MECHANICS
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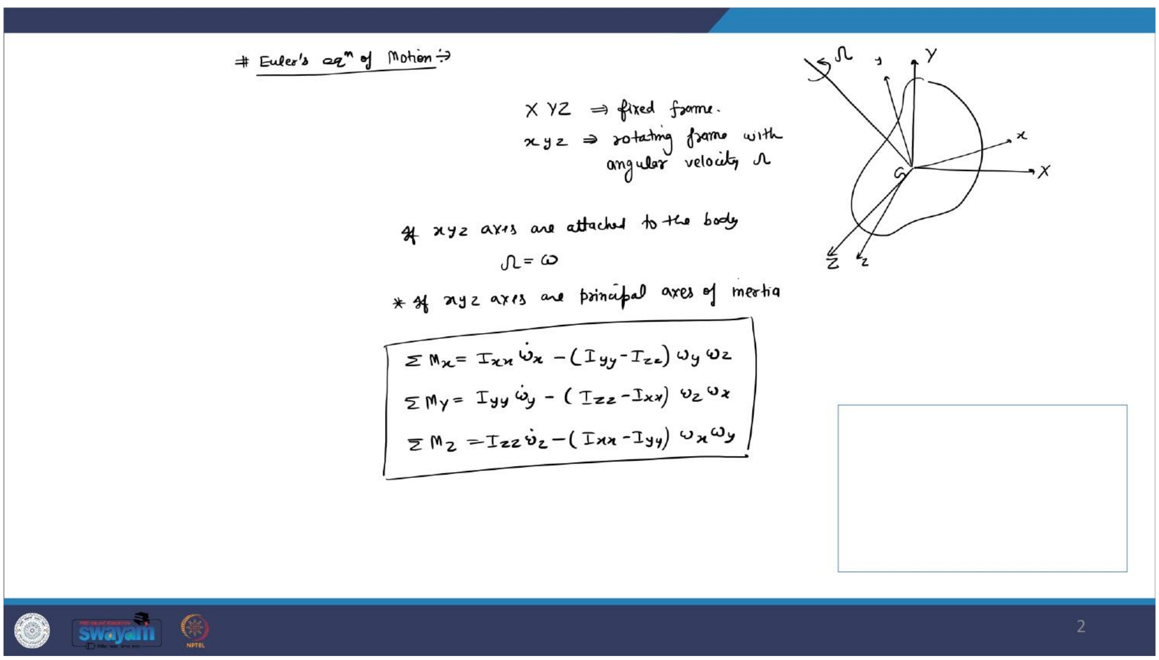
Lecture 59
Euler's equations of motion: examples

Hello everyone, welcome to the lecture again. In the last lecture, we looked at the rotating frame of reference and the Euler equation of motion. Today, we are going to look at couple of examples of the Euler equation of motion, but before that let us revisit the equation.



