Micro and Smart Systems
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Lecture - 27
Energy Theorems and Weak Form of the Governing Equation

So in this lecture, we will talk about some of the energy theorems and weak form of the governing differential equation.

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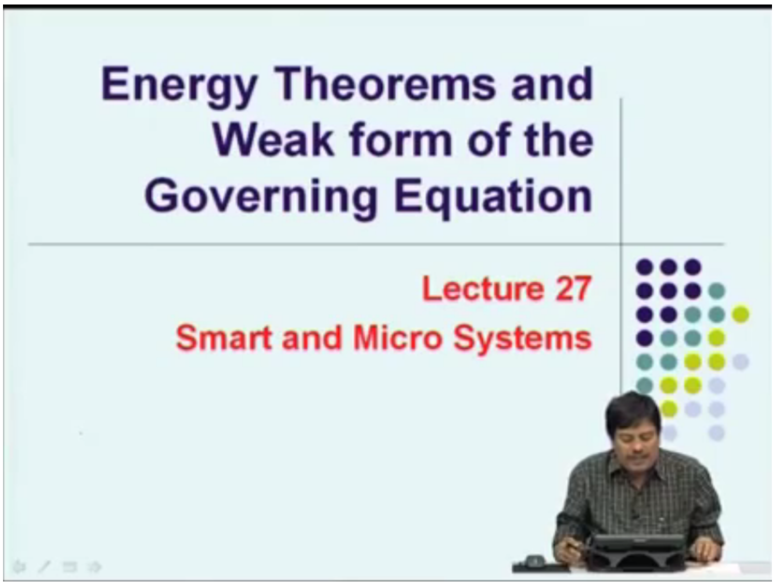

In the last class, we dealt on the theoretical basis for FEM and we briefly introduced the weak form on a general differential equation. So today we will talk about the theorems that lead to the weak form and also the energy theorems based on which the weak form will be constructed. For some structures we normally use in microsystems such as beam structures.

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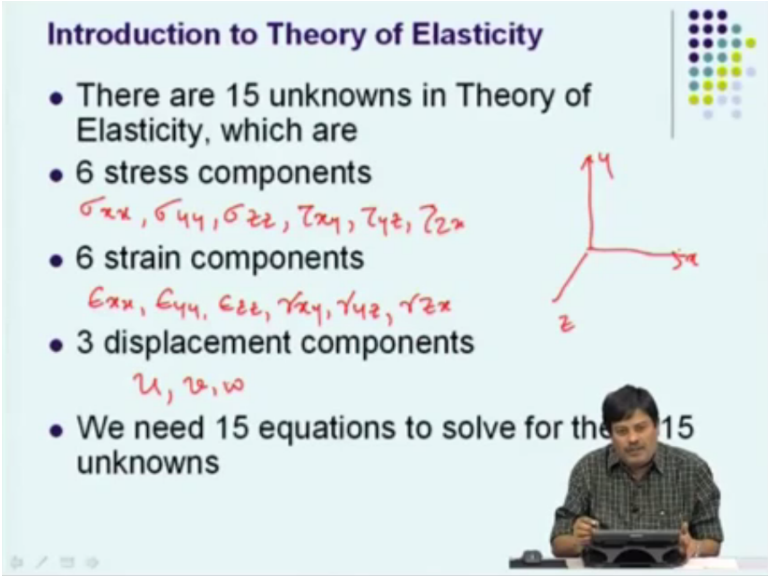

So before we go into the energy theorems, we need to understand basically the theory of elasticity. In theory of elasticity, there are 15 unknowns, which are the 6 stress components, the 6 strain components and the 3 displacement components. So if you have a coordinate system which is given by say x, y and z direction and the stress is a tensor having both the direction and the plane of applications.

The six stress components are sigma xx, sigma yy, sigma zz, tau xy these are shear stresses, tau yz and tau zx and we have 6 strain components corresponding to stresses, which are epsilon xx, epsilon yy, epsilon zz. We have gamma xy which are shear strains, gamma yz and gamma zx and we have 3 displacement components in the 3 coordinate directions, which is u, v and w. So in all we have 15 unknowns and we need 15 equations to solve.

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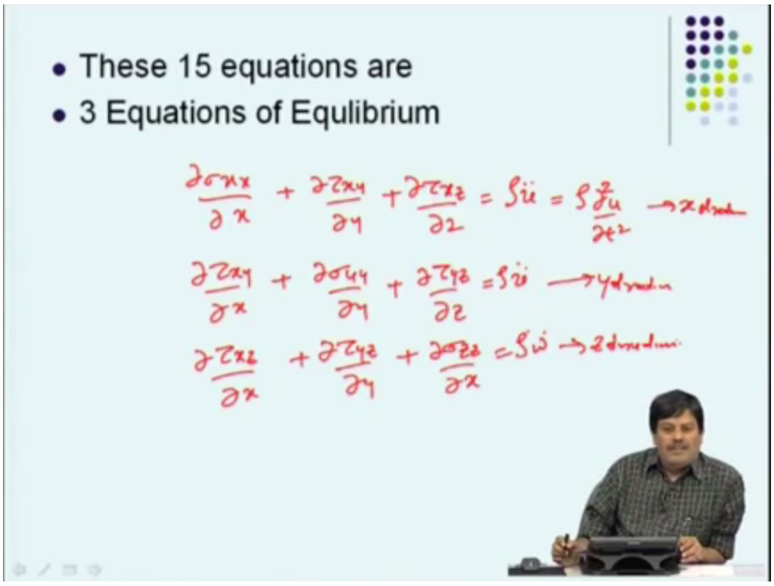

Let us see where we get these 15 equations from. Out of which we have 3 equations of equilibrium which are given by d sigma xx/dx+d tau xy/dy+d tau xz/dz=rho*u double dot, u double dot S=rho*d square u/d t square. This is the equilibrium in x direction. So similarly we have d tau xy/dx+d sigma yy/dy+d tau yz/dz=rho*v double dot. This is the y direction.

And we have dou tau xz/dx+dou tau yz/dy+dou sigma z/dz=rho*double dot. This is the z direction equilibrium. So these are the fundamental equations of equilibrium which we have to solve.

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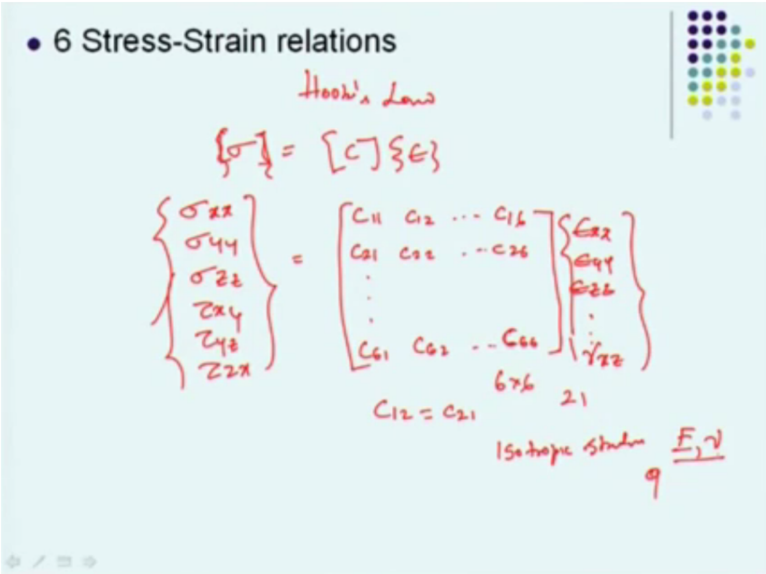

Next set of equations are the 6 stress strain equations, which is given by what is called the Hooke's law, which basically relates the stresses and strains with a matrix sigma will be equal to which will be a vector, sigma will be = some c matrix*epsilon where we have the 6 stress components sigma xx, sigma yy, sigma zz.

We have tau xz, tau xy, tau yz, tau zx. So these are the stress vectors, which is related to the constitutive matrix this is called c11, c12 to c16, c21, c22, c26 and then you have c61, c62, c66, which is related to the strain which is given by epsilon xx, epsilon yy, epsilon zz, gamma xy, gamma yz, and gamma xz. So there are 6 equations here and the c11 this is the 6 by 6 matrix and in this matrix we always have c12=c21 so it is symmetric.

So there are only 21 independent constants. By taking different planes of symmetry we can reduce it to only 2 constants in the case of isotropic structures, which are basically the Young’s modulus and Poisson’s ratio, nu is the Poisson’s ratio. If it is an orthotropic structure we have 9 constants and if it is a transversely isotropic, we have 13 constants like that we can actually reduce depending upon the plane of symmetry, but in this lecture we will be talking only about isotropic structure.

So basically we will be dealing with only 2 constants that is the Young's modulus E and the Poisson's ratio nu.

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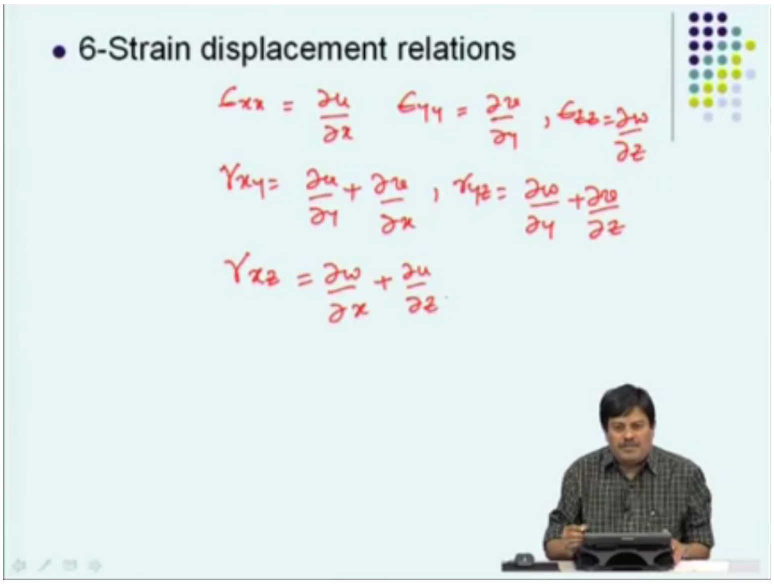

Then we have this 6 strain displacement, for example we have epsilon xx which is the strain is related to displacement du/dx, epsilon yy=dv/d y, epsilon x=dw/d z. Then we have the gamma xy=du/dy+dv/dx. Then we have gamma yz=dw/dy+dv/dz. Then we have gamma xz=dw/dx+du/dz. So we have 6 strain displacement relationships. So in all we have about 15 equations for determining the 15 unknowns.

