MECHANICS

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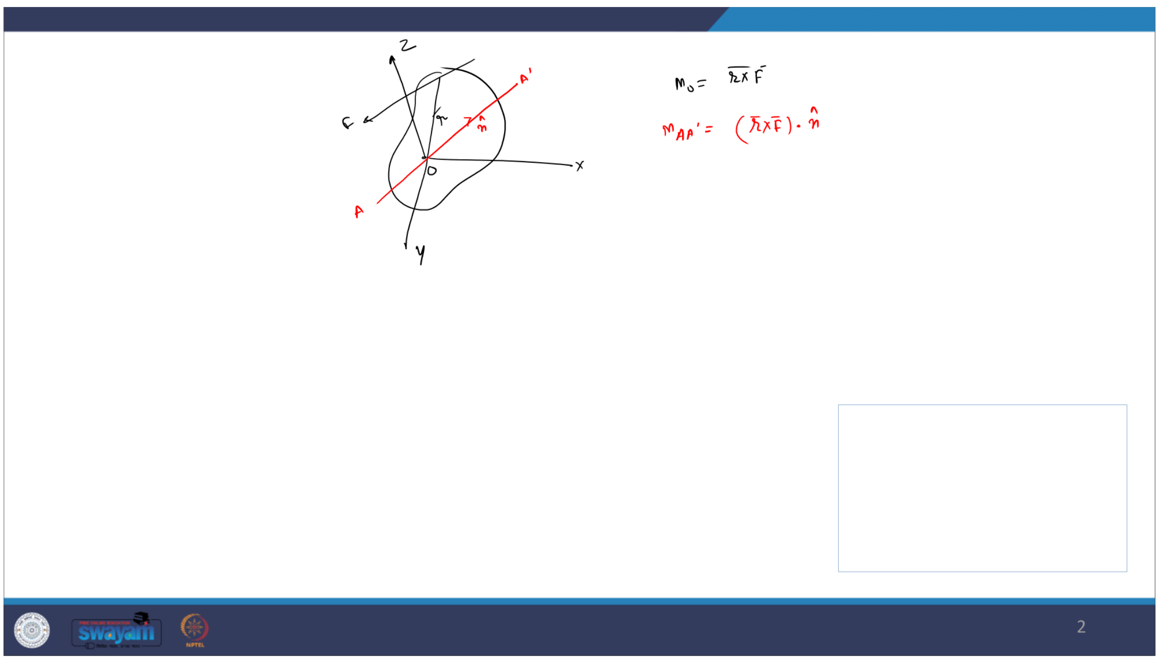
Lecture – 9
Examples: Equilibrium of rigid bodies in three dimensions

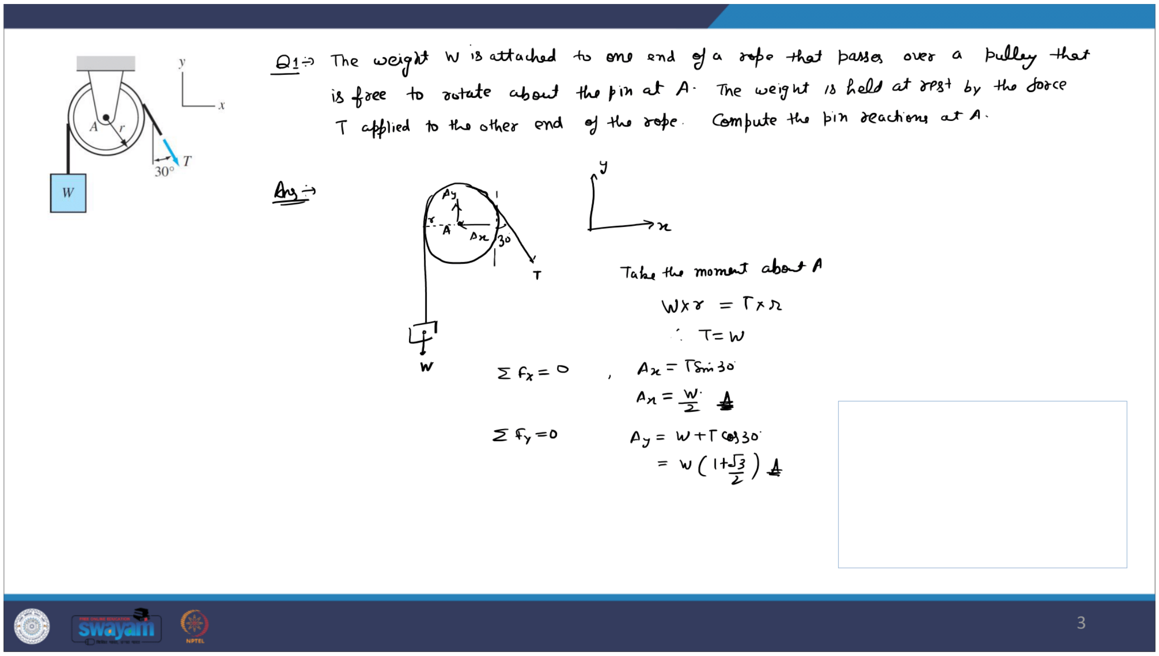
Hello everyone, welcome to the lecture again. In the last lecture, we looked at the equilibrium of the rigid bodies in two dimensions. So, in that case, the forces that were acting on the body, they were coplanar. Today, we are going to look at the examples of the equilibrium of rigid bodies in three dimensions.



