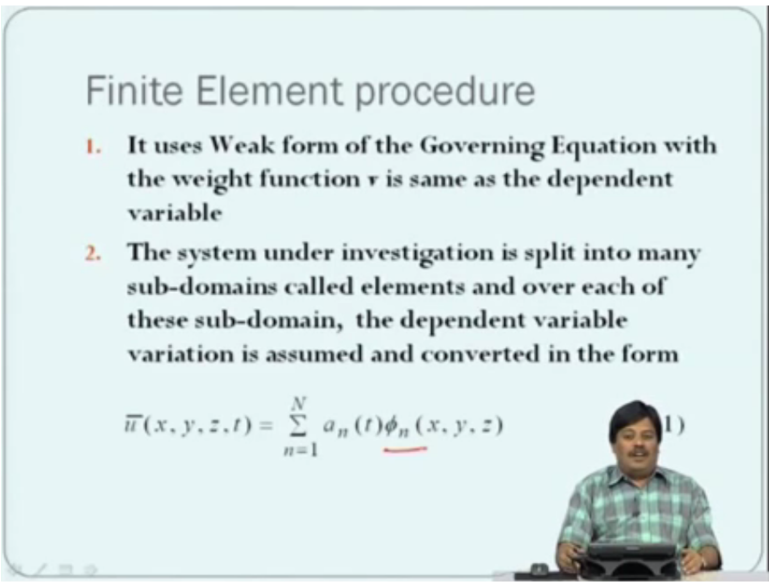
So this is lecture number 28 of the micro and smart system course.

(Refer Slide Time: 00:30)



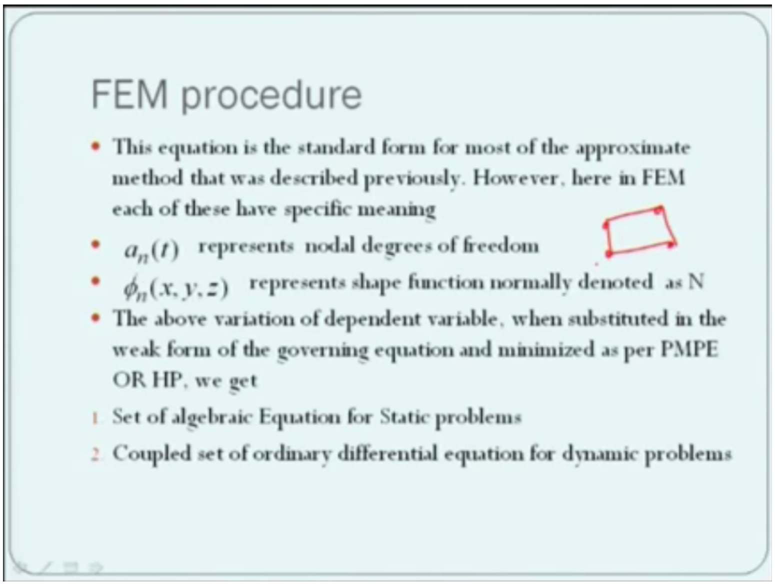
And here we would be talking about the finite element method development as such and as a part of it we will develop the finite element equations and the shape functions that are required for the development of many elements. So just to give a summary of what we talked about in the last class.

(Refer Slide Time: 00:36)



We said that finite element uses the weak form of the governing equation with the weight function v which is same as the dependent variable function, which is given in equation 1 here. So the system and the investigation is split up into many subdomains, which we call it elements and each element has a number of notes. Over each of these subdomains, the dependent variable is assumed in certain form of a series, which is given by here equation 1 where an of t and phi n has certain meanings.

(Refer Slide Time: 01:22)

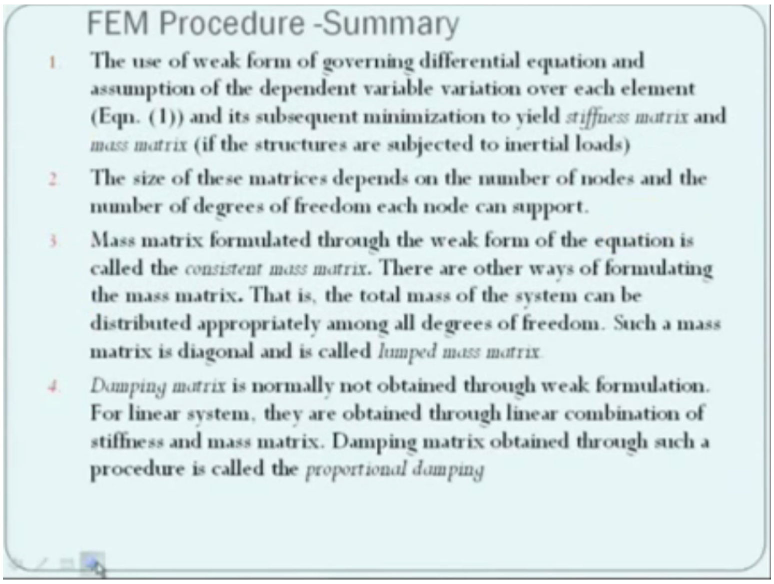


Let us see what these meanings are. So the equation 1 is the standard form for the most of the approximate methods that were described previously; however, here in FEM each as I said earlier has specific meaning. For example, an of t represents the nodal degrees of freedom that is each element will have a node. For example, if it is a rectangular element, we have 4 nodes and the degree of freedom in each of this node represents this an of t.

And the phi n of x, y, z represents the shape function, which is normally denoted by N in finite elements. The above variation of the dependent variable when substituted into the weak form of the governing equation and minimized it we get 2 sets of problems. If the problem is simple static that is the variables where an is just a function is a constant and the phi is not a function of t.

Then we get a set of algebraic equations, which is normally for static problems. If the problem is dynamic, then we get a couple set of ordinary differential equation that is basically a governing differential equation is converted into algebraic equation for static problems and a couple set of ordinary differential that is the PDE is transformed into a set of ODEs for dynamic problems.

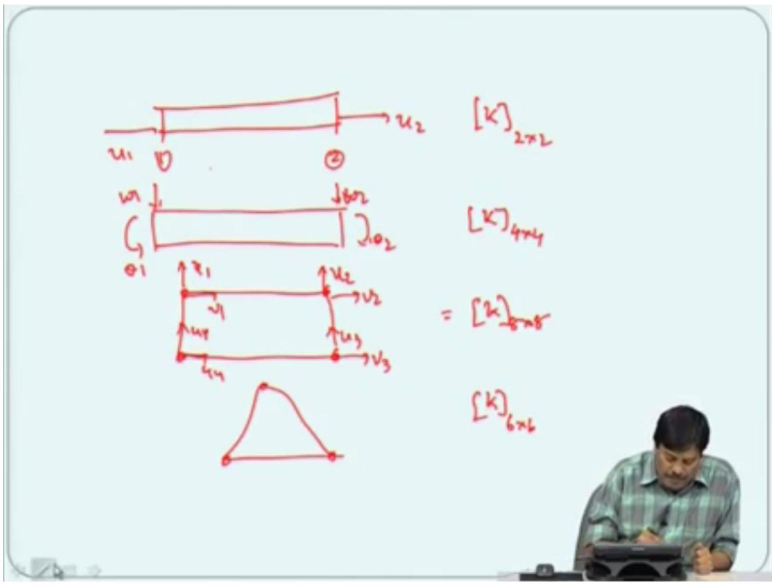
(Refer Slide Time: 02:57)



Let us summarize the procedure now. So basically what it says here is the substitution of the assumed function into the weak form of the governing equation and the subsequent minimization will give us a set of 2 matrices what is called the stiffness matrix and the mass matrix and the mass matrix will be there only if the structure is dynamic in nature that is it is subjected to inertial loads.

And the size of these matrices depends upon the number of nodes and the number of degrees of freedom the each node can support. For example, in order to explain this in little more detail let us take a simple rod problem okay.

(Refer Slide Time: 03:40)

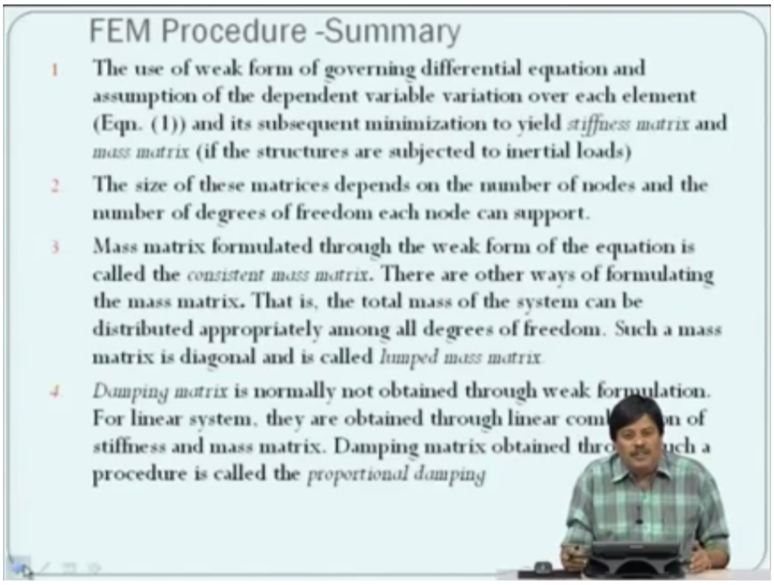


So the rod problem basically will have 2 degrees of freedom that is the axial displacement u1 and u2 and this is node 1 and node 2. So in this case, the stiffness matrix k which we represent will be 2/2. If the same problem is now a beam problem that is when the loading is in the transverse direction in this direction, then it will take 2 degrees of freedom that is the w1 and theta 1 the slope and w2 and theta 2 slope.

So basically in which case this stiffness matrix will be 4/4. The other possibility is these are all 1-D elements. Suppose we have a rectangular element, then this has 4 nodes and each node can support u1, v1, then this is u2, v2, this is u3, v3, and u4, v4 and stiffness matrix will be 8/8. Suppose we have a triangular element other possibility then we have 3 nodes and if each node can support 2 degrees of freedom then the matrix k will be 6/6.

So basically it depends upon how many degrees of freedom it can have. We will come to the definition of degrees of freedom little later.

(Refer Slide Time: 05:26)



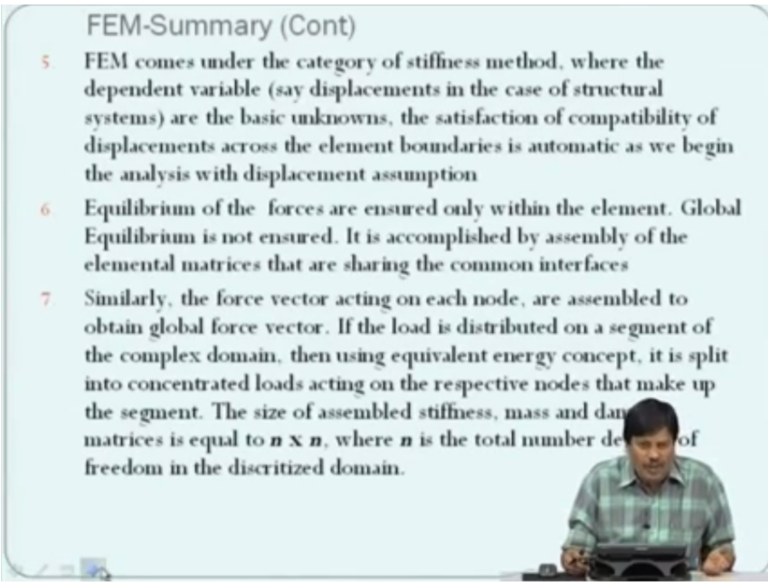
Now the mass matrix formulated through the weak form as I said what we get there is called a consistent mass matrix. There are other ways of formulating the mass matrix, which we will not go in detail in this course, it is beyond the scope of this course so in which case if that is done, the consistent mass matrix is a completely full matrix depending upon the what structure it is 2/2 for a rod, 4/4 for the beam etc.