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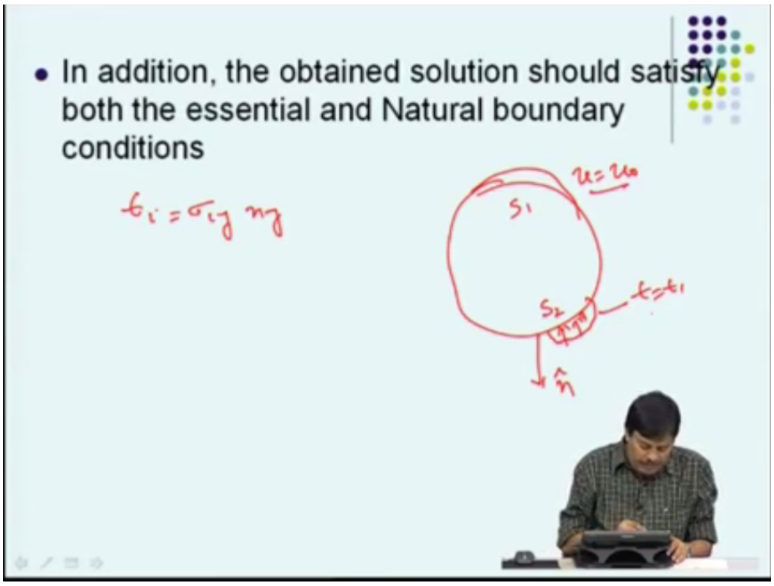

So in addition to having these 15 equations, the equations have to satisfy the boundary conditions. In the last class, we talked about essential boundary condition, which is on the primary dependent variable. If we say displacement, it will be on displacement and if it is depending upon the derivatives of displacement, which is called the natural boundary condition or the Neumann boundary conditions.

So if you have any structure you may have a boundary, which I call s1 where you have u=u naught that is the primary dependent variable is specified or you may have some tractions where we call the t=ti. So this will be on different surface s2. So this traction ti will be related to the stress sigma ij*nj where nj is the outward normal that is n vector, which is an outward normal to the surface.

So in summary, we have 15 equations and for the 15 unknowns coming from the equilibrium equations, strain displacement relation, stress strain relation and in addition it has to satisfy the boundary condition on surface s1, which is the Dirichlet boundary condition and on the traction boundary condition it has to satisfy the Neumann boundary condition.

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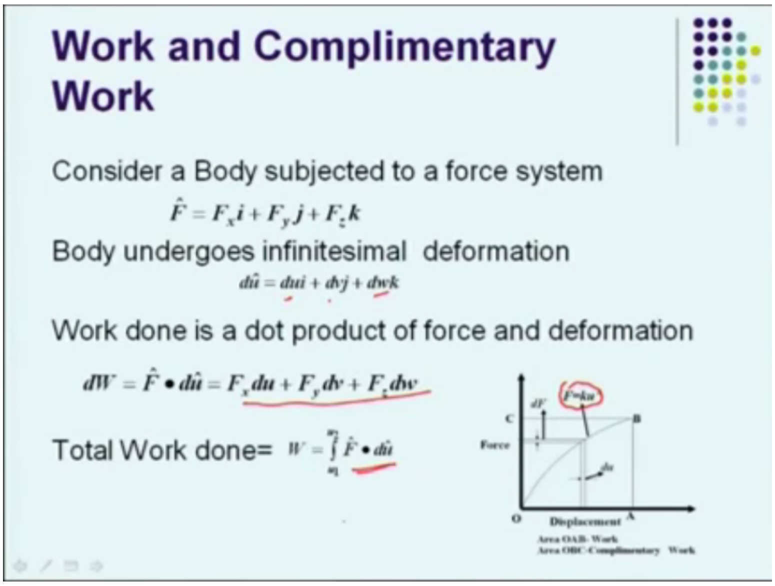

Let us come into the next level. Let us understand because the fundamental objective of this lecture is to bring in the energy theorem so let us actually explain what does energy and work mean? Suppose you consider body under action of some forces and the force vector will have components in both x, y and z direction, which is called F xi, F yj and F zk where i, j, k are the unit vectors in the 3 coordinate direction.

If the body undergoes small deformation, which I call du which has 3 components again du in x direction, v in y direction and w in z direction and the work done is nothing but force into displacement. So in terms of vector, it is a dot product of F*du, which is given by this. So this is the infinite decimal work done what we have based on, total work done is integral between the displacement u1 to u2 the dot product of that.

So if you plot this in a pictorial view, if you have the force displacement relation given by OB here the area under OB is what the work done. So we can actually fit in the curve as force is related to displacement by this relation F=k*u where k is some proportionality constant. So basically it amounts to taking a small strip within this region and integrating this. So this basic expression is basically says the area under OAB.

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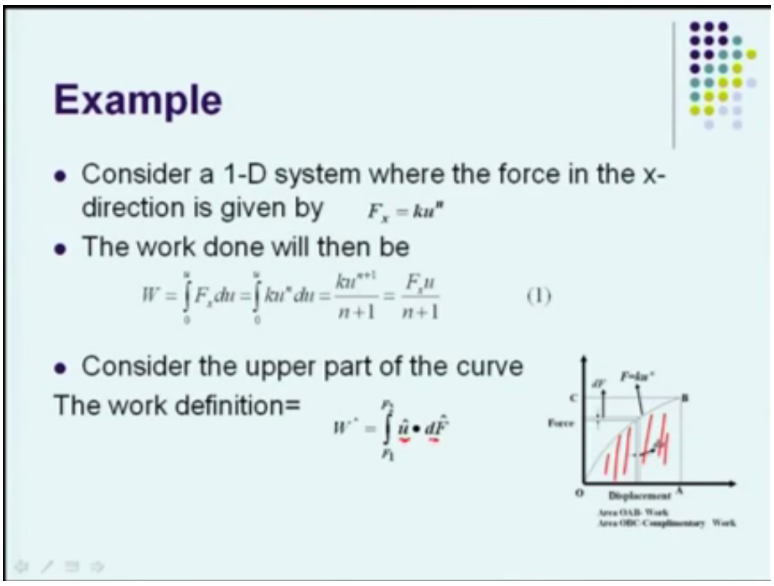

So let us take a system, suppose you have only 1-D state of system where the force Fx is related to k*u to the power of n and now we write this equation by using this definition and substitute for Fx k*un and integrate it we get this. So the total work done depends upon the exponent n. If n=1 that is if you have a linear variation, then the area under this curve is just a triangle, but if n is other than 1, it has a complex variation and it depends upon the value of n of total work done.

On the other hand, what we have done right now is we have taken only the area under this. Suppose we take the area over this, we can again write the work in a totally different notation not in terms of the variation in terms of displacement, but in the variation in terms of forces. So we can have u*dF. So we take a small strip here that is of dF force and integrate between the two force level F1 and F2.

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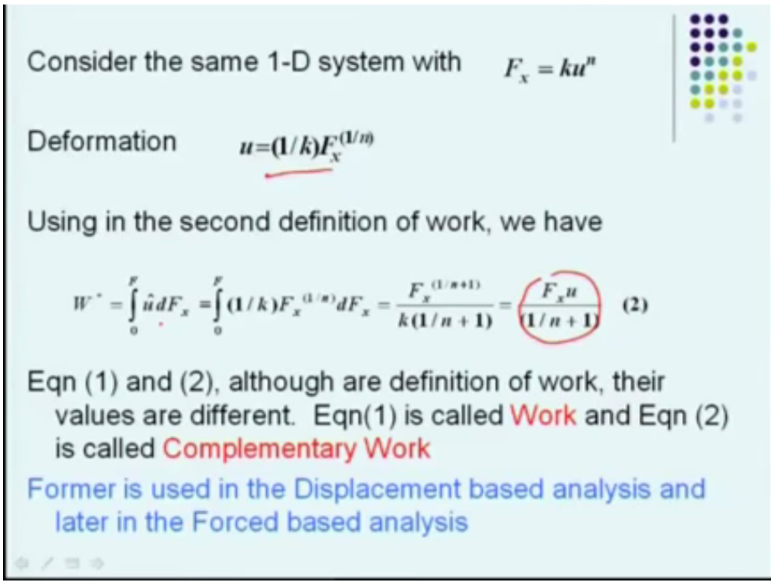

So what do we get here? So we will consider the same Fx*k*un. Since we are varying F so we have to express u in terms of F so we will rewrite this equation in this form. When we do that and we plug in this equation into this equation and integrate between starting 0 to F, we will get totally a new equation, which can be written in this form. Note that when n=1, the W star based on force = the W which is based on displacement whenever the force relation is linear.

When the force relation is nonlinear, these 2 values W and W star are totally different. So what do we conclude from here? So equation 1 and 2 although are the definition of work their values are different. Equation 1 is called work whereas the equation 2 that is this is called complementary work. So we use W for the displacement based analysis and we use W star for the forced based analysis.

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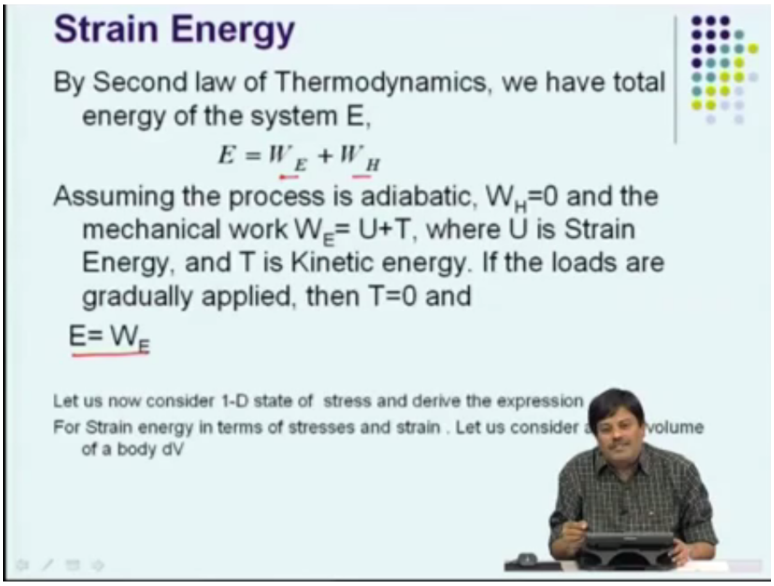

So let us come to the second important aspect of the energy theorems that is the strain energy. So we all know the second law of thermodynamics, which says that the total energy E = the energy due to the mechanical work that is WE and the heat dissipated due to the performance of work that is WH. In most of our analysis, we consider that our thermal process is essentially adiabatic that means we assume that WH is 0.

And in addition we also assume that the mechanical work is done by forces, which are gradually applied not suddenly applied. So those forces contributed due to inertia that is the kinetic energy is 0, so the total energy E = the energy done by the mechanical work. So we start with this premise and we continue our analysis. Now let us consider a 1-D state of stress and try to relate this energy in terms of the stresses and strains.