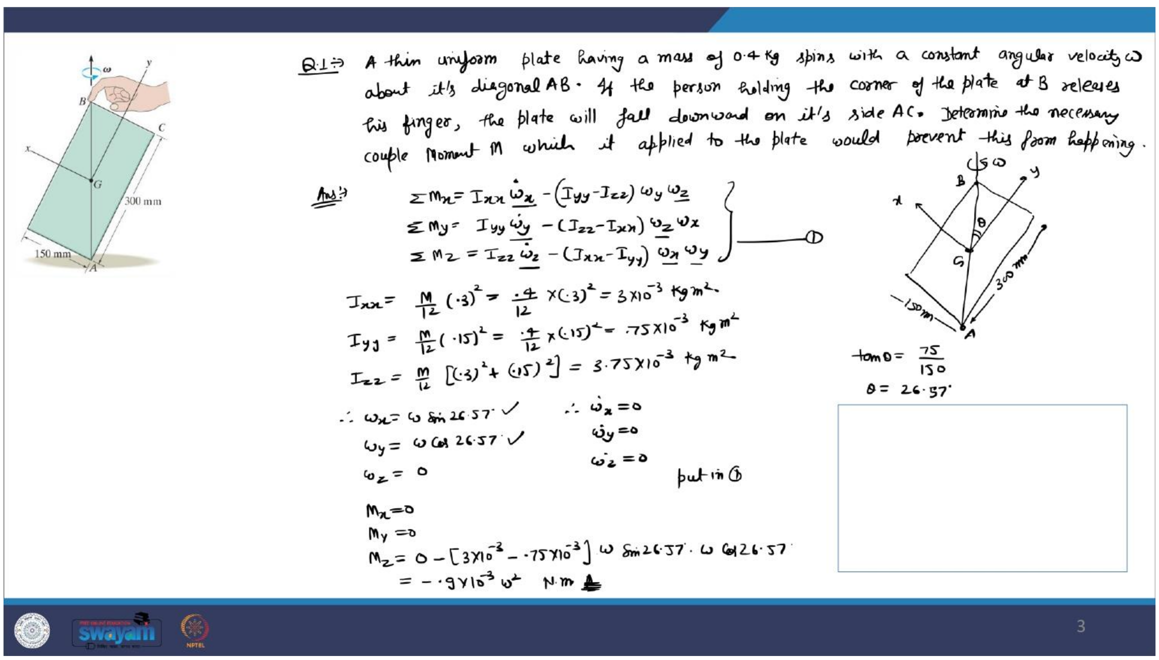
So, let us say we have a rigid body and on this rigid body, we can either have a fixed point or let us say the center of mass G and X, Y and Z are the fixed frame. So, that means they do not rotate. So, we have XYZ. These are the fixed frame and we have a xyz axis. So, these frame are the rotating frame with let us say an angular velocity Ω. So, it is rotating about some axis and let us say the angular velocity is ω. So, x, y, z, these are rotating frame with angular velocity ω. Now, if the xyz axis are the body frame, so that means that xyz axis they are attached to the body. In that case, if the body rotates, then the xyz axis also rotate. So, in that case, ω becomes the angular velocity of the rigid body and if our xyz axis, they are the principal axis of inertia, in that case, we have the Euler equation of motion which is

$$\sum M_x = I_{xx}\dot{\omega}_x - \left(I_{yy} - I_{zz}\right)\omega_y\omega_z$$

$$\sum M_y = I_{yy}\dot{\omega}_y - (I_{zz} - I_{xx})\omega_z\omega_x$$

$$\sum M_z = I_{zz}\dot{\omega}_z - \left(I_{xx} - I_{yy}\right)\omega_x\omega_y$$

Here, this I_{xx}, I_{yy} and I_{zz} are the moment of inertia about the x, y and z-axis respectively and ω_x, ω_y and ω_z are the component of the angular velocity along the x, y and z-axis.



Now, let us look at an example and the problem statement is following. A thin uniform plate having a mass of $0.4\ kg$ spins with a constant angular velocity ω about its diagonal AB. If the person holding the corner of the plate at B releases his finger, the plate will fall downwards on its side AC, determine the necessary couple moment M which it applied to the plate would prevent this from happening. So, in this question, we have this uniform plate, this length is $300\ mm$, this is $150\ mm$ and we have the x and y-axis which are attached to the body and its centre is the centre of mass G. So, this is y-axis and the x-axis and this plate is rotating with an angular velocity ω, this point is B and this point is A. Because the x and y-axis, they are attached to the body and x and y-axis are also the

principal inertia axis of this body. Therefore, we can use the Euler equation of motion and the Euler equation of motion is
$\sum M_x = I_{xx}\dot{\omega}_x - (I_{yy} - I_{zz})\omega_y\omega_z$

$\sum M_y = I_{yy}\dot{\omega}_y - (I_{zz} - I_{xx})\omega_z\omega_x$

$\sum M_z = I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\omega_x\omega_y \qquad ----------- (1)$