However, there are alternate ways of formulating this mass matrix. Suppose you take the total mass and length only to the translation degrees of freedom then you will get a lumped mass matrix, which will be completely diagonal. There are various advantages of using the diagonal matrix when you do a dynamic analysis, which we are not going through in this course.

The damping matrix another matrix that is coming from the weak form is many times not used in weak forms for many reasons because damping is a very complex phenomenon, which is not well understood even today, but that has to be there. The damping comes because whenever there is time dependent force then this force will not could be sustaining the same amplitude it will die down after some time because of the various reasons such as the environment, such as the material in which it is, the responses traversing etc.

But however to represent it they formulate the stiffness and mass matrix and take the damping matrix as a combination of the stiffness and mass matrix and such a matrix is called the proportional damping matrix.

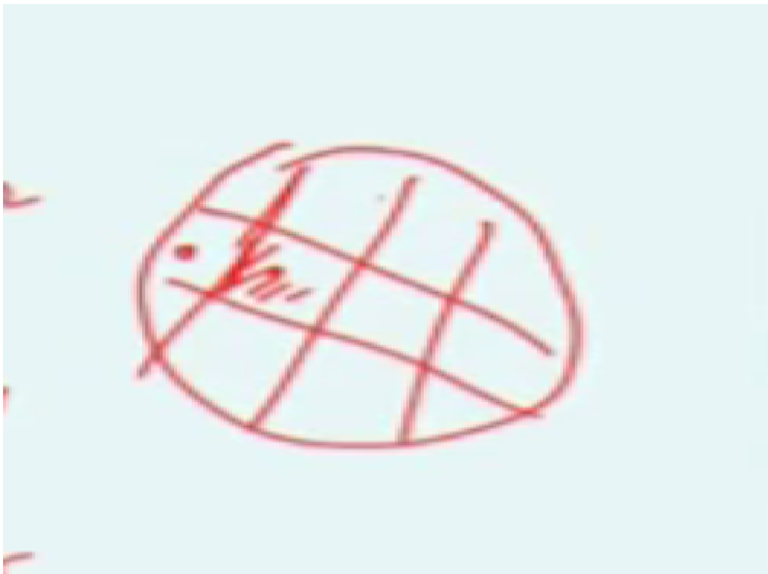
(Refer Slide Time: 07:16)



There are two categories of numerical method of solving under FEM, one is based where the forces are basic unknowns these are called the force method which is also a kind of FEM method, but it is not very popular, but the conventional FEM method which is extensively used is called the stiffness matrix or the stiffness method where the dependent variable say the displacement in the case of structures or it is current and magnetic field in terms for the Maxwell's equations are the electromagnetic problem are the basic unknowns.

And the satisfaction of the compatibility of the displacement across the element boundaries is automatic as we begin here.

(Refer Slide Time: 08:07)

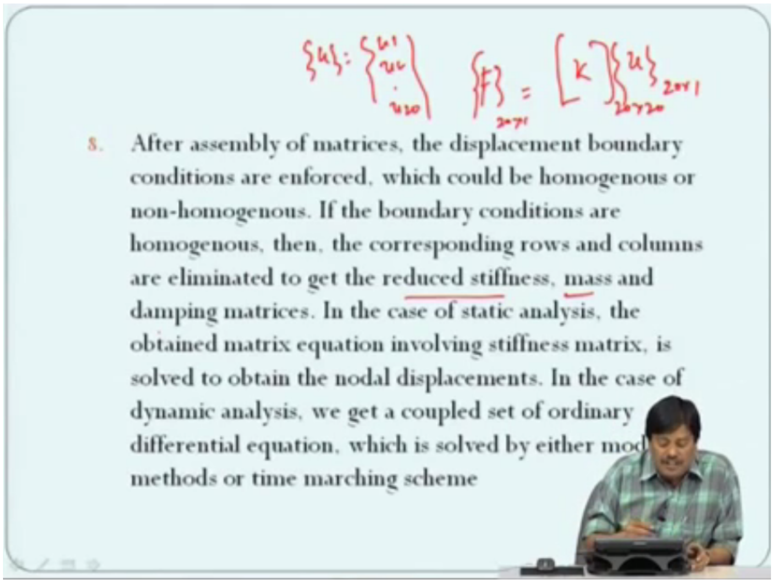


So in order to make this very clear let us again take a domain and this domain is split up into say number of element as shown here. Let us isolate here so in this case the displacement coming from this element and the displacement coming from this element are compatible at this edges because we start with the compatibility. However, because of the applied load somewhere else there will be forces generated and the forces across this inter element boundary will not be compatible, will not be in equilibrium.

So this equilibrium has to be in force. How do we do that? So we generate the matrices here for each of the stiffness matrix when we assemble it, the assembly process ensures that the forces here are in equilibrium. So basically there are two aspects, which we said in the elasticity, one is the compatibility and other is the equilibrium. Here the compatibility is ensured whereas the equilibrium has to be enforced.

The assembly of matrices will basically ensure that the equilibrium is satisfied. So when we assemble the whole matrices so in the case of a rod we have a 2/2 when we assemble all the matrices say it has 10 degrees of freedom so the total assemble matrix will be 20/20. So it will be n/n. So if there are n degrees of freedoms, the assemble stiffness matrix will be of the order n/n.

(Refer Slide Time: 09:53)

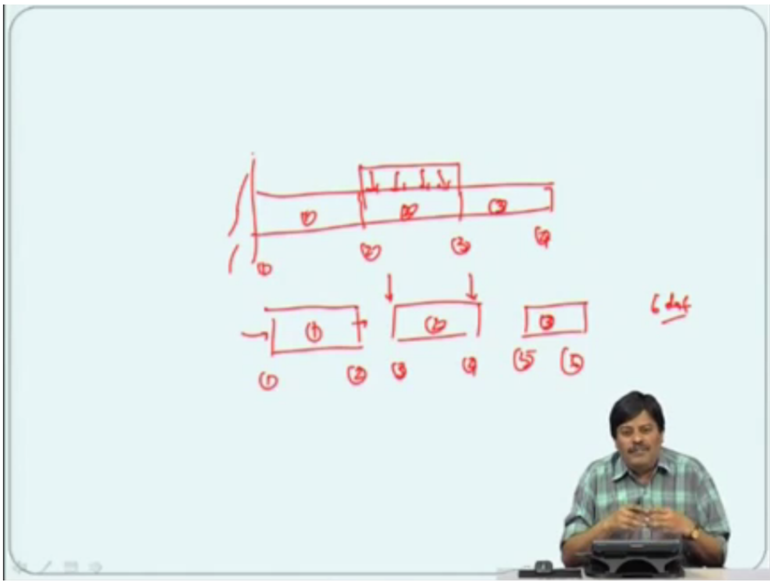


Then what we do? After the assembly process, then we get a stiffness matrix say your k will be say 20/20 for a 10 element rod matrix let us say and this 20/20 will corresponds to the 20 degrees of freedom in the rod. Each element will have 2 degrees of freedom; there are 10 elements so there are 20 degrees of freedom. So in that 20 degrees of freedom, you cannot solve it without applying the boundary condition.

Because this matrix will be singular so basically you will have your force vector F which is 20/1 will be equal to k*u which is 20/1 where u will have basically u1, u2, to u20. So out of which we need to say which are the boundary conditions are known, suppose u1 is 0 so that has to be eliminated. So when we eliminate it, we get a reduced stiffness matrix and similarly we can get a reduced mass and damping matrices if you are doing a dynamic problem okay.

Then basically the reduced stiffness matrix is solved for the applied loads. So many times the load would be basically distributed.

(Refer Slide Time: 11:23)



For example, if you have a structure say we have a rod where some portion is having some distributed loading. So in the finite elements, we can only handle concentrated loads. So basically we split this element into sub elements so this will be element 1, so this will be element 2 and this will be element 3. So I will name it as 1, 2, and 3. So each will have say 2 degrees of freedom.

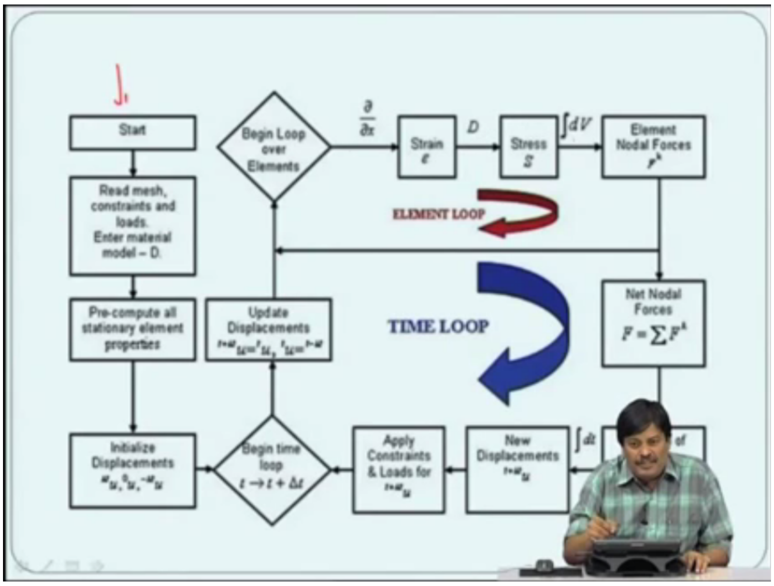
So this will have node 1, 2, 3, 4 so you will have 1, 2, 3, 4, 5, 6, so 5 and 6 has a common node 3 in the global direction. So now we have 6 degrees of freedom okay. So now what we do basically is this distributed load has to be converted into concentrated load acting in these 2 element edges and when we assemble it corresponding to the degree of freedom 2 and 3, we have the nodal vector going into the matrices.

So essentially what we are saying is the distributed load has to be transformed into equivalent concentrated load when we do a finite element procedure. So the resulting equation will then be solved using the standard algebraic solution methods and we solve for the displacement. Once we get the displacement, we now use the elemental equation. For each element F=ku then get the forces and hence the stresses and whatever quantities you want beyond displacement can be got by post processing the results.

So as I said earlier in my earlier lectures, the finite element procedure has the preprocessing that is meshing the system then the solution of the equation and the post processing. So the 3 steps have to be followed in the finite element procedure and in the case of dynamic analysis we have a PDE that is partial differential equation is converted into an ordinary equation, which are coupled together to many differential equations.