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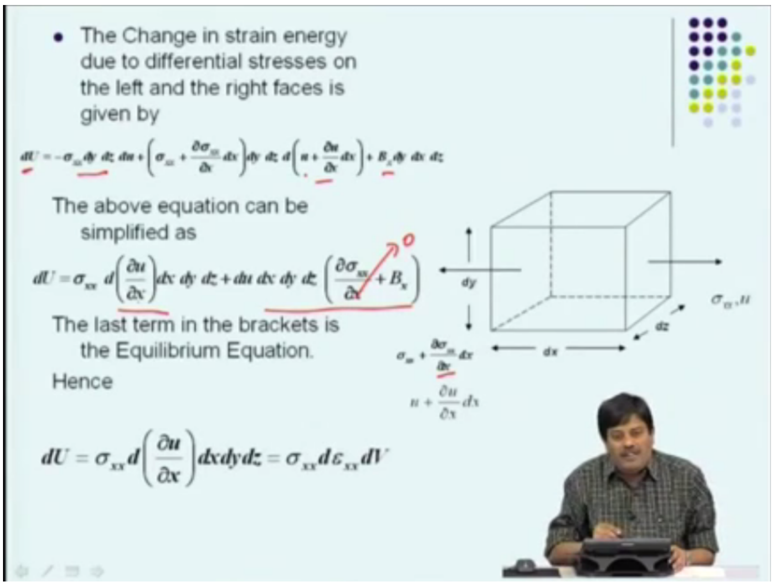

So here we take a body, a small representative volume of the body of length dx, dy and dz and write the free body of the stresses. So we have the stress on the right phase sigma x and the displacement is u whereas on the left phase we have an incremental so this is not same, it is changing and it is changing by a small factor here that is d sigma x/dx*dx and similarly the displacement is also changing from u+du/dx*dx. So this is the incremental change between the right phase and the left phase.

Now let us find the total work done by these stresses on both the left phase and may show what is the difference and these difference in work what we call is the change in strain energy dU. So on the right phase, we have stress multiply the area of the cross-section which is dz*dy, which is here multiply by dU. On the right side, we have sigma xx, this component of stress multiplied by dy and dz again and multiplied by the change in the displacement which is given by this.

We also assume there is a body force B and this body force has the unit of force per unit volume. So we take this body force multiplied by the volume of the element, which is dx, dy and dz and we simplify this. We throw out all the higher order terms and when we do that we get this quantity okay. We group this so this is got by simplifying this expression and if you look at this if the body is in the 1-D state of stress this is nothing, but the equilibrium equation so this goes to 0.

This is not present so only body within the expression this exist and if you look at it du/dx from our strain displacement relationship is nothing but epsilon x so which can be written as du=sigma x*d epsilon xx.

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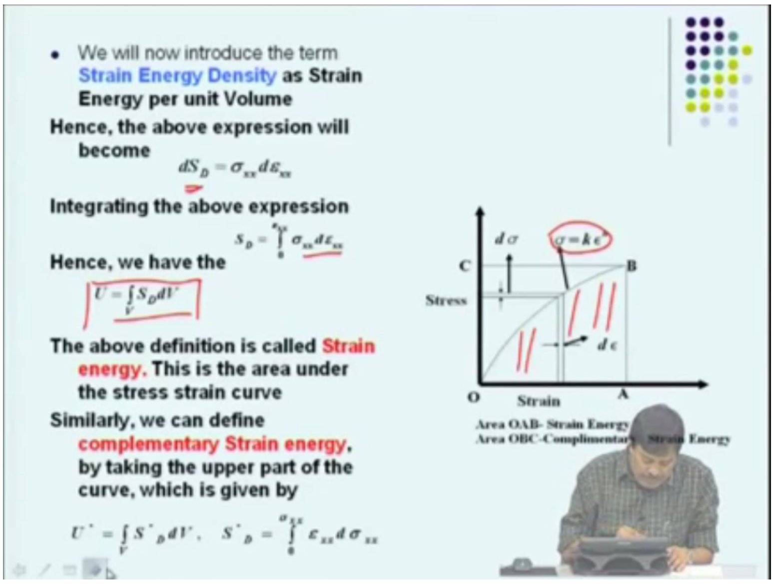

Now we introduce the concept of what is called strain energy density, which is nothing but strain energy per unit volume. Hence from the above expression dU, we can say which = the change in the strain energy density = sigma xx*d*epsilon xx. So the total strain energy density is nothing but integral 0 to epsilon xx sigma xx*d epsilon xx.

And once we determine the strain energy density, we can write the total strain energy = SD*dv. So this is the definition of the strain energy for which we have to integrate the strain energy density over the volume and strain energy density is a function of the stresses and the differential strains and so we need to integrate this for 2 strain levels. So the total strain energy is defined by this equation.

So drawing the analogy between the force and the work we can again draw the analogy between stress and the strain. If stress and strain is related by this and this is the relation that it holds good, obviously there maybe if epsilon=1 we have a linear relationship, which is the constitutive model, which we talked just few minutes before and now if you take this and plot it.

And the area under this curve is basically the strain energy and the area under the curve above this is called the complementary strain energy and one is used for displacement based analysis and other is used for forced based analysis.

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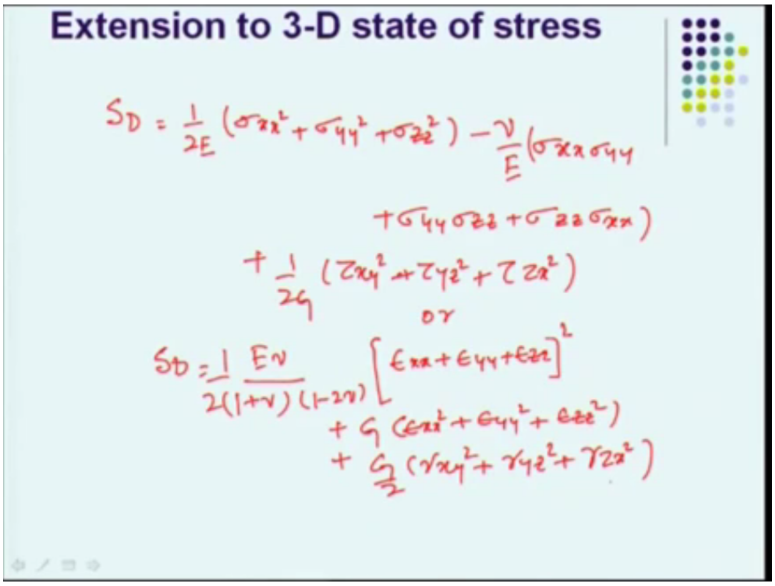

So what we have derived is only for the 1-D state of stress. So we can also extend this by using the constitutive model and I will just write one expression for the strain energy density for a 3-D state of stress would be = 1/2E into we have sigma xx square+sigma yy square+sigma z square-nu/E where nu is nothing but the Poisson's ratio into sigma xx sigma yy+sigma yy sigma zz+sigma zz sigma xx multiplied by 1/2G*tau xy square+tau yz square+tau zx square.

You can also express this in terms of strains by using the constitutive model. We can say SD=E times nu/1+nu*1-2nu/1/2 you will have epsilon xx+epsilon yy+epsilon zz whole square+G*epsilon xx square+epsilon yy square+epsilon z square+G/2*gamma xy square+gamma yz square+gamma zx square. So this if you put the expression of the constitutive relation in the strain energy density equation you can get this.

So you can extend it for that. We for the sake of simplicity and understanding we derived for 1-D state of stress and it can be extended.

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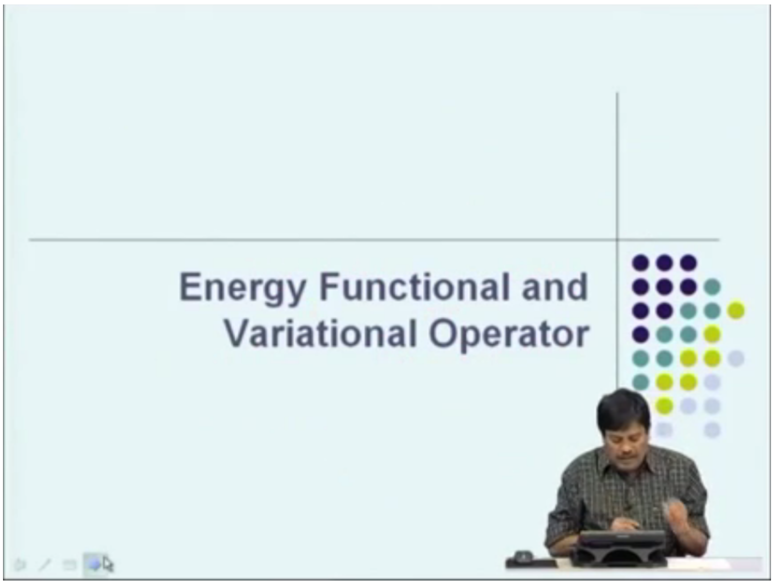

Now we will come to some of the energy functional definition of it. We talked about it little bit in the last lecture. We will go a little more detail in this lecture and see what it means?

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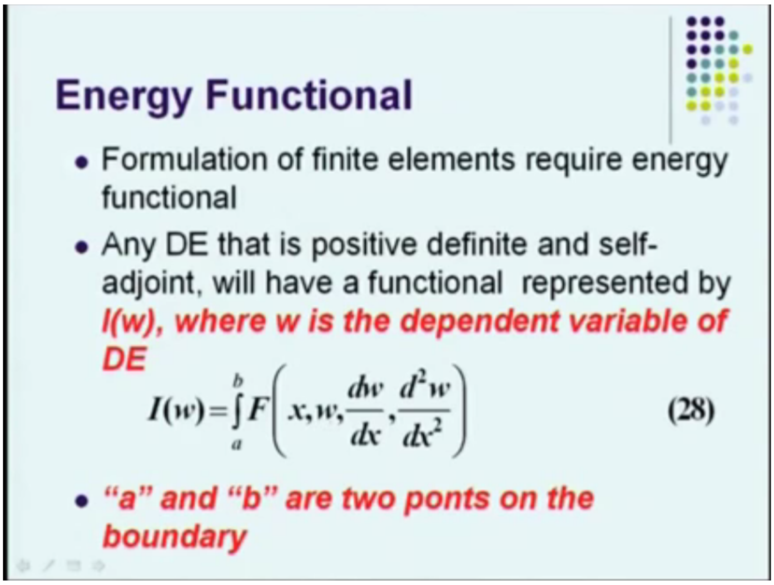

So what is energy functional? The formulation of finite elements requires energy functional. So the energy functional is basically has a broad definition if you take a calculus of variation this is the broad definition and it depends upon the spatial variable or the independent variable, the dependent variable w its gradient with respect to x the second order gradient and so on and where the domain a and b are the two points on the domain. So let us see that how does it translates to some of our conventional structure like beam.

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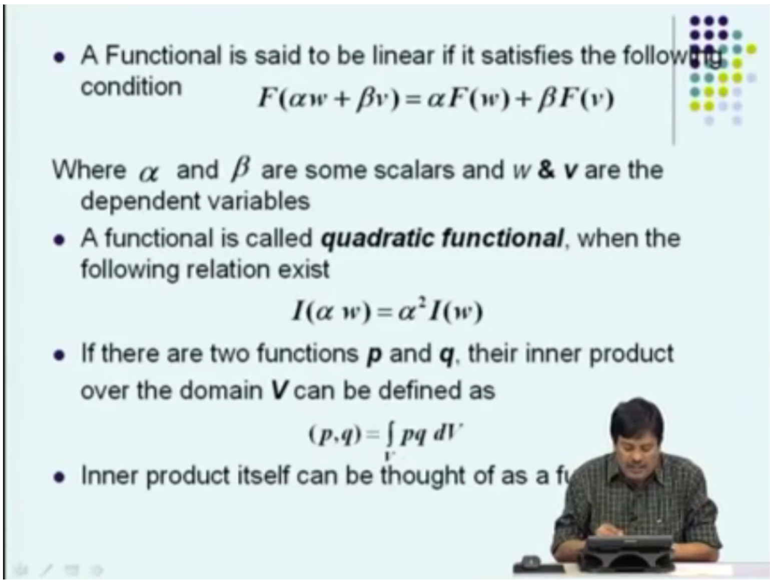

So before going into it let us talk about some of the properties of this functional. The functional is said to be linear if it satisfies the condition. That is if you have a functional alpha w+beta v where w and v are some two functions or the dependent variables. Then if we can write this as alpha*F of w+beta*F of v then we call that such a functional is linear.