So, in the last lecture, we also saw that suppose I have a rigid body and on this rigid body, I have forces that are acting. So, let us first define the axis. Let us say this is x-axis. This one is y-axis and you have a z-axis and let us say I apply a force F and I want to calculate the moment of this force about point O. So, first we draw a position vector r on the line of action of x and the moment about O is $r \times F$. But let's say I want to calculate the moment of this force about a line.

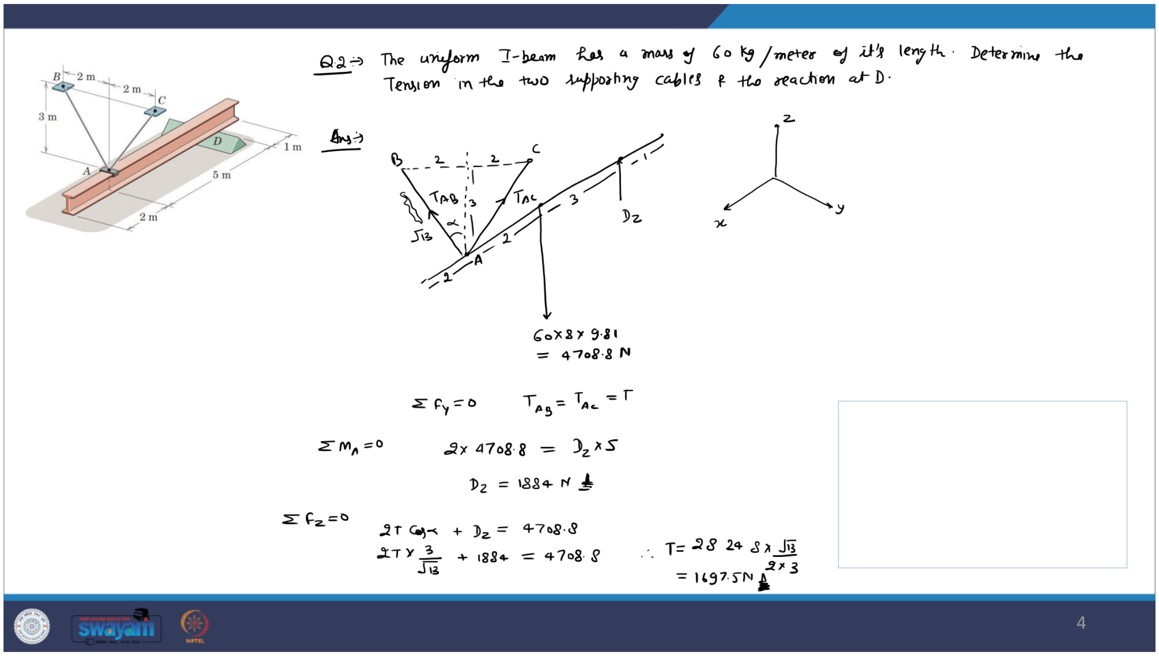
So, let's say this is the axis and about this axis, I want to calculate the moment. So, in that case, the moment about this line, let us say this is some AA' line, then the moment about

AA' will be the moment about O and then you take the projection of this moment along the line AA'. So, let's say the unit vector along AA' is $\hat{n}$, then $M'_{AA} = r \times F$. This is the moment about O and then take the projection along AA'. This concept is very useful when we calculate the moment in three dimension.



Today what we are going to do is first we will look at one example wherein the forces will be coplanar two dimension and then we will look at these concepts okay wherein you know it is convenient to take the moment about a line. So, let us look at the first problem. So, the statement of this problem is following: The weight W is attached to one end of a rope that passes over a pulley that is free to rotate about the pin at A, the weight is held at rest by the force T applied to the other end of the rope. Compute the pin reactions at A. So, first let us make the free body diagram of this problem. So, we have this pulley and on this pulley, we have weight W downwards and we have a force T which is acting at an angle of 30° from the vertical and because of this force and the weight, there is a reaction force at A. Let's say the force is A_y and A_x and we have to find out this force. Let me also put the axes over here.

So, to calculate the values of A_x and A_y, let us take the moment about A. So, if I take the moment about A, then I have $W \times r = T \times r$. r is the radius of this pulley. So, therefore, we see that $T = W$, get the value of T. Now, let us balance the forces along the x and y direction. So, first let us balance the force along the x direction. So, $\sum F_x = 0$ for the equilibrium. Therefore, $A_x = T sin\, 30\,°$ and T is W. So, therefore, this is W and $sin30° = 1/\,2$. Similarly, let us have $\sum F_y = 0$. So, this will give me $A_y = W + Tcos30\,°$ and this will be $W\,(1 + \sqrt{3}/2)$. So, therefore, you know we find out what is A_x, what is A_y and what is T. Now, let us look at the three-dimensional problems now.