There are various methods that we use to actually solve these equations such methods are called the modal methods where we use the Eigen values, we convert the dynamic problem into an Eigen value problem and find the Eigen values and use these Eigen values to find the dynamic response or we can use finite difference scheme, which we talked about little later by actually using the finite difference scheme for the reduced ordinary differential equation, which we call it as time marching scheme. We are not going into details of these right now.

(Refer Slide Time: 14:28)

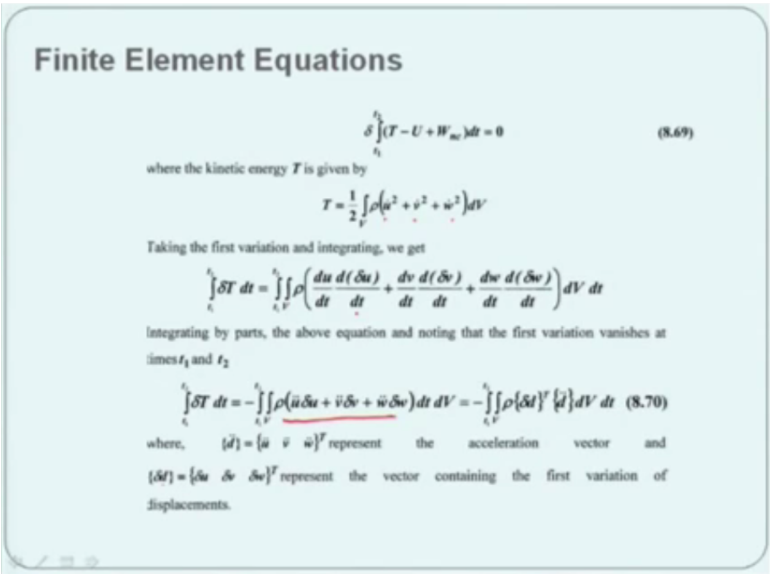


So to understand this we can take this big flow chart. So this is the preprocessing part the left hand part, so this is the preprocessing part where we start, we read the mesh, constraints where the loads are, what are the types of loads, what are the types of materials okay. Then we pre-compute all the elemental properties and then we initiate the displacement. So if you are doing a dynamic analysis, we can ignore this.

So we begin to do loop over the element, we find the strain displacement matrix, we find a material matrix and if it is a nonlinear problem, we have to update the stress, we do not care to do about it. Then we solve for the displacement, compute the element forces and we do this for all the elements. So before this we generate the stiffness matrix, assemble it. After the we generate the strain displacement matrix we generate the stiffness matrix, assemble it, solve for the displacement, find the elemental forces.

If you are doing a dynamic problem, we have to set the initial displacement to 0 then we need to begin a time marching scheme, then we go about doing it, apply the constraint loads, find the new displacement and this has to be done for each of the time step.

(Refer Slide Time: 15:47)



So now let us begin the finite element equations. How do we get the finite element equations? So we again talk about we go back and revisit our energy theorems and one of the energy theorems we derived is the Hamilton's principle because we are now trying to derive the equations for the general dynamic equations and then remove the dynamic part of it and solve only for the static part.

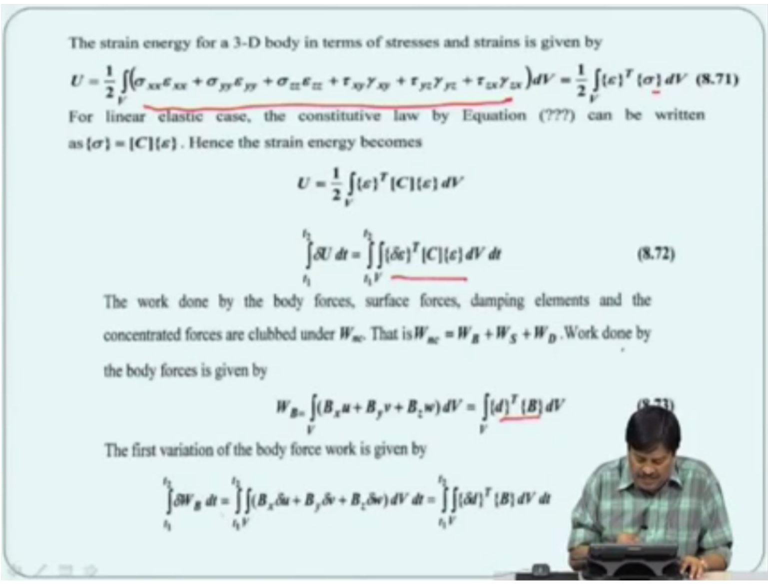
So if you go back to our Hamilton's principle, the Hamilton's principle states that the minimization of the total energy between the time t1 and t2 that is T is the kinetic energy, U is the potential energy and nc is the work done by the non-conservative forces and if you integrate this with respect to t1 and t2 equal to 0 will give us the governing equation. So now let us go and find out each one of them.

Let us take the kinetic energy for a three dimensional system. Kinetic energy is nothing but integral over the volume of the mass times velocity and the velocity has 3 components that is u, v, w in the 3 coordinate directions so we have u dot, v dot and w dot here. So now we take this and you know that the variational operator operates like a differential operator so we take the variation in the 3 respective directions and then integrate by parts.

So now when we take the variation of T, now we have rho*du/dt*d/dt of delta u, dv/dt*d/dv delta u etc. Now you have 2 time dependent functions, we integrate by parts and then make sure that the first variation vanishes at time t1 and t2 we have done this in the last class so I am not going to repeat it here and when we do this ultimately we get this expression where u double dot is basically the acceleration.

So this can be written in the matrix form as variation of the transpose vector into d dot*v where d double dot is nothing but a vector of acceleration and delta d is nothing but du, dv and dw. So we have converted this equation into matrix form with a variation on the displacement vector multiplied by the acceleration vector.

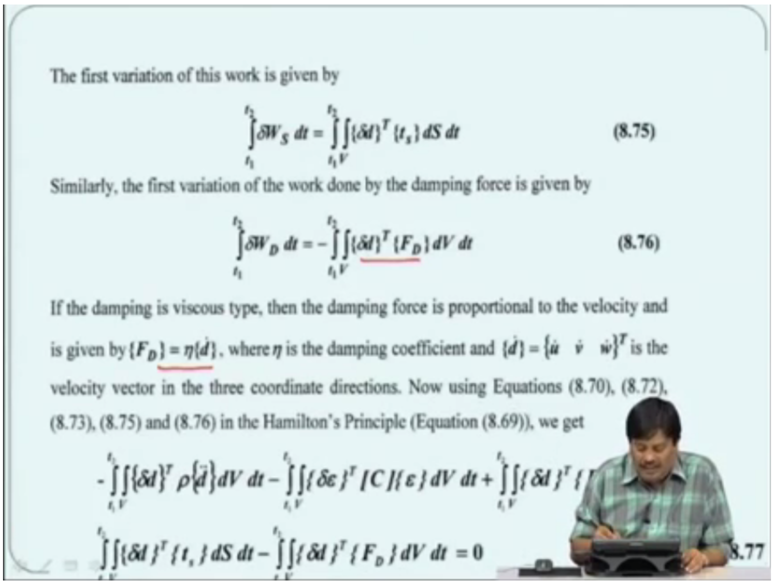
(Refer Slide Time: 18:13)



Next, we take the strain energy part. We know the strain energy, we derive the strain energy for a 3-D state of stress which is nothing but is given by this expression where it is a product of stress and strain in each of these planes, which can be written in matrix form in this form and when we take a variation we have to take a variation on displacement or the quantities dependent on the displacement and the quantity dependent on displacement here is the strain because we know the strain displacement relationship so we write this delta U/this form.

Now we take the work done by the non-conservative forces, which can be split up into 3 different forces, one due to body force, one due to surface force acting on the surface and one due to damping force. So now work done by the body force is Bx*u, By*v and Bz*w and it is a volume integral because Bx, By are all force per unit volume. So this can be written in this form in the matrix form and taking a variation of that we get this in this form.

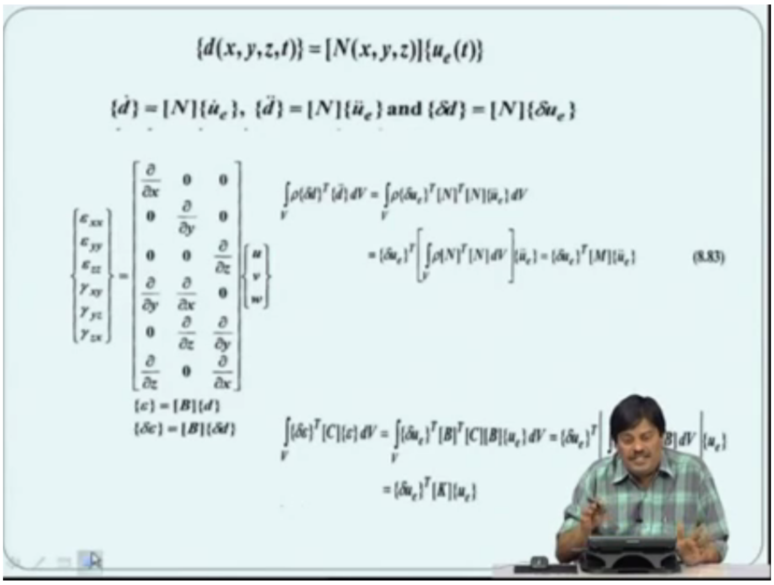
(Refer Slide Time: 19:24)



Similarly we can write the surface forces in this form where ts is the surface vector acting in the 3 coordinate direction and similarly we can write damping in this form. While treating damping, there are different ways of damping, the damping could be frictional, damping could be based on material property, damping could be viscous. The most common type which is mathematically easily tractable is basically the viscous damping.

So we assume that the viscous damping force is given by this expression, which is directly proportional to some constant, which is called the damping constant multiplied by the d dot, d dot is essentially the velocity vector. So now we put all these things together we get this whole thing in the Hamilton's theorem. Now we simplify each one of them. Let us do that.

(Refer Slide Time: 20:25)



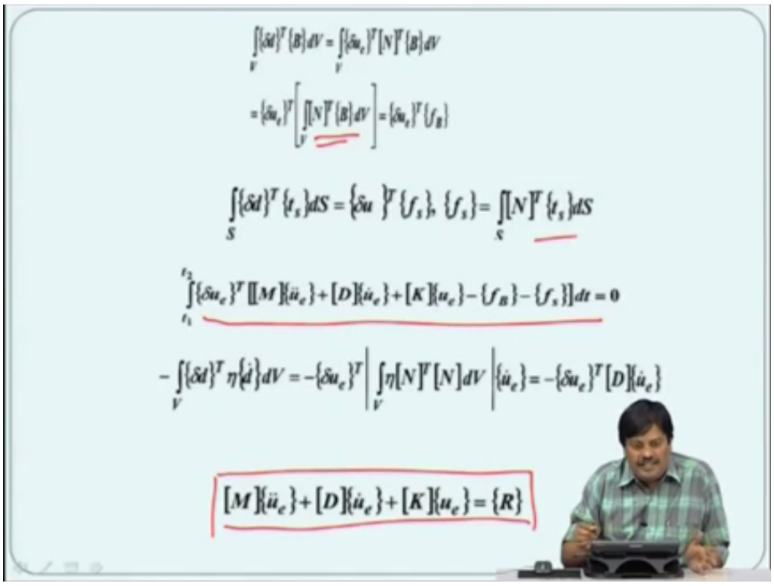
So before we do that now we said that in finite element we assume the variation over each element. That is the variation can be expressed in terms of the shape functions, which I talked about in the first slide multiplied by the an of t, the an of t is the elemental displacement vector of the element, which I call it as ue of t. For the sake of convenience, ue means e is the elemental vector whereas u is the global vector.