A functional is called a quadratic functional when this functional alpha*w where w is the dependent variable and alpha is some constant can be written as alpha square*I of w. What are the key things that we need is in the functional analysis? How do we define an inner product of two functions p and q over the domain V? So the inner product is nothing but the integral of p times q*dV integrated over the domain. So in fact the inner product itself can be thought of as a functional.

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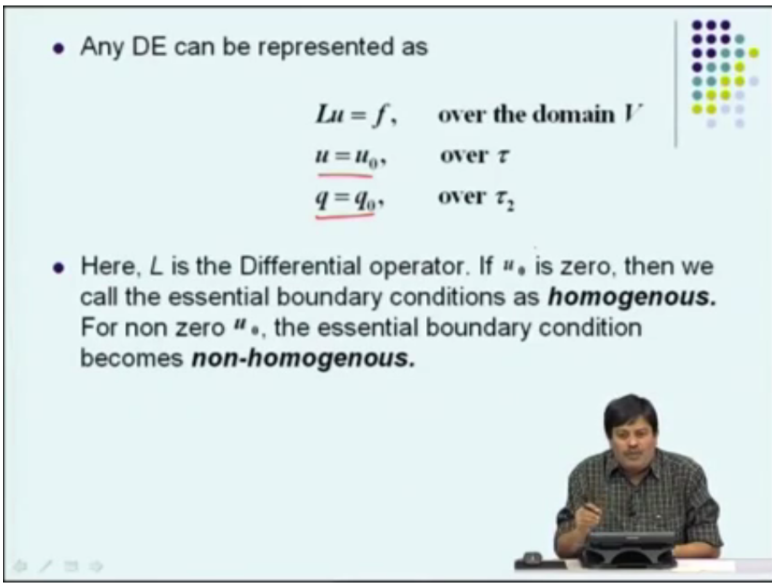

So now having defined this functional when do we say that the differential equation is completely well posed problem once we have this following equations to satisfy. That is if Lu=f where L is a differential operator defined over a domain V and it has two surfaces tau where in one surface the Dirichlet boundary condition is specified and in other surface where the traction boundary conditions that is the Neumann boundary condition is specified.

And L is a differential operator, for a beam it is d4 w, d4/dx to the power of 4, for a rod it is d square/dx square and for many other things it could be different and u naught is the specified value over a domain tau. If u naught is 0, we call it as a homogenous boundary condition, if u naught is non-zero it is called non-homogenous conditions. So that is basically the way you solve is totally different.

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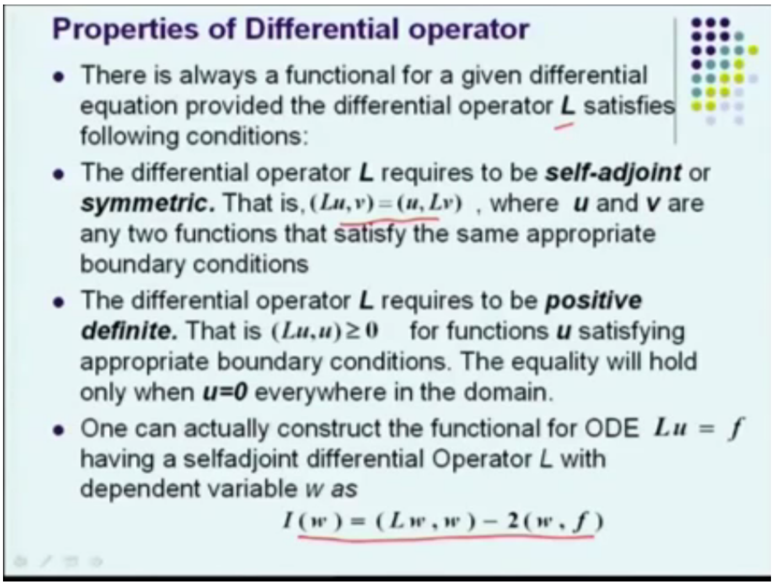

So what are the properties of the differential operator? There is always a functional for a given differential equation provided the differential operator L satisfies the following condition that is the differential operator L requires to be self-adjoint for u to have a functional. What do you mean by self-adjoint? If you take an inner product of 2 functions u and v where u and v satisfy the same boundary condition.

Then Lu v must be = u*Lv. So if this condition satisfied then the functional will exist and such a condition is called self-adjoint or symmetric property. Other thing is the inner product of the function with itself should be positive definite that is the Lu times u should be greater than 0. This is another condition for the existence of the functional for a given differential equation.

If these two conditions are satisfied then we can use this expression in order to get the functional that is if Lu = f where f is the forcing function then the energy functional what we call is Lw times w-2w*f. This is the total expression that gives you the functional. We will actually see this for a beam as we go along.

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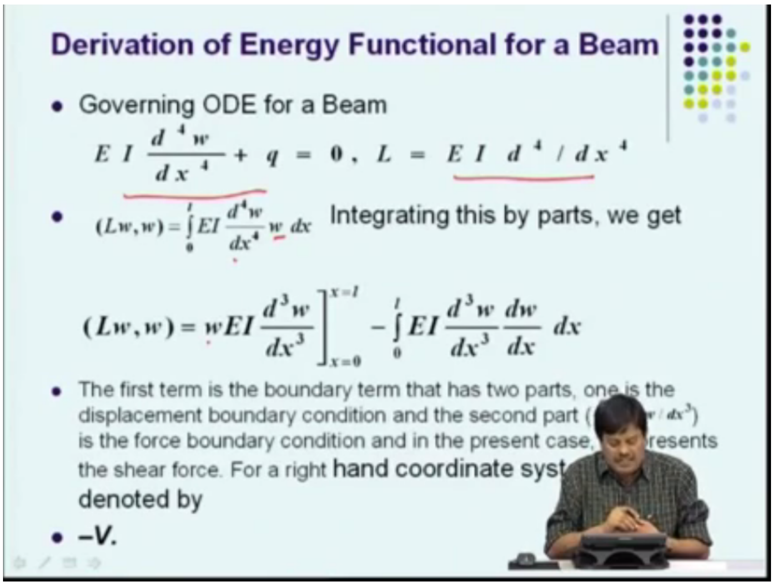

So now let us derive the governing equation for a beam. So we will do things, first we know the governing equation, if we know the governing equation we will see how we can derive the energy functional and the second part we will say if we know the energy functional how will it come back to the governing equation. Both we will do taking beam as an example. So here this is the governing equation and the differential operator is given by this.

That is the fourth order operator here. So now we will take an inner product that is Lw times w which is nothing but EI d4 w/d*w and we will integrate by parts. We have done this in the last class. We will do again here. So when we do this so this is my first function and this is my second function. So the integral is integral of first function into integral of second function is EI d cube w/dx cube evaluated at x=0 and L which is our domain length- the integral of second function that is EI d cube w/dx cube into differential of the first function.

If you look at this boundary terms, the first is the basically the deformation w transverse deformation and second is the force. As I said it always comes in pairs the essential and the Neumann boundary conditions. So w causes the shear force and shear force causes w so it is like cause and effect. We talked about it in the last lecture so similarly this is basically evaluated w of 0, shear force of 0 and w of L and shear force of L.

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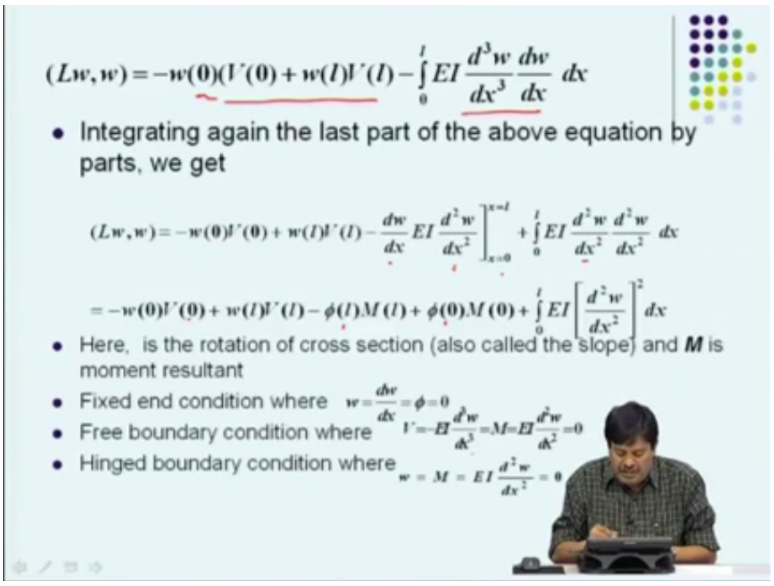

So we can write this expression and again we have this. We again use integration by parts for this portion. When we do that we take this as a first function and this is second function. So we say first function into integral of second function is EI d square w/dx square evaluated at x=0 and L plus we have this term. So if you look at this, this term is the slope that is the rotation of the cross-section for a beam and which is caused by the movement EI d square w/dx square.

So the slope causes moment and moment causes slope. So they are cause and effect as we talked about in the last class. So if you expand this, this is the moment at L, slope at L, slope at 0, moment at 0. So when we put these things together we have this and this is nothing but d square w/dx square whole square okay and if you look at this for any beam either this is 0 or this is 0 so this does not exist.

But this process will tell us what form the natural boundary condition takes which otherwise would not have been possible. So for all these are the probable condition. If the beam is fixed, we have slope is 0, displacement is 0 and if it is a free boundary, we have either the shear force 0 and the moment 0. If it is a hinge boundary condition, we have the displacement and moment is 0.

So either in case either one of them goes to 0 so that the boundary conditions are exactly satisfied and this term becomes the energy functional.

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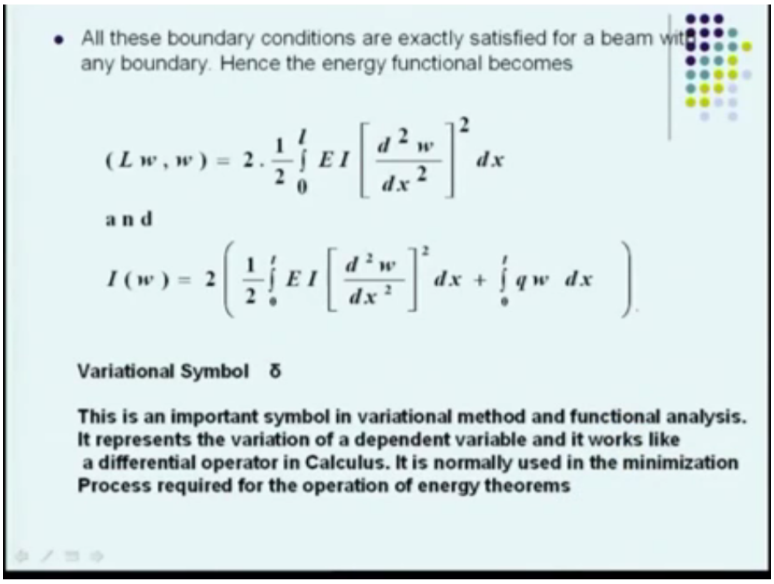

And we have derived the expression that the other way is to get from the calculus of variation. We have shown that Lw is nothing but 2 into this one and the total functional can become this which is exactly what we specified before in the few sections back. So basically what we are saying is if there is a differential equation which is self-adjoint and positive definite, there is a functional which we can use for formulating our energy theorems and also the finite element methods.