So, the problem statement is again following. Question number 2, the uniform I-beam has a mass of 60 kg/meter of its length and it is asked you to determine the tension in the two supporting cables and the reaction at D. So, first let us make the free body diagram of this problem. So, we have this I-beam and at a distance of 2 meter, we have two cables and in these cables because of the symmetry, the tension will be the same.

okay. So, this will be, let us say T_{AB} and this is point C. So, in this direction, it will be T_{AC} , but because it is a symmetric problem, so you can clearly see that T_{AB} will be equal to T_{AC}, let us call it equal to T, okay. Now, at the center of the mass, its weight is going to act. So, this is also 2, its weight is going to act.

So, this is $60 \times\ 8 \times\ 9.81 = 4708.8\ N$. So, the weight is going to act downward and then at point D, which is at a distance of 1 meter from the another end, vertical force is going to act. So, let us call it D_z. Now, this is the free body diagram of that. Also, let me put the axis over here.

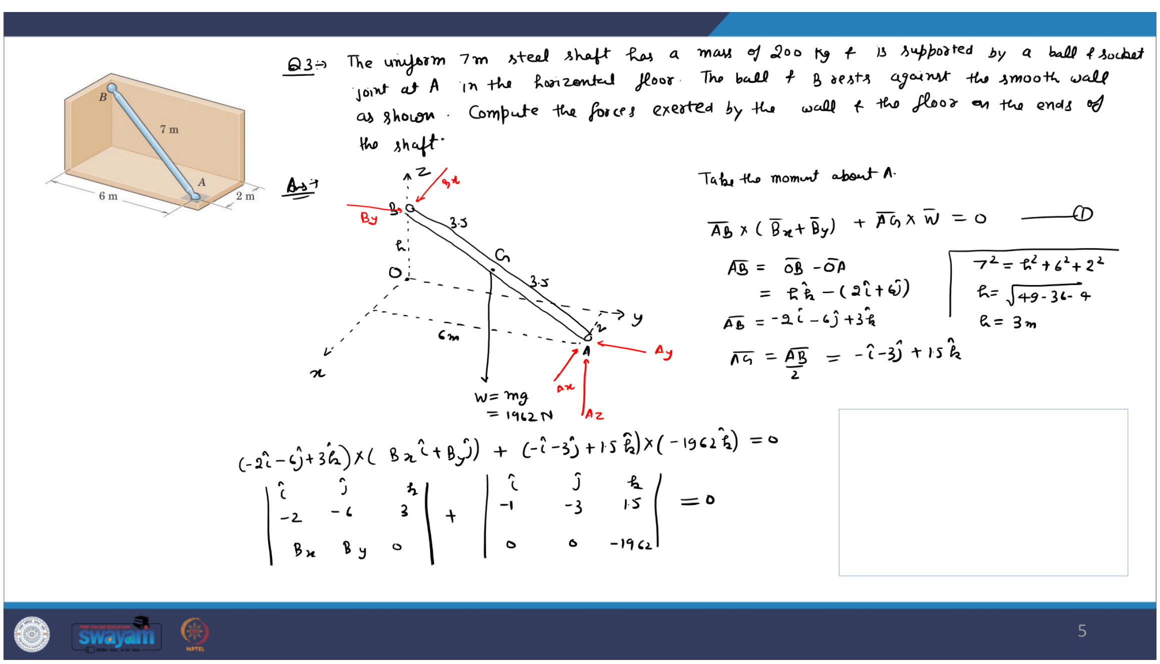
So, we have the x-axis and the y-axis and the z-axis. Now, to calculate the tension etcetera, let us first see that $\sum F_y = 0$. So, summation F_y let me just make this figure. Let us put more detail into the figure. So, let us call this angle as α.

So, $\sum F_y = 0$, this is going to give you $T_{AB} = T_{AC} = T$. This is what I am saying. Also, from the symmetry, you can see or you can use $\sum F_y = 0$ and you can see that T_{AB}, T_{AC} equal to T. Now, let us calculate the moment about A. So, $\sum M_A = 0$. So, for this, we have at a distance of 2, we have $4708.8\ N$ force and the moment of $T_{AB}\ T_{AC}$ will be 0 because that is passing through point A. So, this should be equal to $D_z \times\ 5$ because this length will be 3. $2 \times 4708.8 = D_z \times 5$ .So, therefore, $D_z = 1884\ N$.

Now, to find out T, let us use $\sum F_z = 0$. So, let us look at the component in the z direction. So, we have $2Tcos\alpha + D_z = 4708.8$. This α we can find out from the geometry, equal to the weight which is acting downward.

So, we have $2Tcos\alpha = 2T \times 3\ /\sqrt{13}$ because this is 2, this is 2 and this length is 3. So, therefore, this length will be $\sqrt{13}$. So, therefore, $2T \times \frac{3}{\sqrt{13}} + 1884 = 4708.8$. Therefore, $T = 2824.8 \times \frac{\sqrt{13}}{2} \times 3 = 1697.5\ N$.