So now I can get because here d dot is N*ue, N is only a spatial dependent. So the time dependent comes from the displacement and d double dot is essentially N*ue double dot and delta d=N*delta ue. So basically now we put that into the first expression, which is the inertial expression coming from the kinetic energy. So we can write this and delta d transposes nothing but delta ue transpose into N transpose multiplied by d double dot is nothing but N*ue double dot.

So when I put this I get this and this quantity is called the mass matrix and the way we can do that is the basically is the consistent mass matrix. Now let us take the strain energy portion which has the strain displacement matrix. So strain displacement we know epsilon x=du/dx etc. In matrix form, it is given by here. Now epsilon=B*d. So now I know u, v, w are basically given by the shape function relation, we substitute here u, v, w in this form.

Then we can write epsilon=B*d and delta epsilon is B*delta d. We substitute it into the strain energy expression here that is basically coming from here. Then when we do that we get this form and this form B transpose C*B is basically the stiffness matrix where C is the material matrix. We derive this material matrix for isotropic, orthotropic or whatever the material conditions are in the structures.

(Refer Slide Time: 23:02)

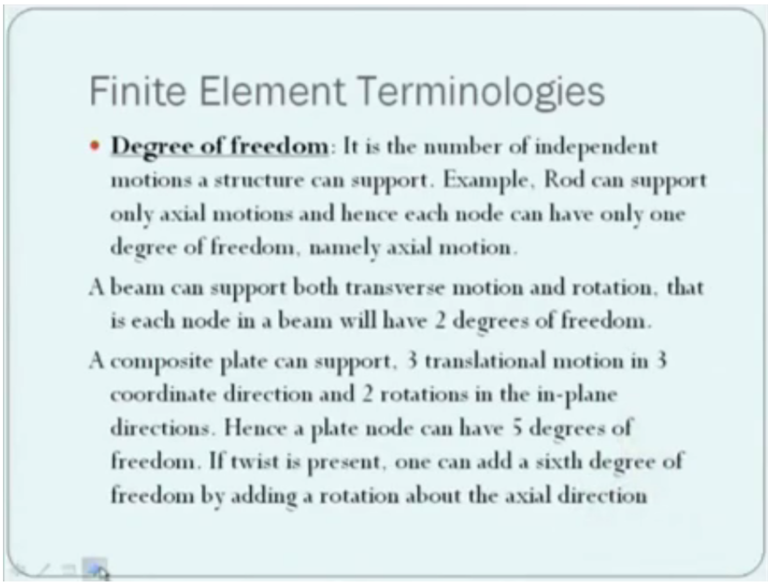


So this can be written in this form mathematically. Similarly we can write the body force delta d using this expression here we can write a delta d transpose is delta ue transpose into N transpose and multiplied by the body force vector and this is the discretization of the body force. So this basically converts the body force, which is distributed into the concentrated force acting on the nodes.

Similarly, we can do the surface force and when we put all these things together we get this equation and here delta ue is the incremental displacement, which cannot go to 0 in this expression so the only thing that can go to 0 is the 1 within the bracket and that is what is the governing equation, discretize form of the governing equation in the FEM. So the mass matrix is completely got from the discretized portion that is N transpose N.

The damping is got here N transpose N*eta and k is B transpose CB integral. So we have everything in the discretized form which we need to solve. If we are solving a static problem, we ignore this portion, we ignore this portion, we solve only Ku=R.

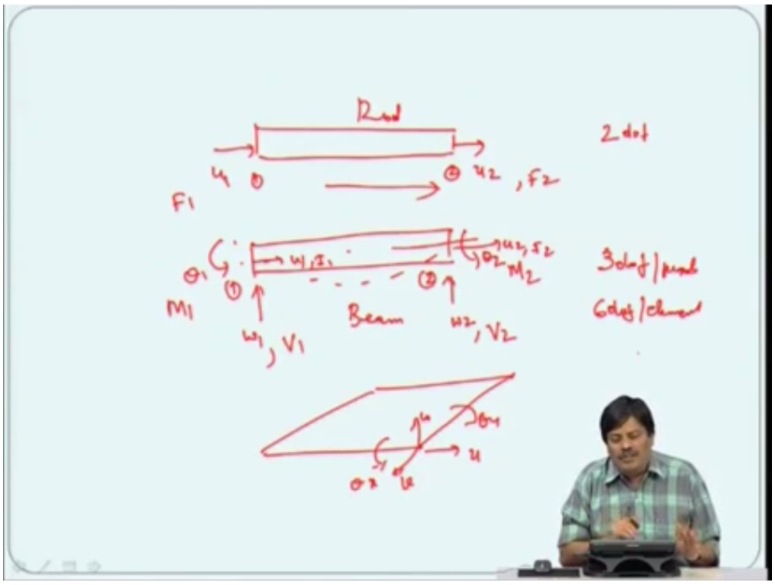
(Refer Slide Time: 24:14)



So now let us introduce certain terminologies. Degree of freedom, what is degree of freedom? It is the number of independent motion a structure can support. Example, a rod can support only axial motion in the 2 nodes and hence it has 2 degrees of freedom. It is a number of independent motions. For example, a beam can support both transverse motion and rotation so that is each node in a beam will have 2 degrees of freedom.

A composite plate can support 3 translational motion that is in 3 coordinate direction and 2 rotation in plane direction. Hence, the plate node will have 5 degrees of freedom. If we need to add a twist about the axis then we can add the sixth degree of freedom.

(Refer Slide Time: 25:13)



To explain this a little bit, so let us take a rod. So rod can move only in this direction, it can move only in this direction so basically that means you have 2, this is node 1, this is node 2.

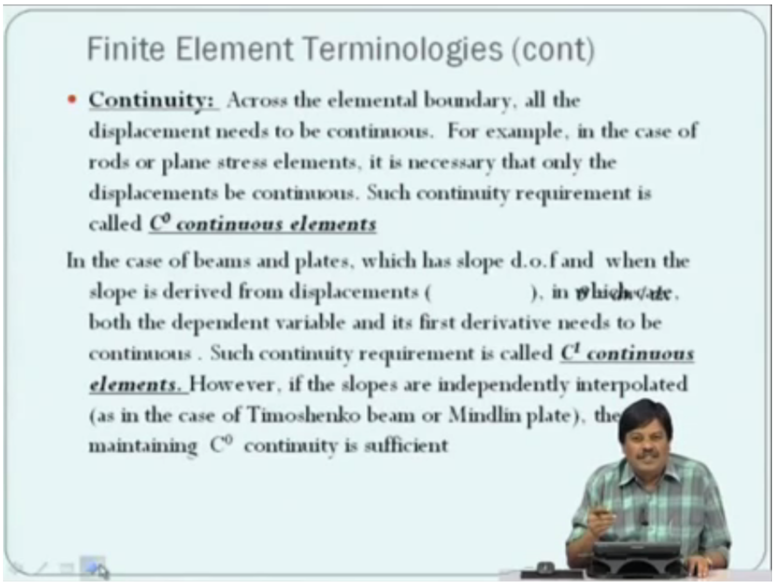
So this is u1 and u2, correspondingly you have F1 and F2 so it can support only 2 degrees of freedom. So this is a 2 DOF model. As I said earlier, this is a rod, so this is a beam, beam can bend so it can bend, it can undergo this bending and this bending is caused by the rotation that is the moment that is applied at the end.

So it can support the transverse displacement w at node 1, this is node 2, w1 and w2 and it can support theta 1 and theta 2 and corresponding to the w1 because everything will have a displacement dependent variable determination and a force variation and what is cause and effect as we found in the variational principle. The w1 is caused by the shear force 1, w2 is caused by shear force 2 and theta 1 is caused by the moment 1 and moment 2 and as I said if it is a composite beam, the mid plane does not coincide with the neutral plane.

That means there will be an additional component u1, u2, F1, F2 so it is basically a combination of both rod and beam. So each will have 3 degrees of freedom per node and 6 DOF per element. So if it is a plate, we have plate so you would basically have it can have the translational degrees of freedom u, w, v. Then it can have the theta x, theta y and you can also have a twist degree of freedom if there is a twisting angle to it.

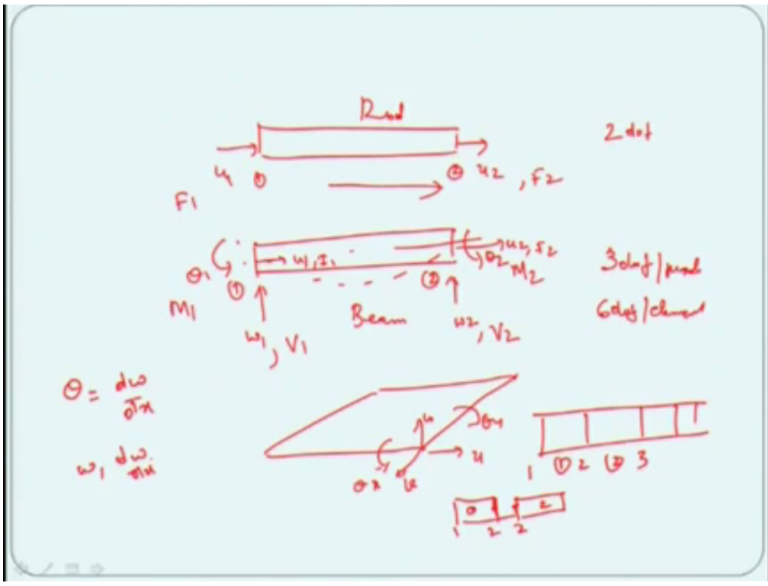
So these are called the independent degrees of freedom that you should understand what is an independent degrees of freedom? So every beam will have a degree of freedom that is defined by the motion that is the structure is undergoing.

(Refer Slide Time: 28:04)



The other one is the continuity. So as I said earlier across the element boundary all the displacement need to be continuous that is a derivative should exist. For example in the case of a rod, or a plane stress element, it is necessary to have only displacement to be continuous such a continuity requirement is called C0 continuous element.

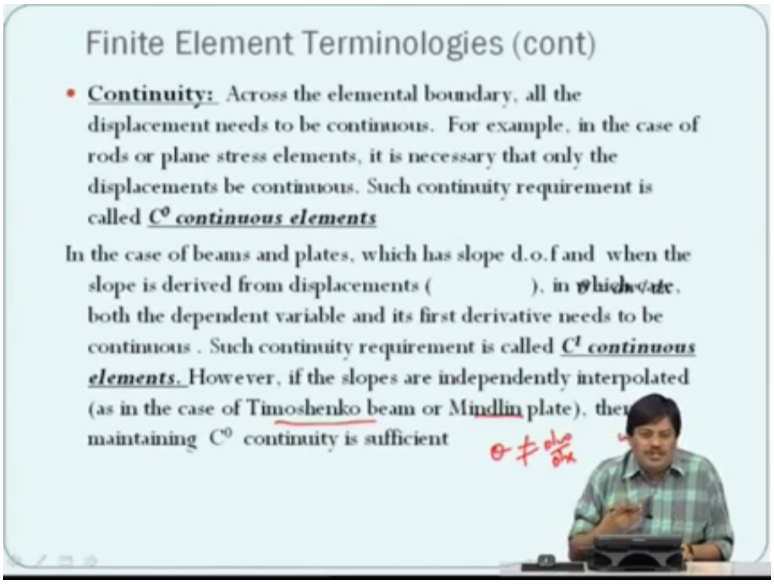
(Refer Slide Time: 28:31)



That is for example if you have a series of rod element and we isolate it, this is element 1 and element 2. So the element 1 this will have 1 and 2, 3 so the axial displacement 2 due to element 1 should be equal to the axial displacement at 2 due to element 2. That is not at all an issue because it can be maintained, but if it is a beam normally in beam the slope theta here is derived from dw/dx from the transverse displacement w.