Now let us go to the important symbol, the variational symbol delta and the delta is basically saying how the particular variable varies over the domain and it acts more like a differential operator in calculus. We will be using this operator more and more as we go along.

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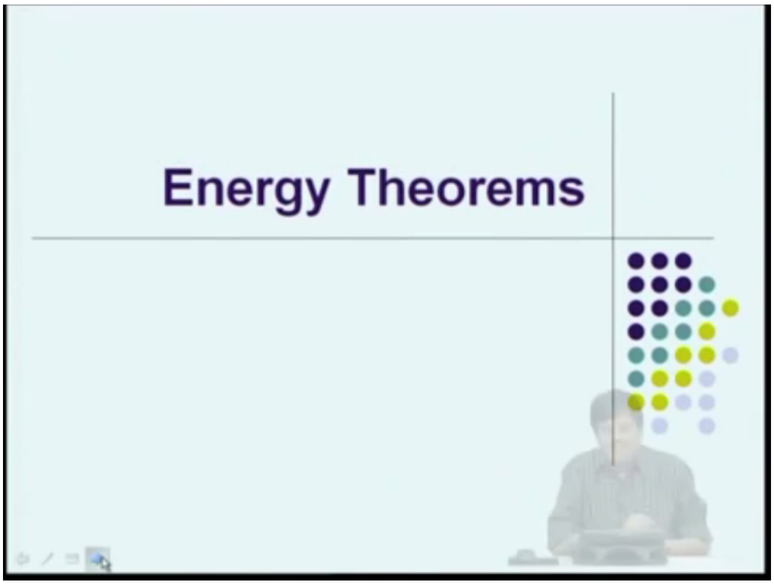

So now come back to energy theorems. So one of the principle things that we are talking about in the energy theorem is the principle of virtual work, which is the heart of the FEM.

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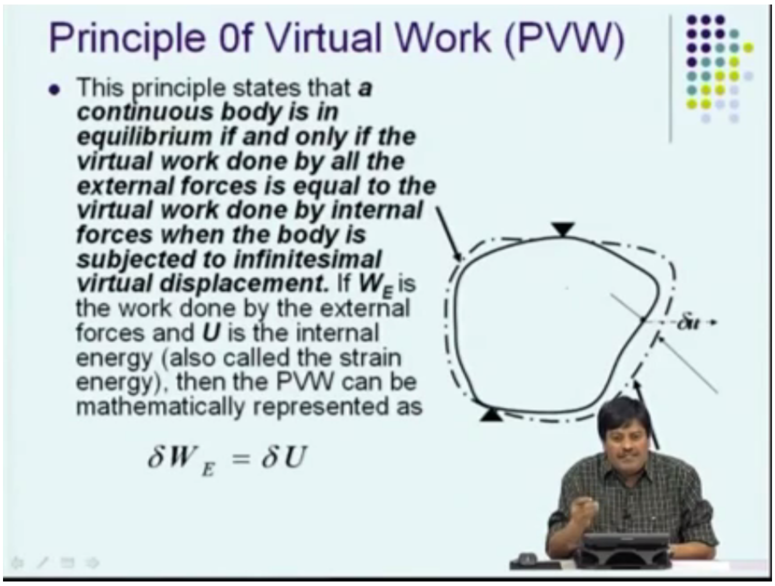

We talked about the principle of the total potential energy in the last class and that is derived basically from this equation. So what does it say? If you have a body having certain constraints and if you give a virtual infinitesimal displacement to certain parts of the body and the body deforms in a manner where the boundary conditions are satisfied then if that is the case, the total virtual energy due to external force = the virtual strain energy. This is the heart of the FEM principle so let us actually derive this.

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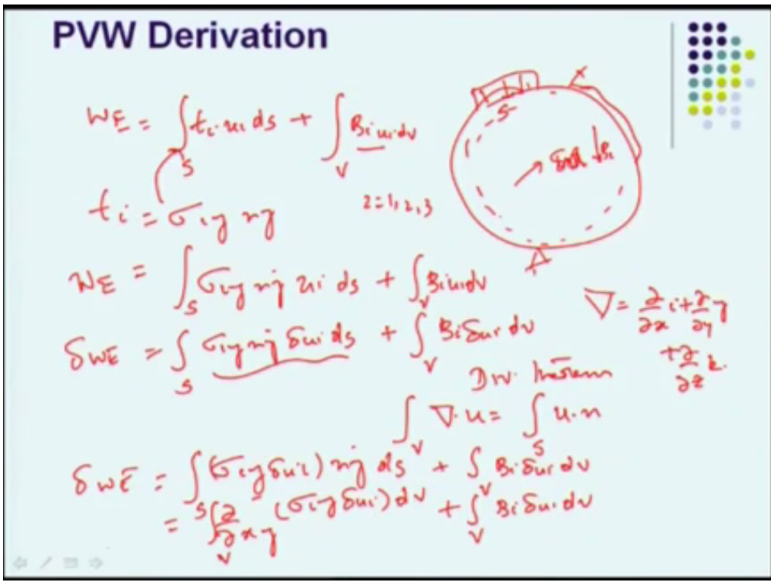

So again I will take a body here and if you have this boundary constraints and you give a virtual displacement some delta u to some position here and you have this body tractions all over acting and this deforms in a manner so that the kinematic constraints are not violated okay. Then we have that the total energy WE will be equal to integral over the surface I call this is the surface where the traction is acting ti*ui, it has 3 components into ds over the surface and if bi is the body force per unit volume, this is over the volume bi*ui dv okay.

So essentially what we are saying is this is the work done by body force. So here i=1, 2, and 3 okay corresponding to 3 coordinate directions. So we said that my ti is related to stress sigma ij*nj initially. We substitute this here and we get this, this is equal to sigma ij nj*ui*ds+integral bi*ui dv. Now this is WE. We take a variation of this as I said which is equal to this is over the surface we have over the surface we have sigma ij nj delta ui because we are varying only the displacement not the stresses into ds+integral over the volume which is bi*delta ui*dv.

Now if you want to proceed further this is the surface integral that needs to be transferred either into a volume integral or this volume integral has to be transferred into surface integral where now we use the divergence theorem, which is most theorem in vector calculus which says that integral over the volume del dot u=u dot n of the surface, where your del is nothing but it is a gradient vector, d/dx of i+d/dy of j+d/dz of k where i, j, k is the unit vector.

Now we use this theorem when we use this so we can write delta WE will be equal to we can say that sigma ij*delta ui*nj ds+integral bi*delta ui*dv. So this is in this form so this I can write it as essentially d/dx j this is over the surface, this can be written over the volume d/dx j*sigma ij*delta ui dv. So this is converted into a volume integral and into v*bi*delta ui dv.

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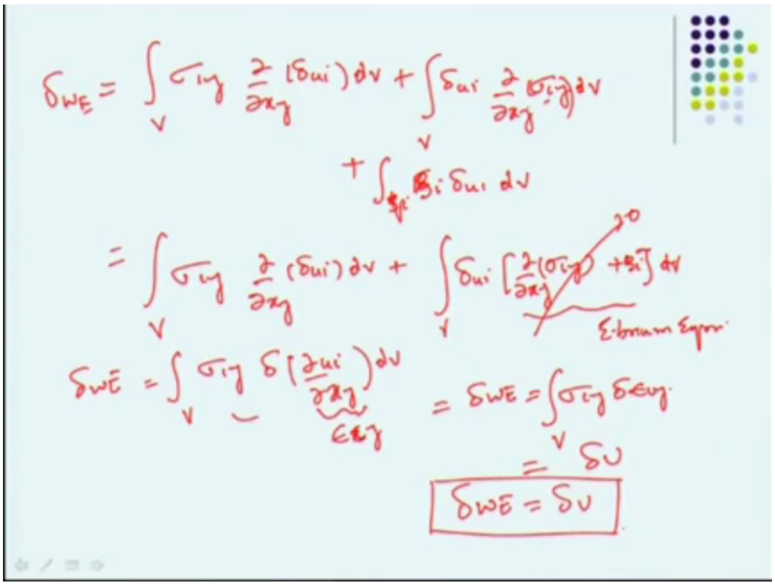

Now I can differentiate this with respect to space and group it together. So when I do that I can write it as integral over the volume, I will have sigma ij*d/dx j*delta ui*dv+integral of volume*delta ui*d/dx j of sigma ij*dv+integral of bi*delta ui. So now what we do here is we club these things together. So this is delta WE so we club these things together and we say sigma ij*d/dx j of delta ui dv+integral over the volume delta ui into we have d/dx j of sigma ij+bi*dv.

So essentially what happens is this is the equilibrium equation, which is equal to 0. So we scratch this out to 0 we are left only with this so delta WE will be equal to integral over the volume which is sigma ij into I can modify a little bit I can take delta outside and I can say this is du i/dx j and we know that this is the definition du i/dx j is nothing but my epsilon ij that is the strain displacement relationship.

So we can write delta WE = integral over the volume sigma ij*delta of epsilon ij which is nothing but delta u. So we have proved that delta WE = delta u. So this is fundamental principle by which we need to do this.

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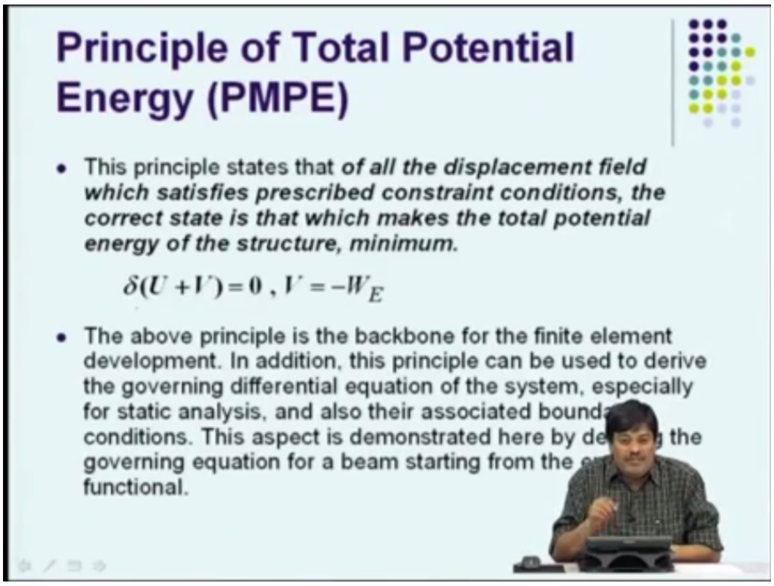

So now we talked about the total potential energy in the last time and we derived that we said that finite elements will be basically derived from this equation. So we know that delta U=delta W and if you designate delta W as V, negative of the V=negative of the total work done by the external forces. We call it as potential of forces then delta U+V=0 which is U+V is the total energy pi we talked about and that is how we get the theorem of minimum potential energy, which we use to develop the theoretical basis in the last lecture.