So, this is how we can, you know, find out the moments in this problem about, you know, z-axis and so on and we can calculate the value of T.



Now, let us look at another, you know, interesting problem in three dimension. So, here the problem statement is following. The uniform 7 meter steel shaft has a mass of 200 kg and is supported by a ball and socket. Joint at A in the horizontal floor. The ball and B, they rest against the smooth wall as shown and it is asked you to compute the forces exerted by the wall and the floor on the ends of the shaft. So, clearly this is a three-dimensional problem and to solve it, let us first make a free body diagram. So, let me first draw the axis. So, we have x-axis, y-axis and z-axis and we have a shaft which is like this. It's center of gravity is going to act in the center and it will be $W = mg$. Now, various dimensions are given. So, for example, this is 3.5 meter. This is also 3.5 meter.

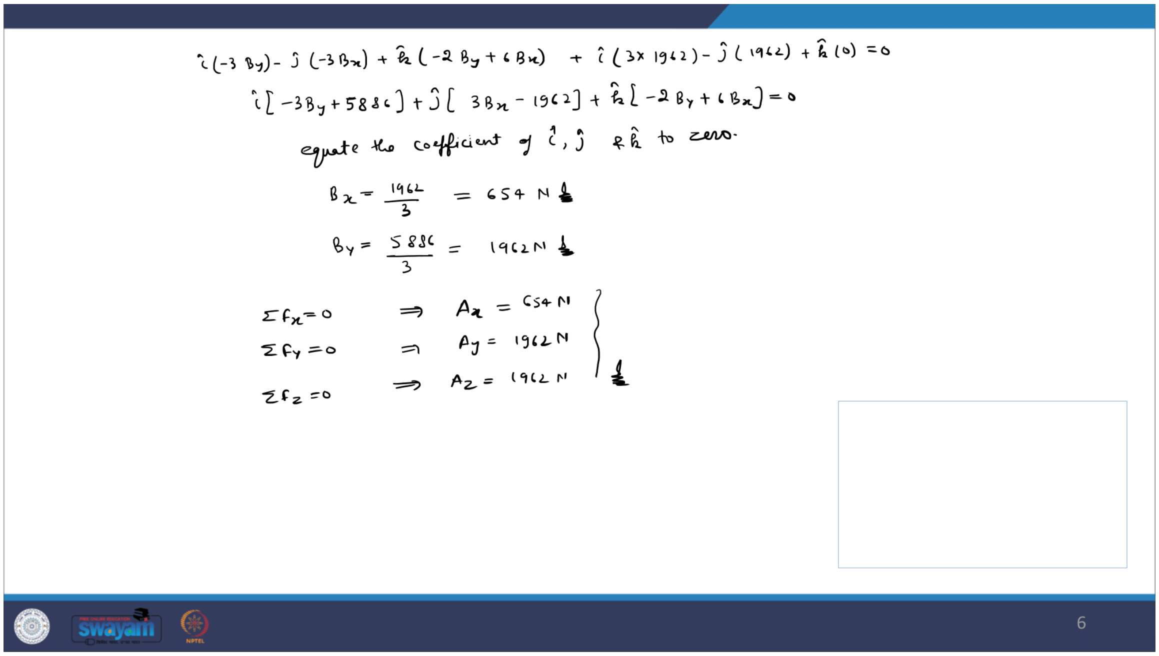
This length is 6 meter. This is 2 meter and this let us say this is h. So, this point is A, this point is B. Now, at point A, again various forces are going to act, various reaction forces are going to act. So, let us say we have the force in the z direction, we have the force in the y direction and we have the force in the x direction.

Because at point A, it is not restricted. It can move in any direction. So, therefore, for the equilibrium, the forces have to act in all the direction. At point B, similarly, we have the force which is acting in the x direction and we have the force acting in the y direction. So, these many forces are there.

To calculate the forces, let us take the moment about A. So, in this case, the contribution from $A_x, A_y\ and\ A_z$ will go away because they are passing through A and the moment will $be\ \overline{AB} \times (\overline{B_x} + \overline{B_y}) + \overline{A_G} \times \overline{W} = 0$ for the equilibrium. Now, to solve this equation, I should know what is $\overline{AB}$ and what is $\overline{AG}$. So, let us first calculate what is $\overline{AB}$. So, $\overline{AB} = \overline{OB} - \overline{OA}$. So, we are in O is the origin and this will be $h\hat{k} - (2i + 6j)$. Now, let us first find out how much is h. So, we can see from the geometry that $7^2 = h^2 + 6^2 + 2^2$ or $h = \sqrt{49 - 36 - 4}$or $h = 3\ m$. So, therefore, $\overline{AB} = -2i - 6j + 3k$and you can see that $\overline{AG} = \frac{\overline{AB}}{2} = -i - 3j + 1.5k$. Now, let us put it in equation number 1. So, we have $\overline{AB} = (-2i - 6j + 3k) \times \left(B_x i + B_y j\right) + (-i - 3j + 1.5k) \times -1962k = 0$