So if it is a beam then we need to see that the slope at 2 due to element 1 should also be continuous due to slope at 2 due to element 2. So in the case of beam, we need both w and dw/dx to be continuous so there are 2 variables. The primary dependent variable and its derivative needs to be continuous, such a continuity requirement is called the C1 continuous element.

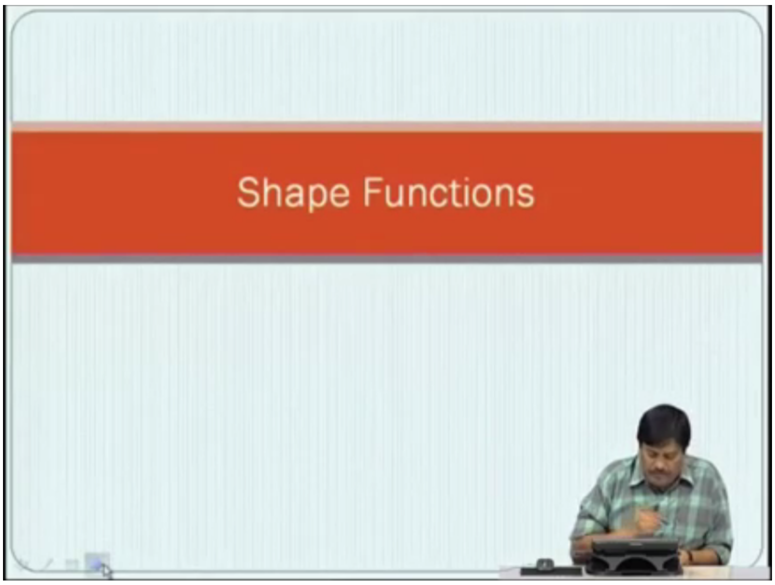
(Refer Slide Time: 29:46)



So basically the C1 continuous element is a problem. It is not easy to satisfy especially when we go from beam to some other higher elements. So that is the problem with the beam so in order to avoid this people introduced what is called shear deformation into the beam formulation and such a beam theory is called the Timoshenko beam theory and if it is introduced in plate is called the Mindlin plate theory.

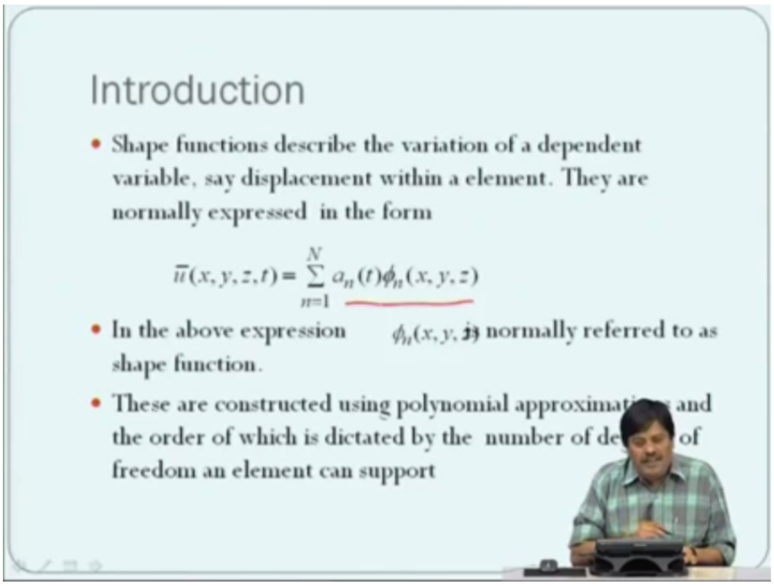
In these beams, theta is not equal to dw/dx and theta can be independently interpolated so then you need only w and theta to be continuous so that you do not need the first order continuity, which is easier to solve. So we can still build in the C0 continuity in beam provided we use Timoshenko beam theory.

(Refer Slide Time: 30:45)



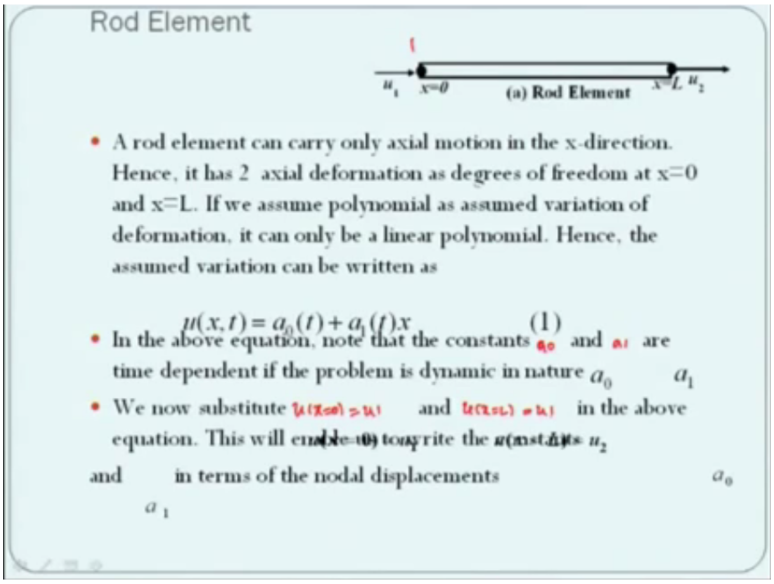
Now let us come to the shear functions.

(Refer Slide Time: 30:50)



So as I said I go back to equation 1 when we said that we approximate the dependent variable or the displacement within the element by a series shown by this equation where an of t is basically the what is called the degree of freedom or the nodal displacement and phi n is the shape function. This phi n are basically the one which satisfies the governing boundary conditions or they are normally used by approximating this whole thing into a polynomial, which is dictated by the order of degrees of freedom.

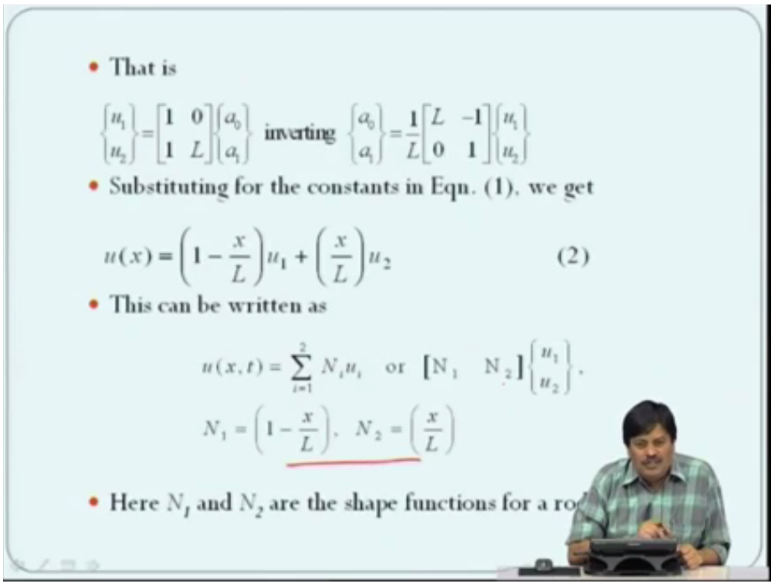
(Refer Slide Time: 31:38)



For example, if it is a rod problem, so for the rod problem we basically have the element has 2 degrees of freedom x=0 and x=L, so if you want to approximate the variation within the element and use polynomial we need to have 2 unknown constants into the variable that we are assuming why because it can support only 2 motions that is x=0 and x=L that is u1 and u2.

So for these two unknowns we need to have 2 constants so we take this equation u of x t=a naught t=a1 t*x. So in this above expression, we know that the constants are here a naught and a1 are time dependent if we are talking about a dynamic problem or it is simply constant in the case of static problem. Now we substitute that at x=0=u1 because we start our axis here this is x=0 and u at x=L=u1 the above equation.

(Refer Slide Time: 33:19)

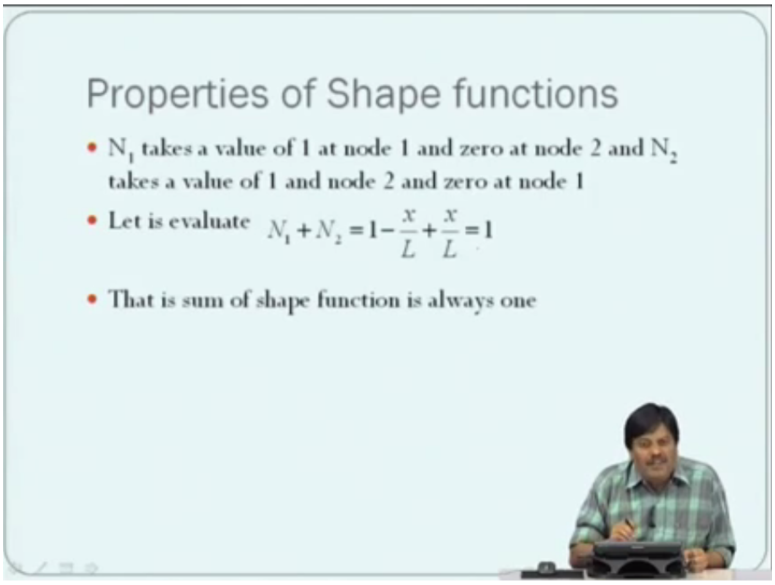


Then we will write the u1 and u2 in terms of the unknown constants so that is given by so now I can say u1 u2=1 1 0 L a naught and a1 and we invert this matrix then we get a naught a1 equal to 1/L L 0 -1 u1 u2. Now we substitute this back into our equation 1 and we can write and factorize all the u1 and u2 separately then we can write u of x=1-x/L*u1 and x/L*u2.

So this 1-x/L and x/L are called the shape functions for the rod element so then we can write u of x, t is sum of i=1 to 2 Ni*ui or you can write this N1*matrix this is N matrix u1 u2 where N1 are given by this as I said these are the shape functions for a rod element. So what are the properties of the shape functions? If we go back x=0 is the left node where displacement is u1 and x=L is the right node where the displacement is u2.