So now we see that where this theorem is coming from? It is coming from principle of virtual work. So basically this principle says that we cannot solve the governing equilibrium equations exactly, we are trying to find an approximate solution and an alternate statement of equilibrium is total minimum potential energy. If this condition if the total variation of the energy = 0, we say that the body is in equilibrium so it is an alternate statement of equilibrium that we will use for this condition.

So that is why this is the backbone of the finite element method that we will be talking about or we talked about little before.

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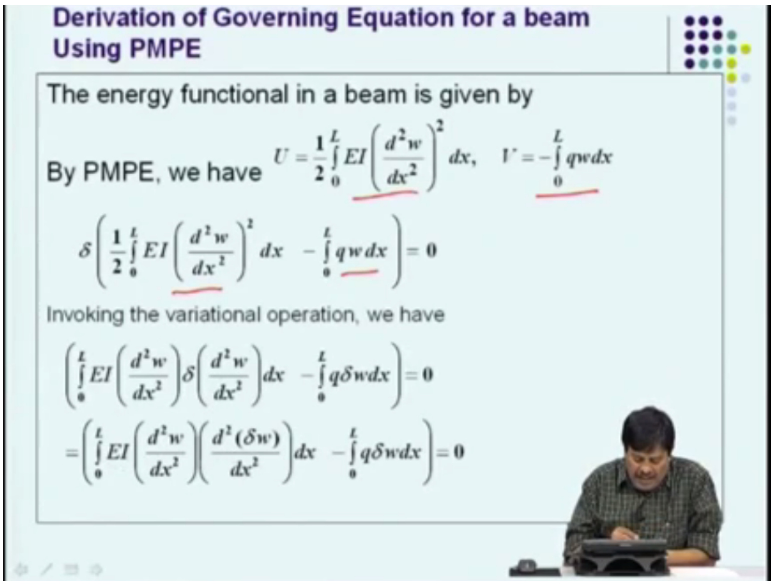

So let us use now the inverse way, suppose we know the energy functional, can we construct a differential equation? and that is where we use the principle of minimum potential energy. So we take the same beam we know this is the energy function, we derived it and this is the potential of external forces because of the uniformly distributed load and the total work done is the negative of the work done by these forces.

So the total energy is the energy due to the internal forces and the energy due to the tractions that is the distributed force we take a variation of this. When we take a variation of this, we have EI into this two gets cancelled because of differentiation we have d square w/dx square into variation of this portion minus into variation of delta w because we are only varying the displacements in the finite element methods.

In the forced based method, will vary forces invoking the variational operation we say we have this so I can tune it I can take this differential out and put the variational operator here by this way.

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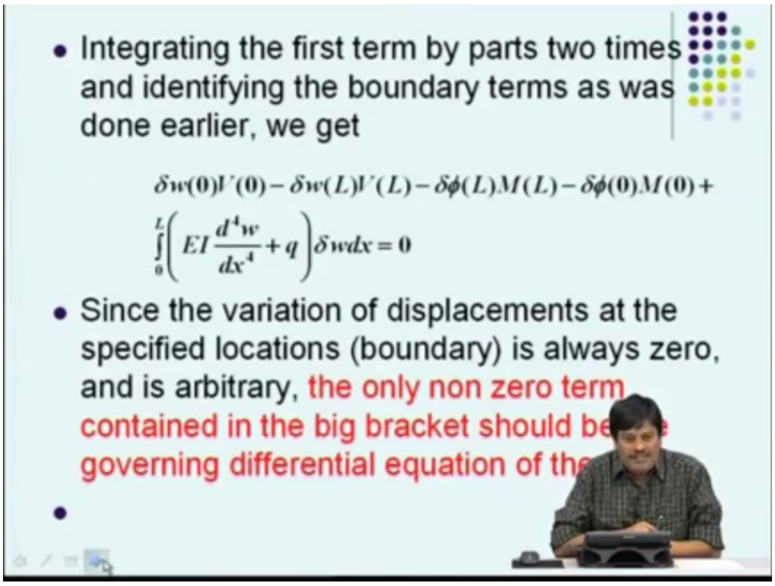

Then we again integrate part by part, we do not want to do that we have done this so we have got this equation, which we will writing in terms of the moment and the shear force, slope and the transverse displacement we get this and we said that for any beam cross section, these terms will be 0. So what is left is only here and since delta w is arbitrary it cannot go to 0, the term inside the bracket should go to 0 and that becomes your equilibrium equations.

So you can derive the energy functional from the governing equation and you can derive the governing equation from the energy functional. We have shown both we can use this variational principles to do that.

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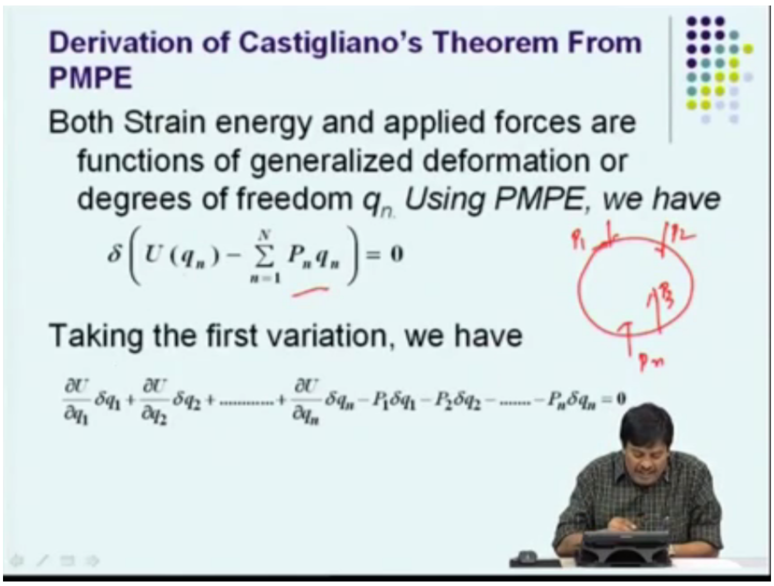

We can also do much more. In many structural mechanics problems, we have used what is called Castigliano's theorem. So Castigliano's theorem is the very famous equations for many civil engineers to solve many of their problems. We can derive this from the principle of minimum potential energy. So what we will do? So basically if there are a body subjected to n concentrated loads, which we call P1, P2, P3 and Pn okay and each of this is dependent upon the displacement qn, then we say that strain energy is a function of qn and according to the theorem of minimum potential energy the total work done by the external forces is –V so we have put this in this form here.

And since strain energy is a function of this qn, we expand this qn by using a chain rule which is du/dq1*delta q1, du/dq2*q2 etc and we expand this in this form.

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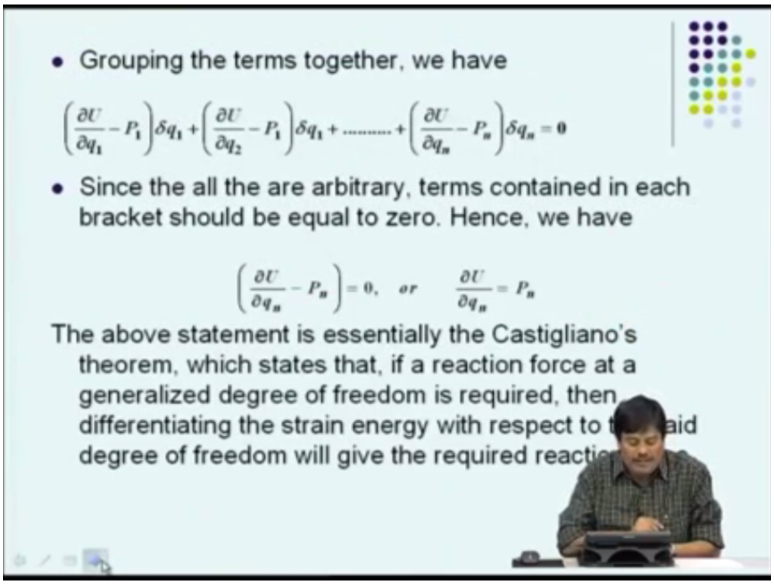

So now we group all q1s, q2s, q3s. When we do that, we get this and in the tensorial form we can say du/dqn-pn=0 or du/dqn is pn. This is the famous Castigliano's theorem, which says that if I want to find out a load at a particular point, you take the total strain energy and differentiate with respect to the displacement at that point.

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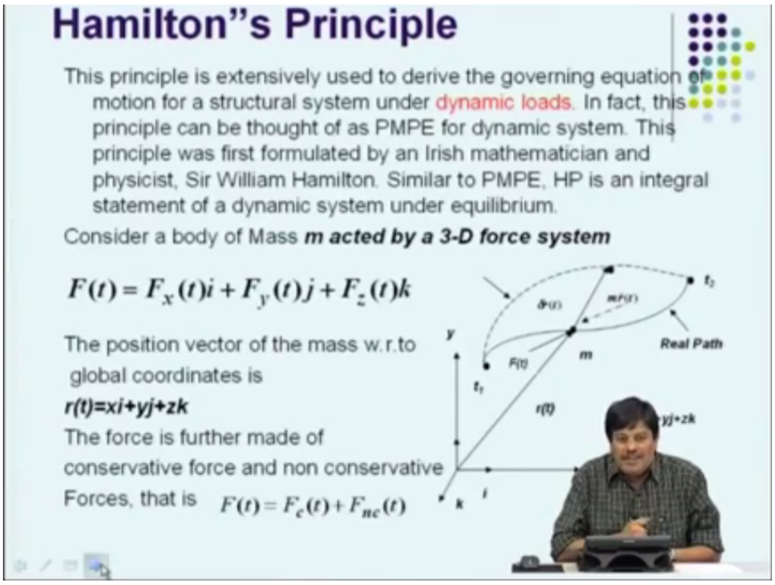

So next important principle what we want to discuss here is the Hamilton's principle. What is this Hamilton's principle? It is nothing but it is the principle of minimum potential energy under dynamic systems. So in the principle potential energy, we derived based on the assumptions that the loads are gradually applied. When the loads are suddenly applied at over a short duration, the dynamic effects will come into picture.

So to handle this it was way back in 1800, Sir William Hamilton propounded this theorem and which has a very fine theoretical basis. We would try to derive this equation by using principle of minimum potential energy and see how we can do this. So consider a mass here which is acted upon by a force F which has 3 components Fxi, Fyj and Fzk, all are time dependent.

And let this mass be located at a position vector r which has a coordinate xi, yj and zk and these forces can be further divided into the conservative force and the non-conservative force. The non-conservative force are because of damping or any other force that is not generated due to the loading because of the phenomenon of the material per se and the conservative forces are those that are caused by the generation of internal forces because of the external loading.