And this is very simple cross product. So, let us first calculate it.

$$\begin{vmatrix} i & j & k \\ -2 & -6 & 3 \\ B_x & B_y & 0 \end{vmatrix} + \begin{vmatrix} i & j & k \\ -1 & -3 & 1.5 \\ 0 & 0 & -1962 \end{vmatrix} = 0$$



So, this is the equation that we have. Now, we can do very simple you know simplification of this and it will be

$$i(-3B_y) - j(-3B_x) + k(-2B_y + 6B_x) + i(3 \times 1962) - j(1962) + k(0) = 0$$

so this is the equation that we have let us collect the coefficient of i j and k so we have

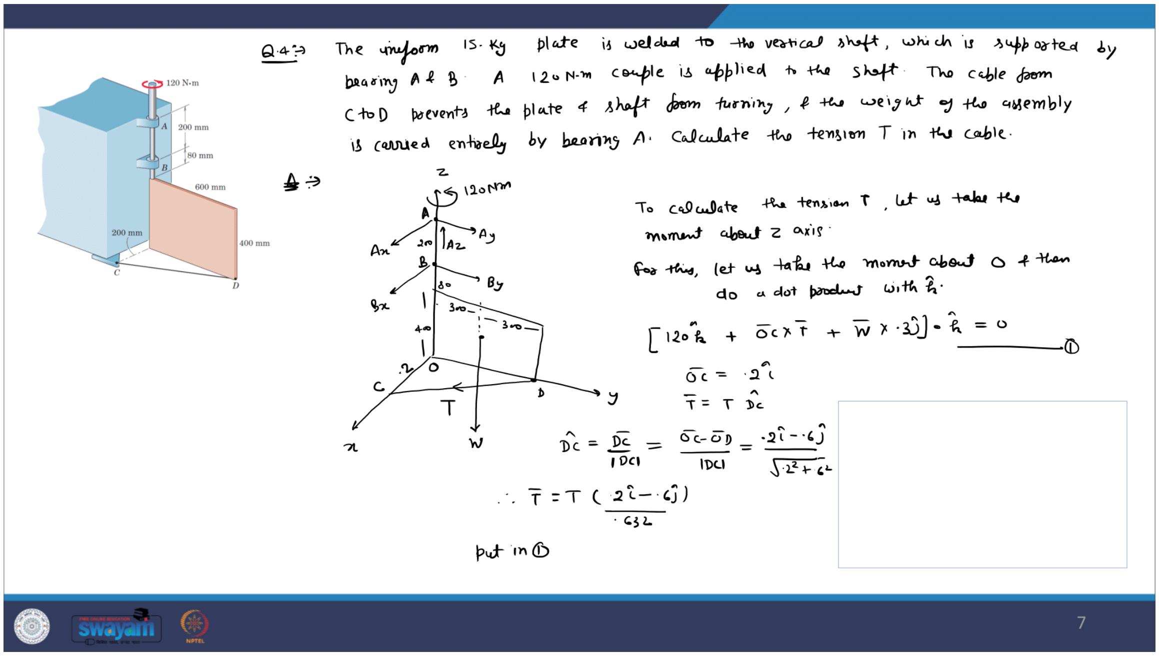
$$i(-3B_y + 5886) + j(3B_x - 1962) + k(-2B_y + 6B_x) = 0.$$

So, now because this equation is true, so therefore the coefficient of i, j and k must be 0. So, let us equate the coefficient of i, j and k to 0. So, we get $B_x = \frac{1962}{3}$ =654 Newton and we have $B_y = \frac{5886}{3} = 1962\ N$.

So, now we got what is B_x and what is B_y. Now, we can use the force equation along the x, y and z direction. So, if I use $\sum F_x = 0$, I will get the value of A_x. So, let us use $\sum F_x = 0$. This will give me A_x.

Let us see. So, here $A_x = 654\ N$. Similarly, $\sum F_y = 0$. This gives you $A_y = 1962\ N$ and $\sum F_z = 0$ should give you $A_z = 1962\ N$.

So, therefore, we are able to find out the reactions which is exerted on this road shaft by the wall and the floor. Now, let us look at more three-dimensional problem so that we get used to it and here the problem statement is following.



The uniform 15 kg plate is welded to the vertical shaft which is supported by bearing A and B, a 120 $N.m.$ couple is applied to the shaft. The cable from C to D prevents the plate and shaft from turning and the weight of the assembly is carried entirely by bearing A and in the question it is asked you to calculate the tension T in the cable. So, first let us look at the free body diagram of this problem. So, first let me put the axis

So we have x-axis, we have y-axis and we have z-axis. The plate is like this and at point D, this is held by a cable and on x-axis, it meets at point C. O is the origin. And at point B, we have the shaft. So, two forces are going to act at point B, force in the y direction and force in the x direction. And at point A, three forces are going to act, A_y, A_x, and A_z.