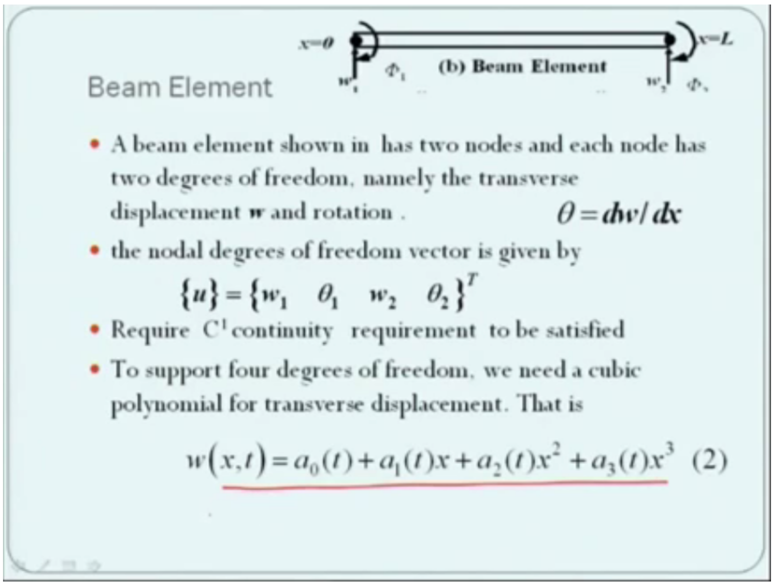
Suppose we substitute x=0 here that is at the left node, N1=1 and N2 will be 0 so it takes the value of unity at node 1 where the displacement has to be satisfied. So at node 1 we need displacement to be u1 so u2 should go to 0 so that is why x=0. At x=0 the N2 goes to 0, N1 is 1. At x=L, N1 goes to 0 and N2=1 so N1 takes the value of 1 at node 1 and 0 at node 2 and N2 takes the value of 1 in node 2 and 0 at node 1 and let us evaluate the sum of the n1+n2 which is equal to 1.

(Refer Slide Time: 35:23)



So the sum of shear functions are always equal to 1. So these are some of the properties of the shape functions. It takes the value of unity at the node where you are evaluating it the coordinate and all other places it is 0.

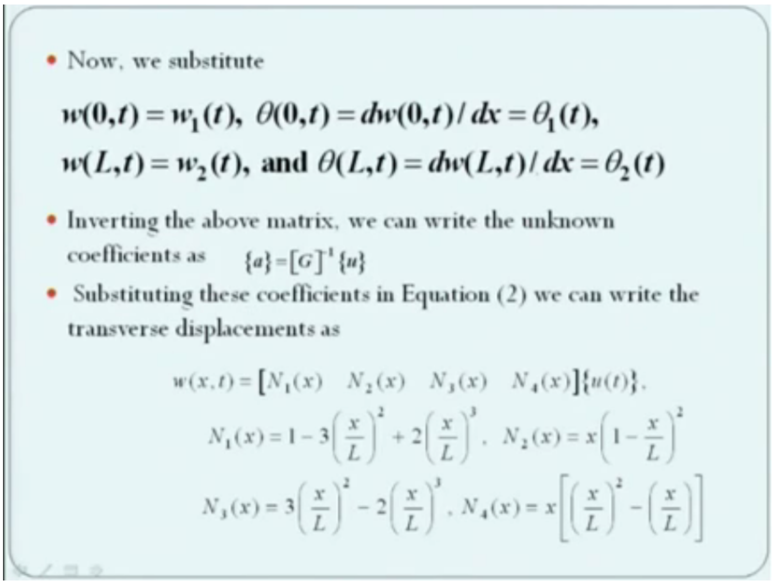
(Refer Slide Time: 35:40)



We will do the same this with beam element. As I said the beam element has 2 degrees of freedom for node, which is w is the transverse displacement and theta is the rotation. So the minimum order that is needed here is it has to have 4 degrees of freedom are there so with the assumed polynomial field if we assume polynomial should have at least 4 constants for determination of this 4 displacement boundary conditions.

So that means minimum we need a cubic so the higher order approximation that is required because of this 4 degrees of freedom will lead to continuity requirement which is C1 mainly because as I said earlier both w and dw/dx have to be continuous at the interelement boundary. So we take a cubic polynomial here. So what we do we substitute at x=0, w=0 so that is given here.

(Refer Slide Time: 36:55)



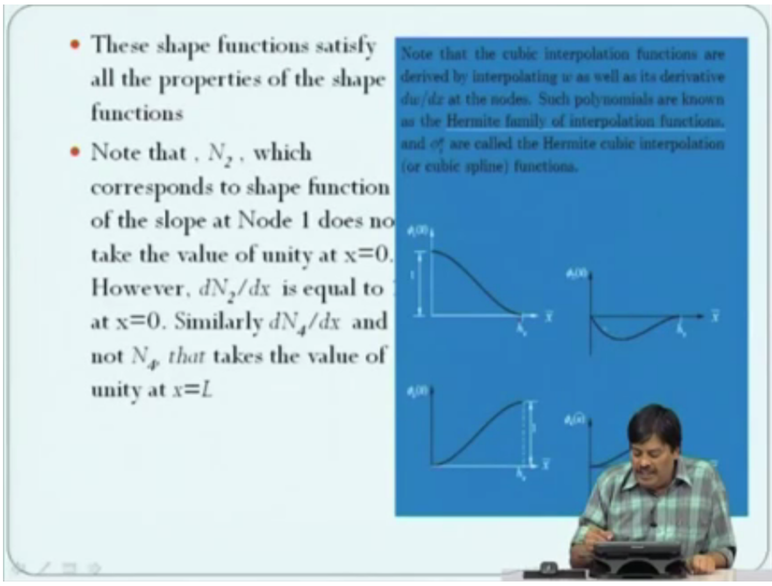
At x=0, you have w1, at x=0 dw/dx=theta 1 because the slope is derived from the displacement. At x=L, the transverse displacement w2 and at x=L theta dw/dx will be theta 2. When we substitute this, we can relate u with theta and inverting it we get the unknown constants a naught, a1, a2 can be related to the nodal displacement by this and when we substitute this back into our original equation here for a naught, a1, a2, a3 we can write w=N1, N2, N3, N4 of u so ni*ui.

That is the standard from for all finite element formulation and we get the following shape functions. So a lot of mathematics has to be done just it is a matrix inversion. Once you do that we get this. So let us look at these shape functions, does it satisfies the shape function properties? That is we need that it takes the value of 1 at where it is evaluated. Suppose we substitute at x=0, we expect that the N1 should go to value 1 and all other should go to 0.

So at x=0, we clearly see N1 is 1 and N2, N3, N4 are all 0. If we substitute x=L, we see that this is x=L we have (()) (38:34) N2 will be 0, this will be 0, N3 will be 1 because it is the transverse displacement that is 3-2 and N4 will be equal to 0. Then what about the theta degrees of freedom. So the N2 will not go to 0 when x=0 because what we are looking at N2 is the shape function for the dw/dx that is the rotation which is got from w.

And similarly N4 it is the shape function for theta at node 2 at right node so we will find that the dN2/dx that is because theta is derived from the w, we need to take dN2/dx and substitute at x=0 you will see that this will be equal to 1 and this will be 0 and when x=L dN2/dx basically this will be equal to 0 and this will be equal to 1. So that is precisely what I have written here.

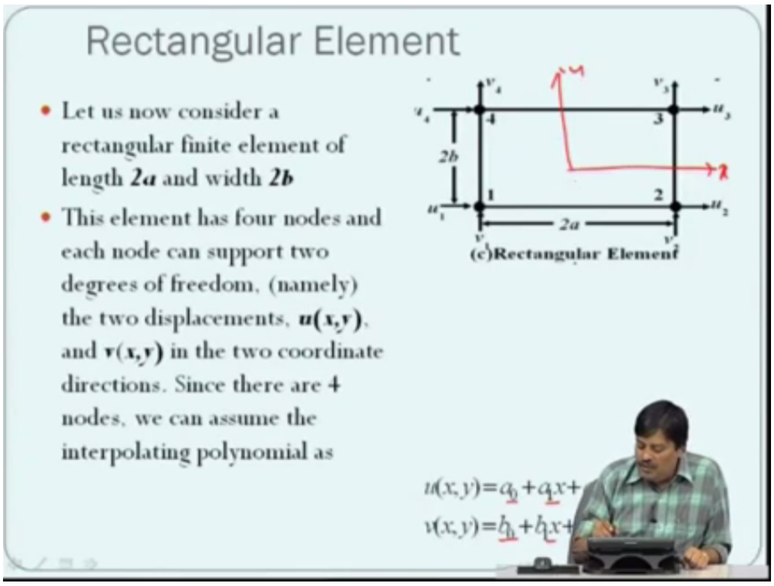
(Refer Slide Time: 39:38)



These shape functions essentially satisfies the properties of the function and I have specifically said that N2 which correspond shape function of this slope and node 1 does not take a value of unity at x=0 even though it is evaluated at that node, only dN2/dx=0 and if you plot this you see that the variation, N1 takes the value of 1 and 0 at 2. The slope dN2/dx takes the value of 1 at node 1 and 0 elsewhere.

This is N3 which is again corresponding to the transverse function so this is how the shape functions vary.

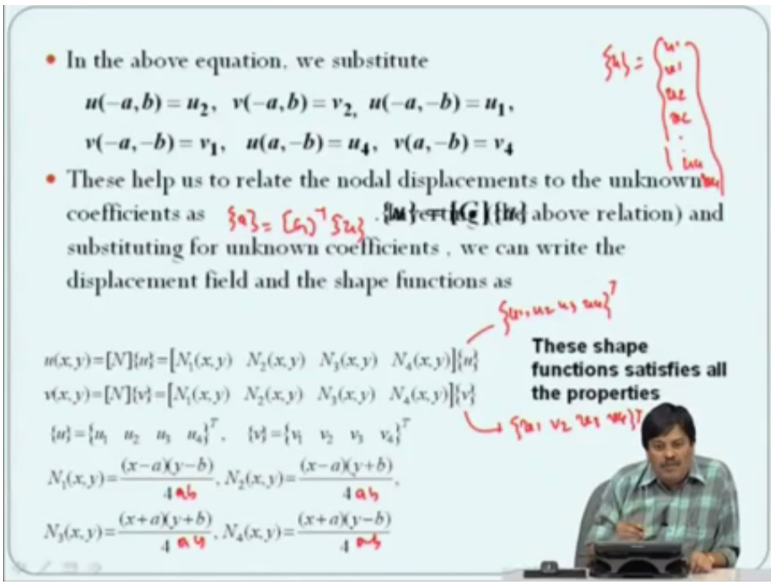
(Refer Slide Time: 40:30)



Now let us go to a 2 dimensional element and suppose we have a rectangular element of length 2a and width 2b, which has 4 nodes and each node can take 2 degrees of freedom that is u and v in 2 horizontal directions so it will have totally 8 degree of freedom element. So we have 4u degrees of freedom and 4v degrees of freedom that means the polynomial is not only x it is also y dependency is there.

It will have x and y and it should have 4 constants both in u and v so we take the form u of xy=a naught+a1x+a2y+a3xy. Similarly, we have b naught+b1x+b2y+b3xy so there are 8 constants corresponding to 8 degrees of freedom here. So now we substitute the coordinates of the element. So here the axis is exactly at the middle so we have a middle axis this is x and y.