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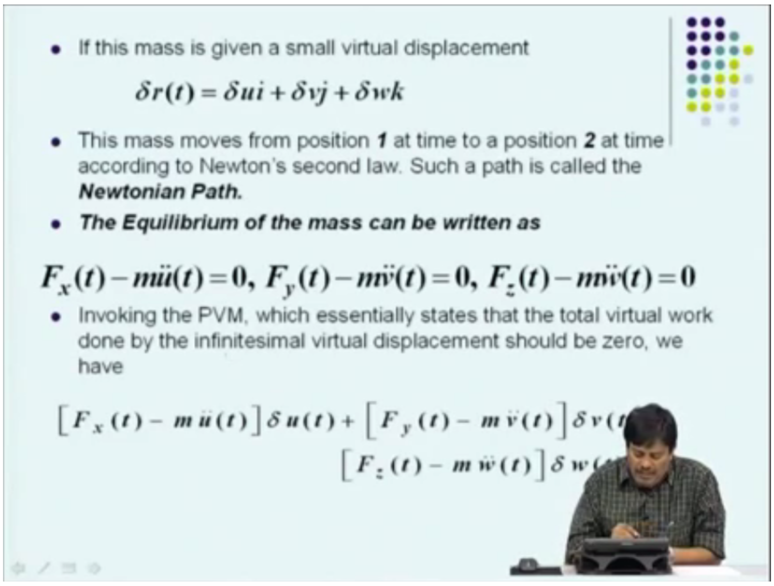

Now if this mass is given a small virtual displacement, then the position vectors become delta ui, vj and wk okay. So when doing so, the mass moves from position 1 at time t1 to position 2 at time t2 according to the Newton second law. So if we do that and such a path is called the Newtonian path. If you write the equilibrium of mass in these 3 directions, we have Fx = by Newton second law F = mass times acceleration, u is the displacement in x so u double dot is the acceleration in x direction, v double dot in y direction and w double dot in the z direction.

So the Fx has to balance each of these. Now we invoke the principle of virtual work so we multiply this with the virtual displacement delta u for the x component, delta v for y component and delta w for z component.

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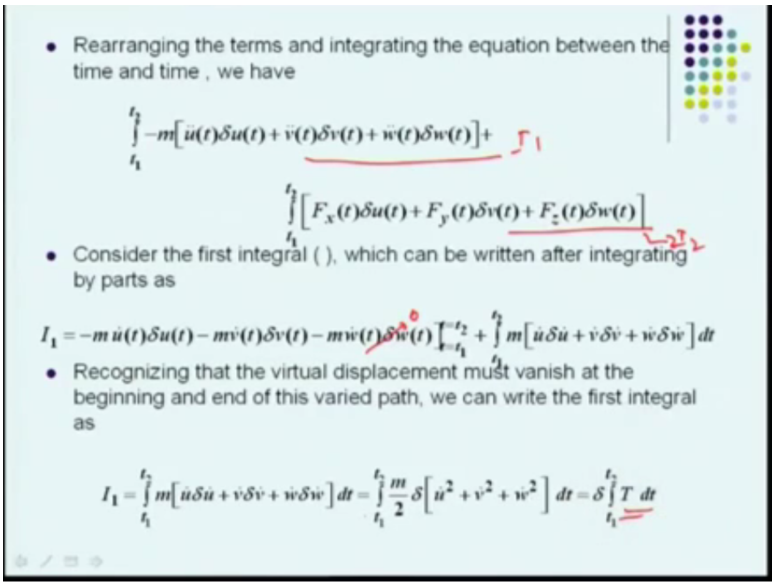

And we rearrange the term by this form, all the inertial components in one region and the forced components in another and we integrate during time t1 and t2. So here u double dot, we use integration by parts we call this integral I1 and this is I2 so when we do that the I1 will basically become first function and integral of the second function, which is u dot like that for 3 and you have u dot and u double dot that is there by integrating by parts.

And if you look at it delta w at each of this point t1 and t2 should go to 0 otherwise the existence of the conservation laws is not valid, so this goes to 0 here so this is left by this. So only this portion is left. When we actually do that we get this one so this is basically the m/2*delta of u dot square v dot square and w dot square so mass into velocity is basically the kinetic energy so we get delta of t*w dot.

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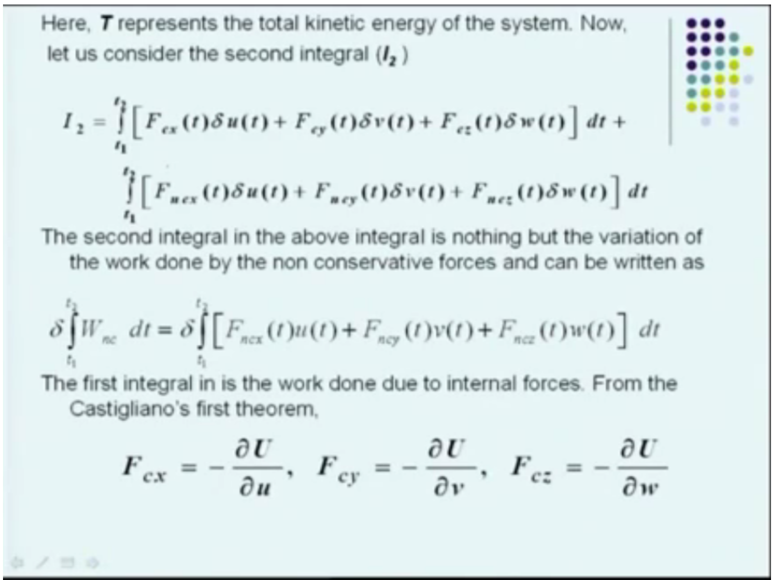

So we take the next portion of it I2, which can be split up into both conservative and non-conservative and each has x, y and z component and we can write it in this form okay. So the second integral here is a non-conservative that is basically delta of non-conservative that is the work done by the non-conservative forces, which we can write it as W nc, but we invoke now the Castigliano's theorem.

The conservative forces are related to delta U/u in the x direction, y direction and z direction. So when we plug this, this portion is nothing but delta U.

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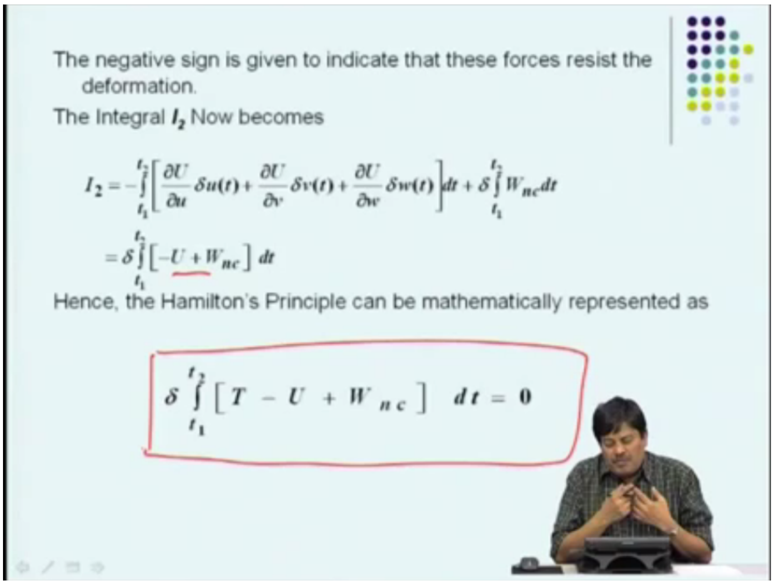

So if you do this so we get the second integral becomes this and the overall Hamilton's principle is given by this. So this is the principle, which we need to use for the cases where we are using the dynamic equilibrium. So it is essentially a principle of minimum potential energy type principle under the dynamic conditions. So when the t is 0 and w is 0, this basically becomes the theorem of minimum potential energy.

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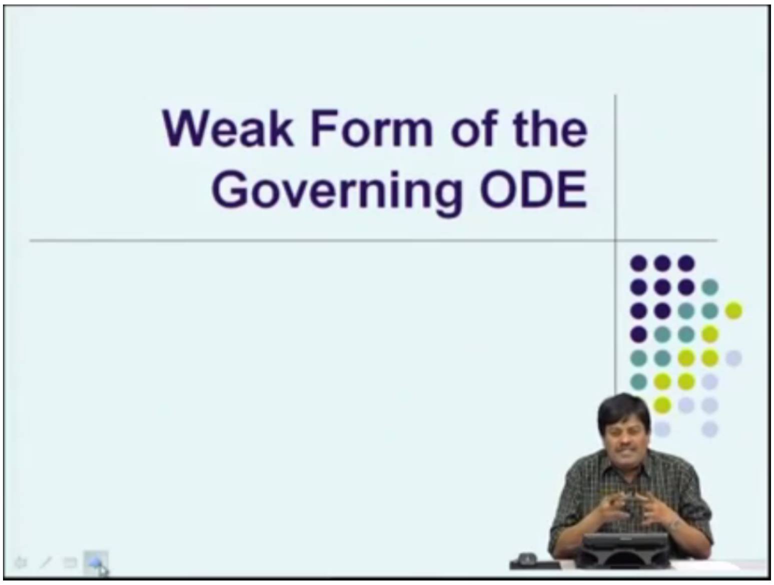

So now let us see how we can derive the weak form of the governing equation. We just dealt with the weak form of governing equation for a general system in the previous classes for a general differential equation. Here we will again take this particular example of a beam, which is a fourth order system and generate the weak form of the equations. So we said that we take the governing equation and weight it with the function and that function and integrate it.

When using we have to use integration by parts when doing so we will be getting a set of equation, which is called a weak form and which is a starting point for our FEM formulation.

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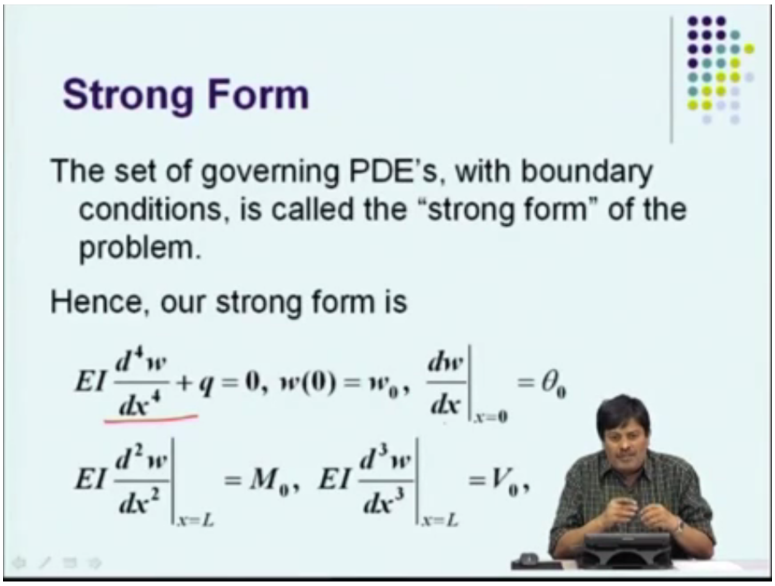

Before going to weak form what is a strong form? A strong form of a governing PDE is the one which is shown here and this governing equation is subjected to a set of the Dirichlet boundary condition and in this case for a beam, it is on slopes as well as displacements that is w and dw/dx and the Neumann boundary condition that is on movement and the shear force. So this is the strong form and you see that strong from is always order higher.