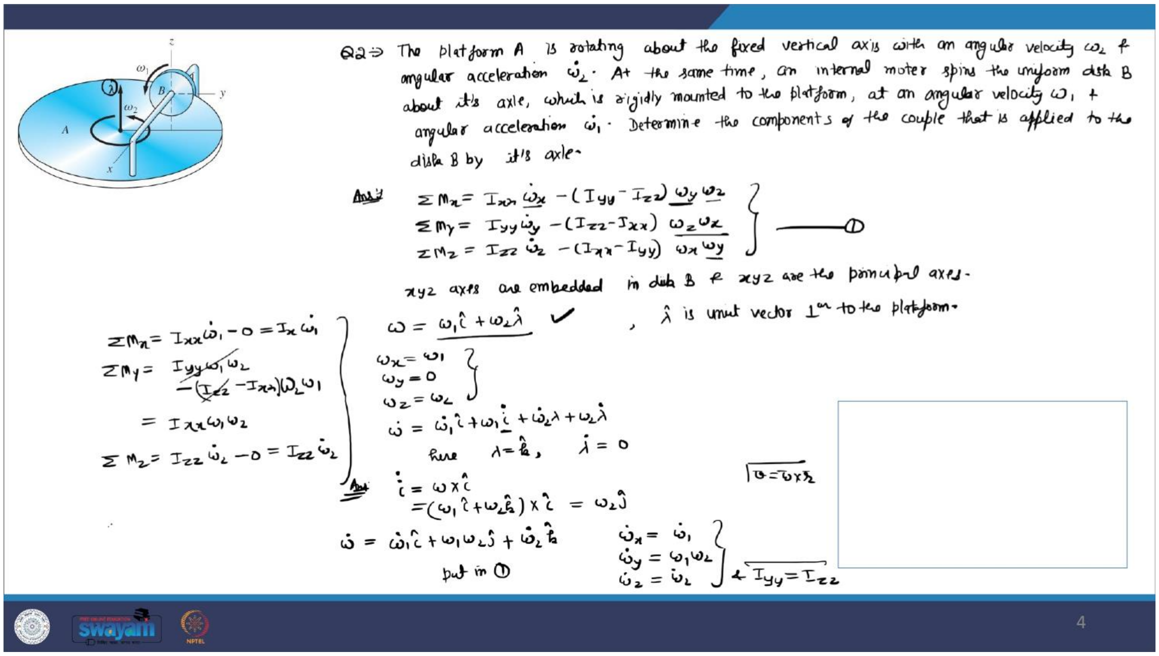
Now, the first thing that we have to, you know, find out is the moment of inertia about the x, y and z-axis, so that we get these values. So,
$I_{xx} = \frac{M}{12}(0.3)^2 = \frac{0.4}{12} \times (0.3)^2 = 3 \times 10^{-3}\ kg\ m^2$
Now, the moment of inertia about the y-axis will be
$I_{yy} = \frac{M}{12}(0.15)^2 = \frac{0.4}{12} \times (0.15)^2 = 0.75 \times 10^{-3}\ kg\ m^2$
And the moment of inertia about the z-axis can be found out by the perpendicular axis theorem. So, $I_{zz} = \frac{M}{12}((0.3)^2 + (0.15)^2) = 3.75 \times 10^{-3} kg\ m^2$.

So, we have the I_{xx}, I_{yy} and I_{zz}. Now, let us find out the values of $\omega's$. Now, let us say this angle is θ. In that case, $tan\theta = \frac{75mm}{150mm}$. So, this gives you $\theta = 26.57°$. And from here, because we have $omega$, we can find out its component along the x, y and z direction. So, therefore, $\omega_x = \omega\ sin26.57°$, $\omega_y = \omega\ cos26.57°$ and $\omega_z = 0$. And you can see that ω_x, ω_y and ω_z, they are fixed values. So therefore, $\dot{\omega}_x = 0$, $\dot{\omega}_y = 0$ and $\dot{\omega}_z = 0$. So now let us put these values in equation number 1. So, we have $\dot{\omega}_x$, $\dot{\omega}_y$ and $\dot{\omega}_z$, they are equal to 0. So therefore, the first term will not contribute, and ω_z is also 0. Therefore, $M_x = 0$. Similarly, $M_y = 0$ and $M_z = 0 - (3 \times 10^{-3} - 0.75 \times 10^{-3})\omega\ sin26.57.7\ \omega\ cos26.57$ and this comes out to be $-0.9 \times 10^{-3}\omega^2\ Nm$. So, this much amount of moment needs to be applied to prevent this plate from falling down.