(Refer Slide Time: 42:00)

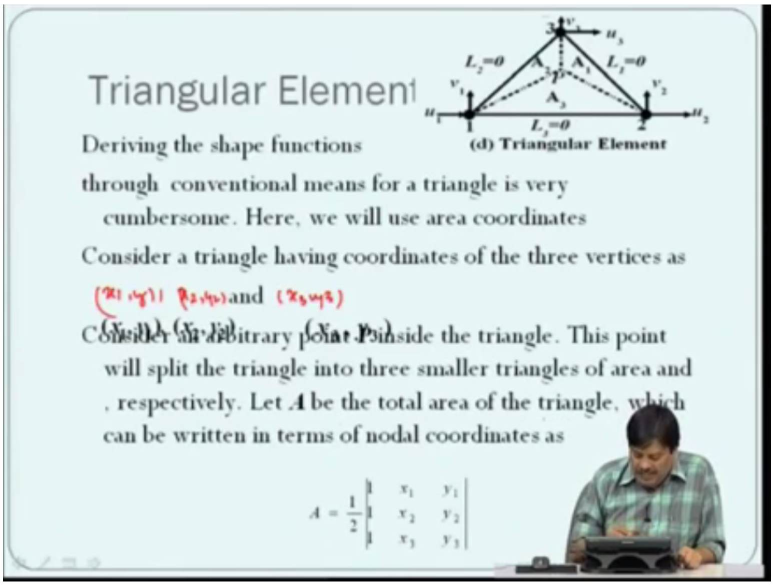


So this corresponds to -a and this is +a and -b and +b so we substitute this equation into the assumed function here in this function and we evaluate the coefficients we can write a will be equal to this G inverse*u where my u will be a vector u1,v1,u2,v2 to u4 v4. So this is the vector of my u displacement. So when I substitute this back into this relation and simplify we can write u=N1 N2 N3 N4 into only u degrees of freedom which will have u1, u2, u3, u4 this will have only u1, u2, u3, u4 transpose.

And this will have only the vertical degrees of freedom that is v1, v2, v3, v4 okay and each one can be given x-a*y-b/4, x-a*y+b/4, x+a*y+b/4, x+a*y-b/4. If you look at this carefully let us substitute at x=-a*b suppose we want u1 which is -a*-b this is 4ab, everywhere you have a 4ab here. So when you substitute x=-a*y=-b here, N1 it takes a value of unity whereas N2 will be 0 at y=-b, this will be 0.

And similarly when we take the value of say u2 at -a and b so N2 takes a value of unity and if you take the sum of N1+N2+N3+N4 it will always be equal to 1 so all these properties are followed and if this is not followed then they are not called shape functions.

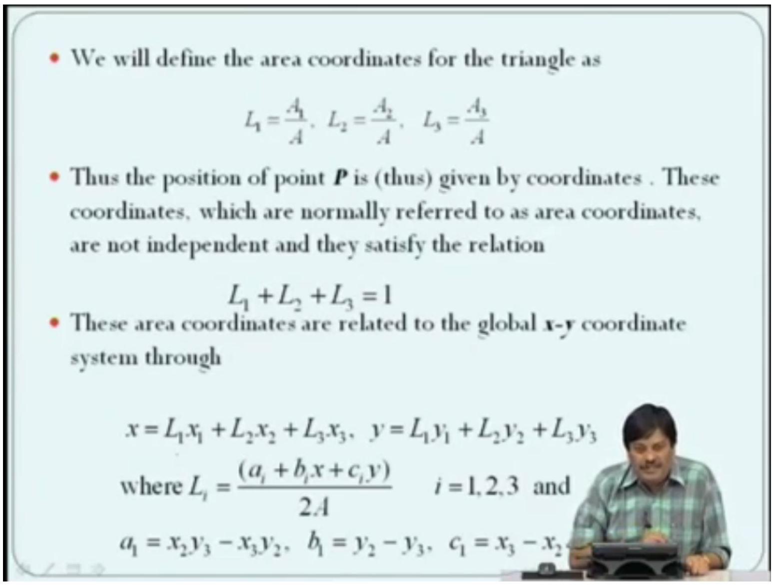
(Refer Slide Time: 44:21)



Now for the triangular element, how do we do it? So now let us look at it is very important element but inconvenient to generate the shape function by the procedure we adopted before even though it can be done. So here we take a slightly different approach where we will not use the conventional coordinates, which is given by you have x1 y1, x2 y2 and x3 y3. What we will do here is we will take any point for which we require a coordinate.

We split up this into 3 areas, we call them A1, A2, A3 okay and if I want the area, area can be got from the coordinates if A is the total area of the triangle they can be got from the expression given here which is the determinant of this where x1, x2, x3 are the rectangular coordinates of the original element.

(Refer Slide Time: 45:31)



So now we will define the area coordinate for this problem or the point P it is located by L1, L2, L3 where L1, L2 is the sub area A1/A so if you look at it the sub area A1/A is the L1, L1 is the one where the opposite node is the node 1. Similarly, we take L2 which is the sub area A2/A, L3 is A3/A. The position of the point is thus given by these coordinates, which are normally refer to as area coordinates and or not independent and they satisfy the relation L1 L2+L3.

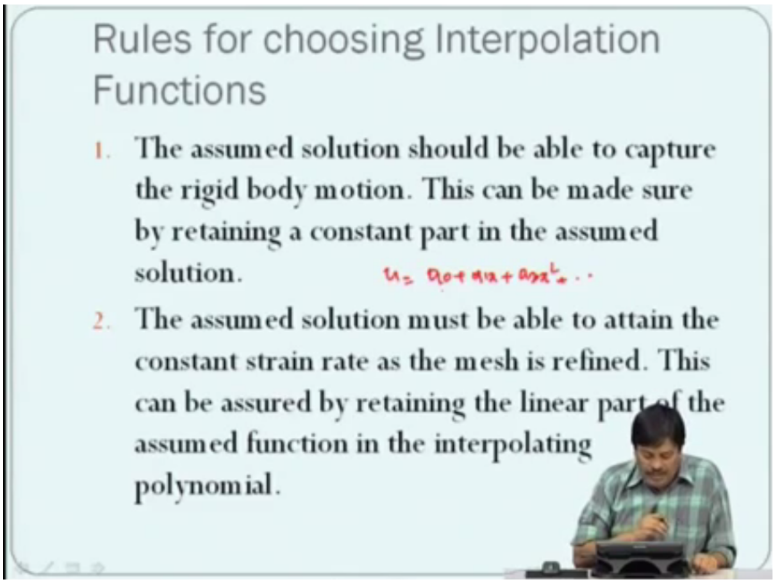
So the moment you solve that it is similar to sum of shape functions satisfying the equation N1+N2+N3=1 where the coordinate x can also be related. I am not going into the derivations of this. Through this relation L1 x1, L2 x2, L3 x3 so from this we can derive the expression between L and the coordinates that is here we say Li will be Ai+Bi x+Ci y/2A where A, B, Cs are given in terms of the coordinates.

So you can get L1 will be A1, B1, C1, L2 will be A2, B2, C2 and these are all cyclic permutations which we can get. So basically the above equation requires to be used we will use this when we want to find the derivatives because in most of our B matrix we need dN/dx with respect to x when we have the area coordinate L we need to use this expression to get the derivative with respect to x.

So that is why we need this relations so this is more convenient to do that. So now we can write this shape functions for the triangle as u=N1u1, N2u2, N3u3 where N1 is nothing but your area coordinates L1, L2 and L3. So here these shape functions also follow the normal rules that is it takes the value of A where L1=1 at the node 1 and 0 elsewhere and L2 and L3 is 0 at node 1 so in order to fix this we can go back here.

So at this point L1=1, at here L1=0 and if you come here L2=1 and L2=0, if you come here L3=1 and L3=0. So basically when you say here in this point L3 is 0, L2 is 0 but L1 is 1. If you look at here, L2 is 1 but L1 and L3 are 0. So it satisfies the property of the shape function and at the same time elegantly we can able to formulate the element in more convenient manner. The conventional method will give us lot of problems.

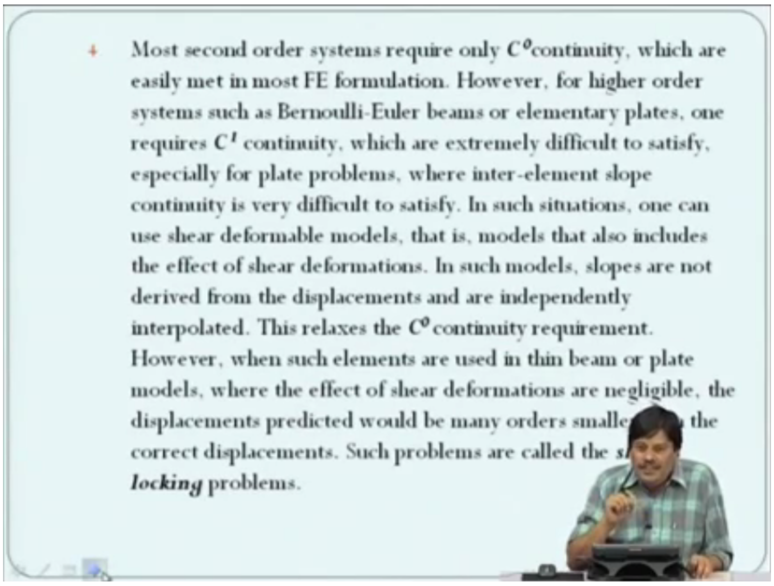
(Refer Slide Time: 49:05)



So what are the rules for choosing the interpolating functions? The assumed solution should be able to capture the rigid body motion and this can be made sure by retaining the constant part of the assumed solution that is basically what we are saying here is so if you have a function u=a naught+a1x+a2x square etc the constant part make sure that the rigid body motion can be present in the assumed function, which is absolutely necessary for convergence for the solution.

The assumed solution must be able to retain the constant state of strain as the mesh is defined and this can be assured by retaining the linear part that is the linear part a1x in the interpolating function. This is absolutely necessary. These are some of the rules that we need to follow.

(Refer Slide Time: 50:05)



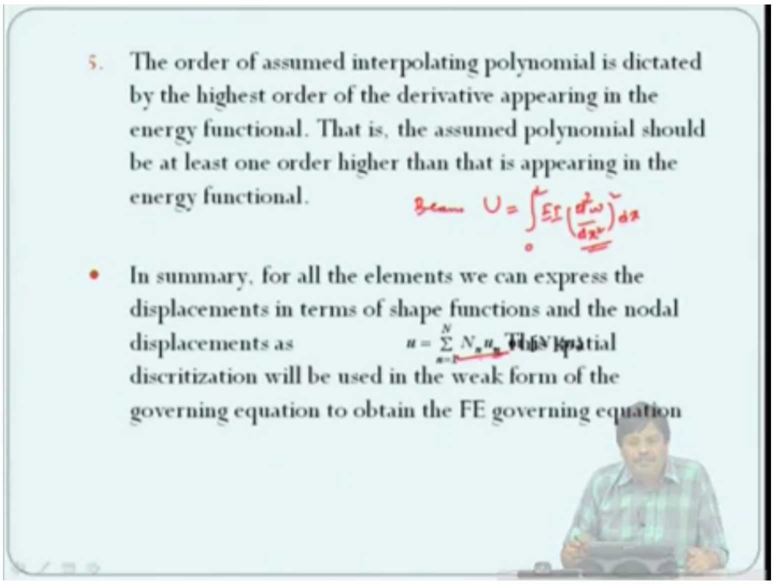
Most second order systems that is the second order system is where the differential equation is order 2 requires only C0 continuity so you want only displacement to be continuous. We explained this which can be easily met in most finite element formulation. However, for higher order systems such as the Bernoulli-Euler beams where the elementary beam or the plates where the slopes are derived from displacement.