We are having a fourth order system here. So we can solve it exactly we get 4 constants and the 4 constants corresponds to the 4 possible boundary conditions that is 2 Neumann and 2 Dirichlet boundary conditions.

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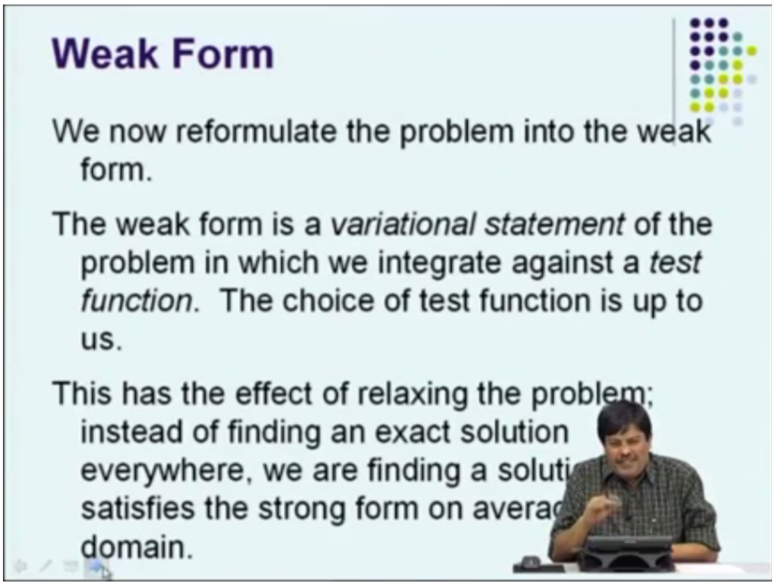

Now we reformulate the problem into a weak form. A weak form is a variational statement of a problem in which we integrate against a test function or a weight function under choice of weight function is up to us we discussed this in length in the last lecture. In doing so we are relaxing a problem a little bit instead of finding an exact solution we are trying to find a form of the equations, which is the governing equations or the equilibrium equation which is amenable for numerical solutions.

So in short we can say that a weak form satisfies the strong form in an average sense over the domain.

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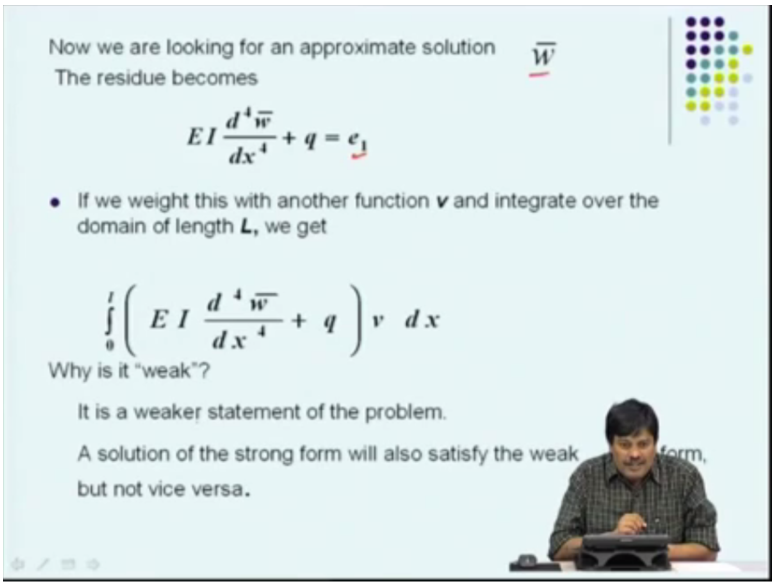

So now what we are looking is not w but an approximate solution W bar so when we substitute this into the equation we said that we get an error function, which we discussed in the last time and this error function needs to be weighted in order to find the weak form of the solution. Now why is it called weak? Is it a weaker statement of the problem? That we will come a little later once we find out the weak form of the solution but one thing is very clear that the solution of the strong form will also satisfy the weak form, but not vice versa that has to be very clear.

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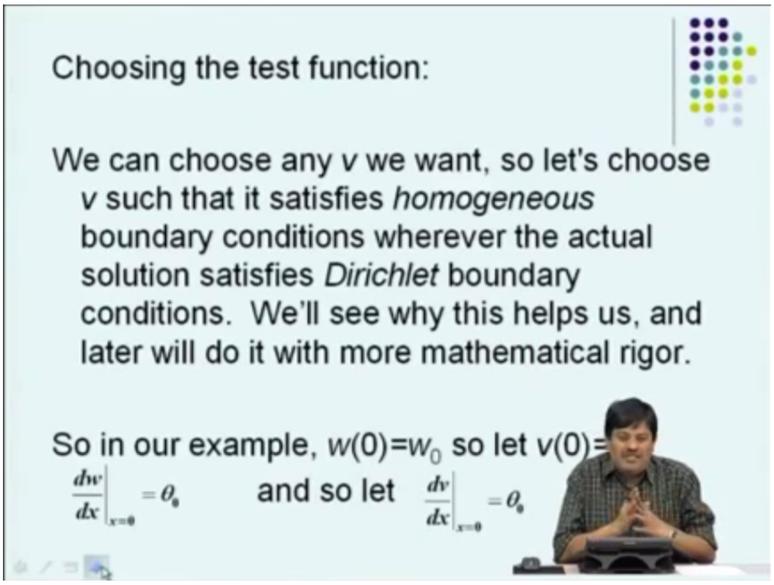

So choosing the test function, we can choose any function we have seen. For a typical differential equation we had chosen a whole bunch of functions in the last class and shown that the variations and the approximations are the accuracy of the solution varies with the choice of these weight functions. So let us choose some v here that also satisfies the homogenous boundary condition wherever the actual solution satisfies the Dirichlet boundary condition okay.

Suppose u=u naught=0 at some boundary the choice of that function when you substitute that coordinate should go to 0 at that function that is fundamental. So we will see why this helps us sometime later. So in our example we need to have w at x=0 and the v at x=0 and the slope at x=0 and the same slope for the test function v=0. They are exactly matching with each other.

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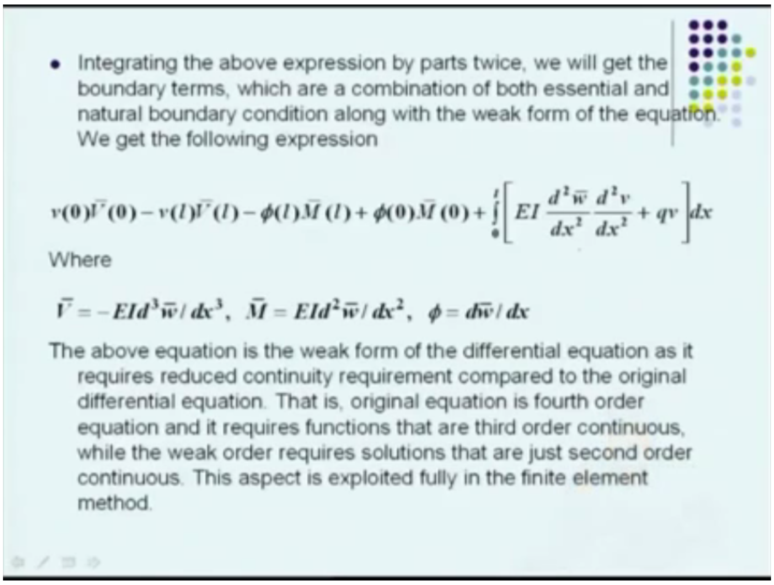

So we plug into the equation we again integrate by parts. We have done this many times we will not do this again here. So we will get a set of functions where V bar is the shear force, M bar is the moment at this location and you have a test function w and v here. So this is in general is the weak form of the solution and this is the starting point. Now to start the FEM what we should do?

We say that W bar=v when we do that this becomes the energy functional, which we will use it in the theorem of minimum potential energy and do that. If we do not want to do that if you want v different from W bar then it becomes a weighted residual technique. Then we come up with a new integration scheme and this is where the link between FEM and other methods are there.

All for everyone we need a mesh and how we define the test function defines what kind of method we have. When W bar equal to V bar, which is essentially the Galerkin method, so when we have such a thing then we are in a position to say we have either constructing FEM or we have constructing different method depending upon what we choose. So the above equation is called the weak form of the differential equation as it requires reduced continuity requirement let us say what it is?

In the original the highest derivative is d square w/dx square here whereas the strong form had d4 w/dx4 so any equations any assumptions we need we need to have at least the fourth order derivative should exist. We have a stronger continuity requirement for the strong from, here we do not have it, we have only a reduced order continuity is required d square w/dx square so whenever we choose the test function d square w/dx square should exist, which is relatively relaxed form, the restrictions on the governing differential is relaxed a bit and that is the advantage of the weak form over the strong from.

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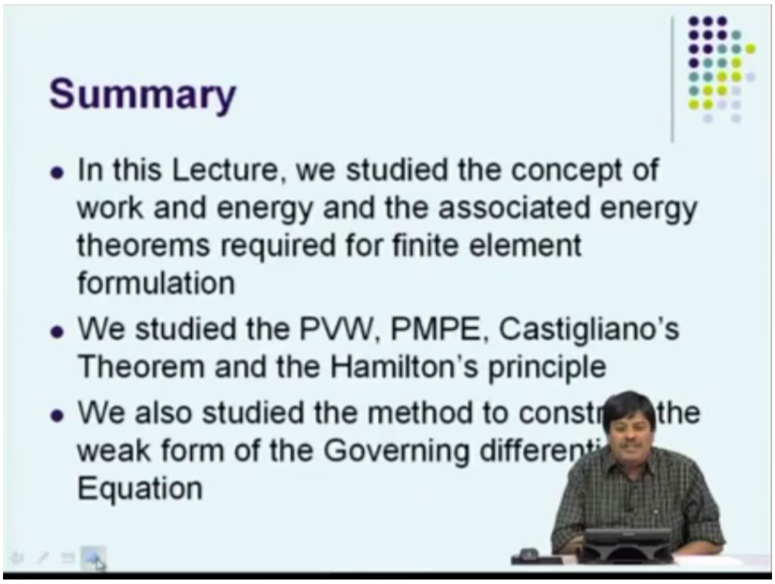

And this aspect is fully exploited in the finite element methods. So in summary what did we studied in this lecture? So in this lecture we studied basically we introduced the concept of elasticity we said that where are the 15 equations coming from. We use these equations of the elasticity to study the concept of work and energy and associated energy theorems required for the finite element formulation.

We studied in detail the principle of virtual work, principle of minimum potential energy, Castigliano's theorem, Hamilton's principle and said how they are linked up to FEM. We also studied the method to construct the weak form of the governing equation for a given differential equation. We also studied some aspects of the calculus of variations and how we can actually construct the functional?

What are the restrictions that differential operator should have in order to have a functional? So in the next class we will actually go into details of FEM formulations and how these theorems can be exploited for a variety of structures to formulate various elements. Thank you.