This force A_z is going to act at A because the weight of the whole setup is supported at point A. It is written in the question that the weight of the assembly is carried entirely by bearing A. So, therefore, at point B, it cannot support the weight. So, therefore, no vertical force is going to act at point B. And also, there is a torque. So, this is 120 Newton meter and this length is 200 mm. This is 80 mm. This is 400 mm and the weight of the plate is going to act downward and the length of this plate is 600. So, therefore, at 300 distance, it is going to act. And also, in the cable, there will be a tension T, which will be away from the body. So, this is the free body diagram of this. Now, to calculate the tension, let us take the moment about z-axis.

So, Why is that? Because if I take the moment about the z-axis, then A_x, A_y, B_x, B_y and A_z, all these forces will go away. So, I will calculate directly T. So, to calculate, let me write it down. To calculate the tension T, let us take the moment about z-axis and as I said the advantage is the forces that are acting at point A and point B will go away.

So, to calculate the moment about the z-axis, let us take the moment about O and then do a dot product with $\hat{k}$ because I will calculate the moment about O and then I will take it projection along the z direction. So, therefore, I will take a dot product with k. Let us write it down. So, I have a couple of $120\ k$ which is acting you know in the z direction and then I will have a OC, so this is my $120k + OC \times T + W \times 0.3j$ and then for the z-axis, I will take the dot product with $\hat{k}$ and this has to be equal to 0.

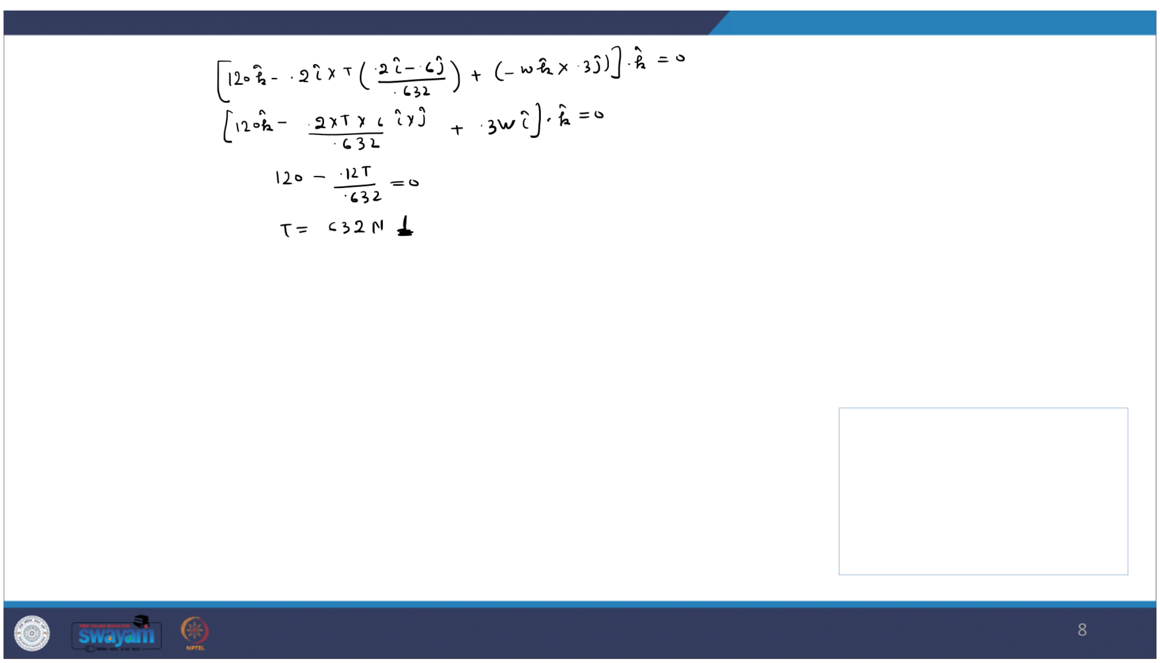
$(120k + OC \times T + W \times 0.3j).k = 0$

I have directly taken the moment of, you know, the forces that are acting at A and B equal to 0 because eventually when you take the dot product, with $\hat{k}$, whatever is the moment that will go away. So, to find out equation number 1, the result from equation number 1, let us first see what is OC and what is T. So, OC, you can clearly see, $OC = 0.2i$ because this is given in the question statement, this is 0.2. So, therefore, it will be $0.2i$ and $T = T\ \widehat{DC}$. So, first let us see what is $\widehat{DC}$.

$$\widehat{DC} = \frac{\overline{DC}}{|DC|} = \overline{OC} - \frac{\overline{OD}}{|DC|} = \frac{(0.2)i-(0.6)j}{\sqrt{(0.2)^2+(0.6)^2}}.$$