Now, let us look at another question statement and it is following. The platform A is rotating about the fixed vertical axis with an angular velocity ω_2 and angular acceleration $\dot{\omega}_2$. At the same time, an internal motor spins the uniform disc B about its axle which is rigidly mounted to the platform at an angular velocity ω_1 and angular acceleration $\dot{\omega}_1$. Determine the components of the couple that is applied to the disc B by its axle. So, we have to find the couple that is applied to the disc B. Again, you can see here that x, y, z-axis they are embedded in disk B and x, y, z-axis are also the principal inertia axis. Therefore, we can use the Euler equation of motion. So, the Euler equation of motion is
$\sum M_x = I_{xx}\dot{\omega}_x - (I_{yy} - I_{zz})\omega_y\omega_z$

$$\sum M_y = I_{yy}\dot{\omega}_y - (I_{zz} - I_{xx})\omega_z\omega_x$$

$$\sum M_z = I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\omega_x\omega_y \qquad ----------- (1)$$



And we can use this equation as I said because xyz axis they are embedded or attached in disk B and x, y, z-axis are the principal axis. So, let us find out these quantities. So, from the figure, you can see that the disc A is rotating horizontally with an angular velocity ω_2 and the disc B is rotating with an angular velocity ω_1. So, from the geometry, I can write down the angular velocity ω, which is a vector quantity as $\omega_i\hat{\imath} + \omega_2\hat{\lambda}$, where this $\hat{\lambda}$ is the unit vector, which is perpendicular to the platform. So, it is given here that this one isλ. So, let us take the unit vector along this direction. So, from here, you can see that we already have how much is ω_x, ω_y and ω_z. So, $\omega_x = \omega_1$, $\omega_y = 0$ and $\omega_z = \omega_2$. Now, we have to find out how much is $\dot{\omega}_x$, $\dot{\omega}_y$ and $\dot{\omega}_z$. So, from this equation, let us write down $\dot{\omega}$. So, $\dot{\omega}$ will be let us differentiate this equation. So, we have $\dot{\omega}_1\hat{\imath} + \omega_1 i + \dot{\omega}_2\lambda + \omega_2\dot{\lambda}$. Let me write here again that this $\hat{\lambda}$ is a unit vector which is perpendicular to the platforms. Now, here $\dot{\lambda}$ is not going to change, it is always along the z direction. So, $\dot{\lambda}$, so first of all λ is in the direction of k and $\dot{\lambda} = 0$. Now, let us look at i. So, i will be, so you can use the relation $v = \omega \times r$. Here r is the unit vector which is i. So, therefore, $i = \omega \times i$. Now, we already have ω . $(\omega_1\hat{\imath} + \omega_2\dot{k}) \times \hat{\imath} = \omega_2\hat{\jmath}$. Therefore, $\dot{\omega} = \dot{\omega}_1\hat{\imath} + \omega_1\omega_2\hat{\jmath} + \dot{\omega}_2\hat{k}$ and therefore, $\dot{\omega}_x = \dot{\omega}_1$, $\omega_y = \omega_1\omega_2$ and $\dot{\omega}_z = \dot{\omega}_2$. And also note that here the x, y and z-axis, they are the principal axis and because of the symmetry, the moment of inertia of this disc B about the y-axis and the z-axis will be the same. So, I_{yy} and I_{zz} will also be equal. Let us put this in equation number 1 . So, we have $\sum M_x = I_{xx}\dot{\omega}_1 - 0 = I_x\dot{\omega}_1$ and $\sum M_y = I_{yy}\omega_1\omega_2 - (I_{zz} - I_{xx})\omega_2\omega_1 = I_{xx}\omega_1\omega_2$. Now, let us look at the moment about the z-axis. So, it will be $\sum M_z = I_{zz}\dot{\omega}_2 - 0 = I_{zz}\dot{\omega}_2$. So, these are the component of the couple that are applied to the disk B by the axle.

With this, let me stop here. See you in the next class. Thank you.