Theta is dw/dx we need to see that both w and dw/dx are continuous across the element these are extremely difficult to satisfy in the interelement continuity. So in such situations what we do is we introduce shear deformation as I said earlier and Bernoulli-Euler beam is converted into Timoshenko beam where the theta is not derived from the slope that is theta is not equal to dw/dx where now we need to have only w and theta continuity that is we go back to C0 continuity.

Now if you look at it if the beam is very thick shear deformation is very, very predominant we can use Timoshenko theory. Suppose the beam is very thin, there is no shear deformation, you cannot make a C1 continuous to become C0 just by relaxing this. So in which case the displacement predicted would be many, many orders smaller than the correct displacements and such a problem is called the shear locking problem.

So we have to be careful well actually migrating from C0 to C1 by using different beam theories.

(Refer Slide Time: 51:53)

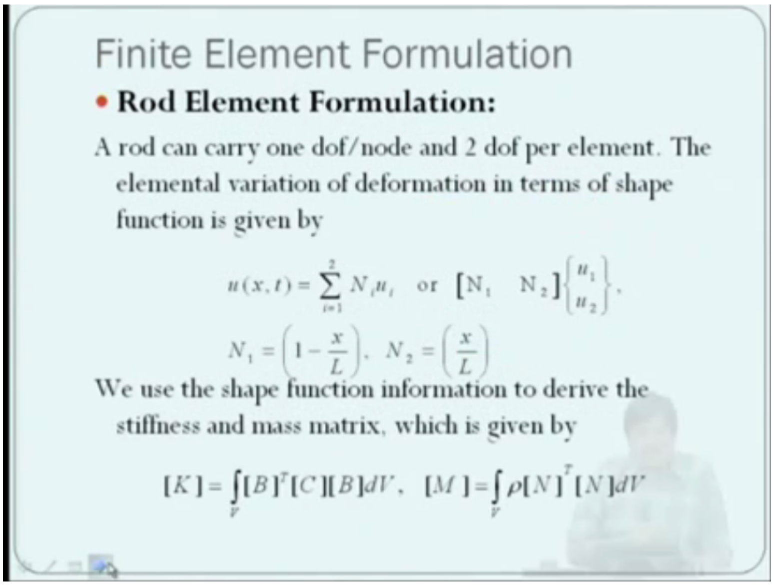


The order of assumed polynomial is dictated by the highest order of the derivative appearing in the energy functional that is the assumed polynomial should be at least one order higher than what is appearing in the energy function. To understand this now if you take a beam you have u will be equal to 0 to L EI d square w/dx square whole square to dx and suppose you assume a linear polynomial, which you cannot do anyway if you assume it then the energy will go to 0.

Because d square w/dx square will not exist and suppose you use a quadratic polynomial then d square w/dx square will only be a constant. So we at least make sure that in the case of a beam we need to have something, which is where one more higher order should be there. So what we need to choose is choose a polynomial where d cube w/dx cube exist. So basically that is what is required here.

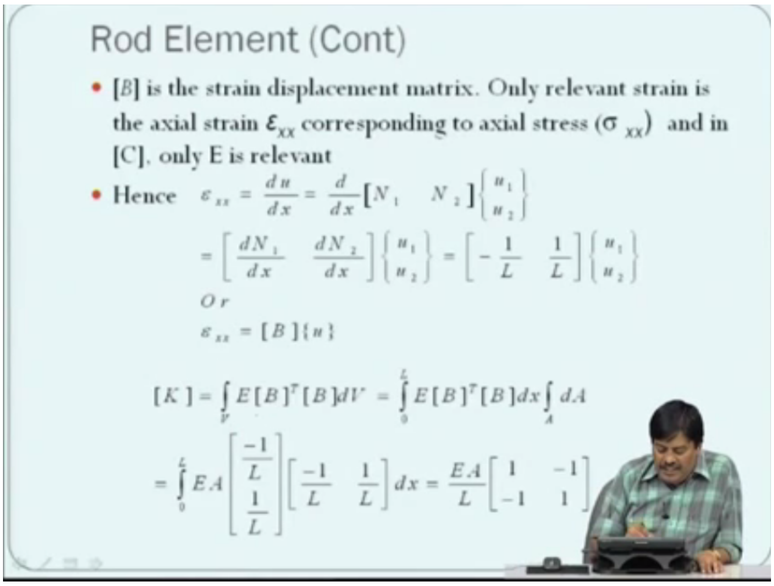
So in summary what are we doing here. For all the elements, we can express the displacements in terms of shear functions and nodal displacement, which are given by u=uN into which is given here ni*ui and the spatial discretization will be used in the weak form of the governing equation substituted there and then we formulate the stiffness and mass matrices then these matrices will then be used, assembled, solved for forgetting the solution of the structure.

(Refer Slide Time: 54:01)



This is in summary about how the finite element process takes place. Now let us look at some of the element formulation, this is a rod element formulation where you have N1 and N2 given by here where N1=1-x/L and x/L we use this shear functions then we know this is the stiffness matrix, we have derived this so we put this into this system.

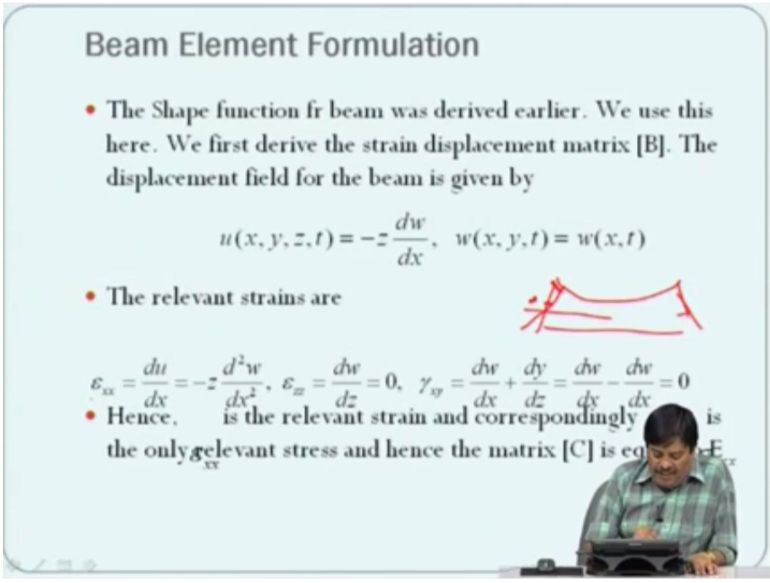
(Refer Slide Time: 54:20)



So B is the strain displaced only relevant strain is epsilon xx. So only you will have C present here then we put the C back here. So once we use this we can actually get the strain displacement matrices and the C matrix will contain only E the Young’s modulus. We substitute all these things into this equation replace C/E, once we do that we get this equation.

And we put this back here and this is the stiffness matrix for a rod we see that it is symmetric and it is a function of the material property E, sectional property A and length.

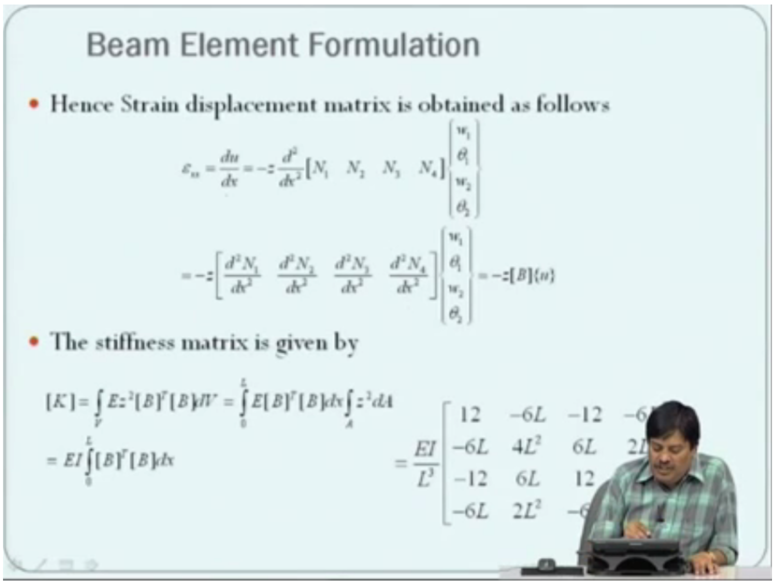
(Refer Slide Time: 55:10)



Similarly we can do the beam element formulation where we have the displacement relation between the axial displacement is given by theta times slope. So basically when the beam bends we have so this is the mid plane so we draw this and this is the axial displacement, which is the theta so that is the z times theta is nothing but the axial displacement and this one.

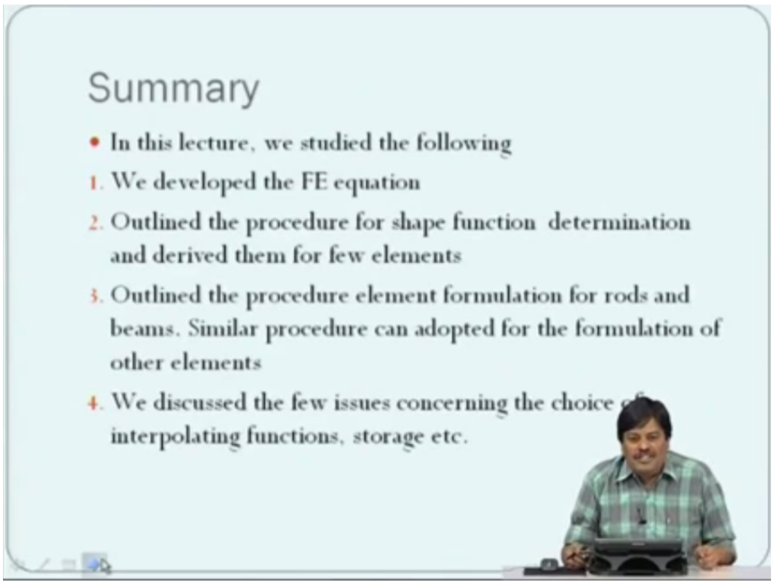
And on using the strain displacement relation we find that this is second order strains and we have the normal strains in the y direction and this is x and this is z so the sigma z in this direction is 0 shear strain is 0 so only relevant material matrix here is again E.

(Refer Slide Time: 56:05)



Here we plug this back into this equation and derive the strain displacement matrix, which is du/dx which is given by there and when we substitute for u we get this is the second order matrix and we substitute back into the governing differential equation and integrate between 0 to L we get this is the governing equation.

(Refer Slide Time: 56:30)



So basically we have done this and now in summary what we have understood here is we have developed the FE equations, outlined the procedures for finding out the shape functions and also derived some of the elements that can be directly go into our analysis like rods and beams and we also discussed few issues concerning the choice of the interpolating functions etc. Thank you.