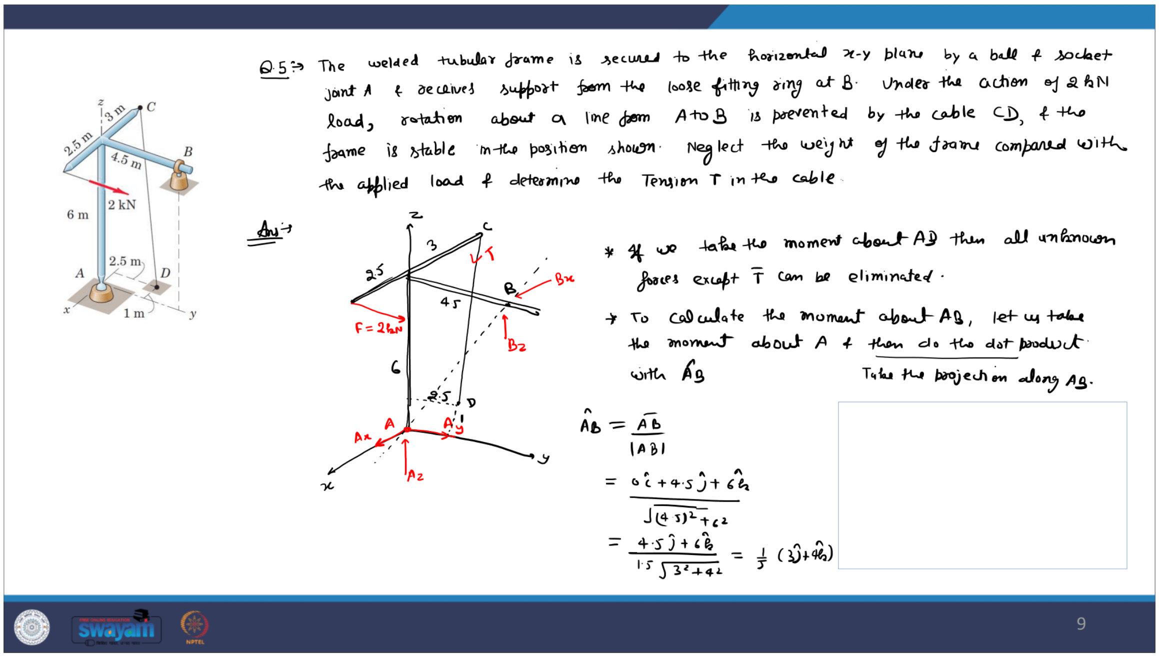
$$\bar{T} = \frac{T(0.2i - 0.6j)}{0.632}$$

So, we have, you know, everything that is required. Now, let us put it in equation number 1.



Then we have $\left(120k - 0.2i \times T\frac{0.2i - 0.6j}{0.632} + (-W)k \times 0.3j\right).k = 0$. Now, let us simplify that this is $\left(120k - 0.2 \times T \times \frac{6}{0.632} i \times j + \; 0.3Wi\right).k = 0$ and this gives $120 - \frac{0.12T}{0.632} =$ 0 and this gives you $T = 632\ N$.

So, just by using a single equation, note that we were able to find out T and the trick was to calculate the moment about the z axis.



Now, let us look at one more problem in three dimension and here the problem statement is following. The welded tabular frame is secured to the horizontal xy plane by a ball and socket joint A and receives support from the loose fitting ring at B under the action of $2\ kN$ load rotation about a line from A to B is prevented by cable CD and the frame is stable in the position shown and it is given that neglect the weight of the frame compared with the applied load and determine the tension T in the cable. So, this is the problem statement.

Now, to solve this, first let us look at the free body diagram. So, for that, let us first fix the x, y and z-axis. So, it is already given that this is the z-axis. We have y-axis and x-axis and in this, we have a this frame.

This is parallel to the x-axis. So, this is the frame that we have. Now, at point B, we have a weight and this length is 3 meter, this is 2.5 meter. At this point, we have a force.

So, this force is parallel to the y-axis. So, this is the force F equal to 2 kN and at point B, we have the reaction forces B_z and B_x that is going to act. Now, there will be no force along the y direction because this ring at B, it can move in the y direction. So, it is free to move.

Therefore, there will not be any reaction along the y direction. Now, also at point A, we will have three reaction forces. So, let us say A_z, then the force along the x direction and the reaction force along the y direction and also there is this cable. So, therefore, there is a tension force away from the body.

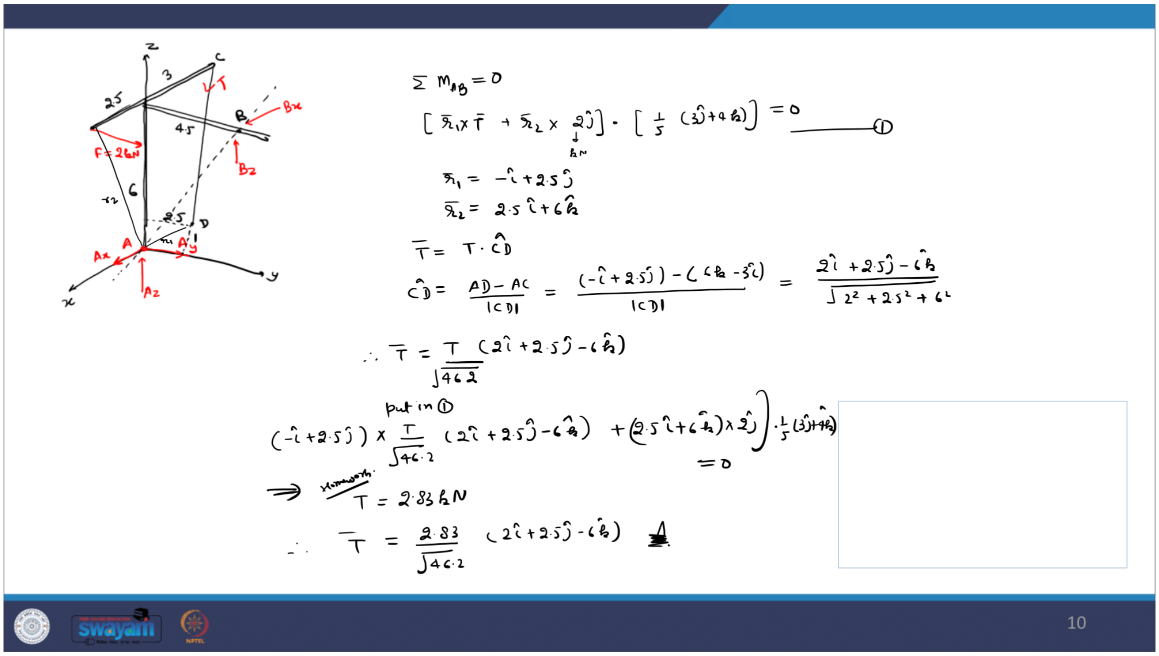
So, it will be along CD Now, in this problem, we have to determine what is T, okay? So, there are various unknown over here, but if I take the moment about AB, then five unknown forces I can get rid of. So, for example, A_x, A_y, A_z and B_x and B_z, these forces will be gone if I take the moment about AB, okay? So, only the unknown force T, I can find out.

So, what I am saying is if we take the moment about AB, then all unknown forces except T can be eliminated. Now, to calculate the moment about AB, what I can do is I can take the moment about A and then take its projection along the line AB. So, to calculate the moment about AB, let us take the moment about A and then do the dot product. So, with $\widehat{AB}$.

So, when I say dot product, I mean you are taking the projection, right. Take the projection of the moment along line AB. So, to find out the moment about AB, first I should know what is the unit vector $\widehat{AB}$. So, first let us find out what is the unit vector $\widehat{AB}$.

So, $\widehat{AB} = \overline{AB}/|AB|$ and AB I can find out from the geometry. So, this one is 6 meter, this length is 4.5 meter, and this is 2.5 meter and that is 1 meter. So, let us first see what is AB. So, $\overline{AB} = 0i + 4.5j + 6k$.

So, this is you know x, y and z of the point B divide by its modulus.
So, $\widehat{AB} = \frac{0i+4.5j+6k}{1.5(3^2+4^2)} = \frac{1}{5}(3j + 4k)$. So, therefore, I know what is $\widehat{AB}$.So, this is the $\widehat{AB}$.



Now, let us calculate the moment about AB. So, for the equilibrium, $\sum M_{AB} = 0$. So, we have let's say this is r_1 and let us say this is r_2.
So, we have $r_1 \times\ T + r_2 \times 2j$. So, this is kilo Newton acting in the y direction. We have to take the dot product with a $\hat{b}$, which is $\frac{1}{5}(3j + 4k)$ and this would be equal to 0.

$$(r_1 \times\ T + r_2 \times 2j).(\frac{1}{5}(3j + 4k)) = 0$$

Now, let us see what is r_1 and r_2. So, $r_1 = i + 2.5j$ and $r_2 = 2.5i + 6k$. Now, let us put all this thing in equation number 1, and also we should know how much is T. So, first let us find out what is T. So, T is its magnitude of dot product with the unit vector along T. So, this is $\widehat{CD}$. Let us see what is $\widehat{CD}$. $\widehat{CD} = AD - AC$ divided by its magnitude.

So, $AD = -i + 2.5j$, $AC = 6k - 3i$ and divided by C D.

$$\widehat{CD} = (-i + 2.5j) - \frac{6k - 3i}{|CD|} = \frac{2i + 2.5j - 6k}{\sqrt{2^2 + (2.5)^2 + (6)^2}}$$

$$\bar{T} = \frac{T}{\sqrt{46.2}}(2i + 2.5j - 6k)$$

And now we have, you know, all the ingredients that is required to put in equation number 1. So, let us put in 1.

$$\left((-i + 2.5j) \times \frac{T}{\sqrt{46.2}}(2i + 2.5j - 6k) + (2.5i + 6k) \times 2j\right).\frac{1}{5}(3j + 4k) = 0$$

And this is a very simple equation which can be solved by taking the cross product.

And if you do that, you will get $T = 2.83\ kN$. You can take it as a homework. And therefore, your $\bar{T} = \frac{2.83}{\sqrt{46.2}}(2i + 2.5j - 6k)$. So with this, let me stop here and thank you once again.