MECHANICS

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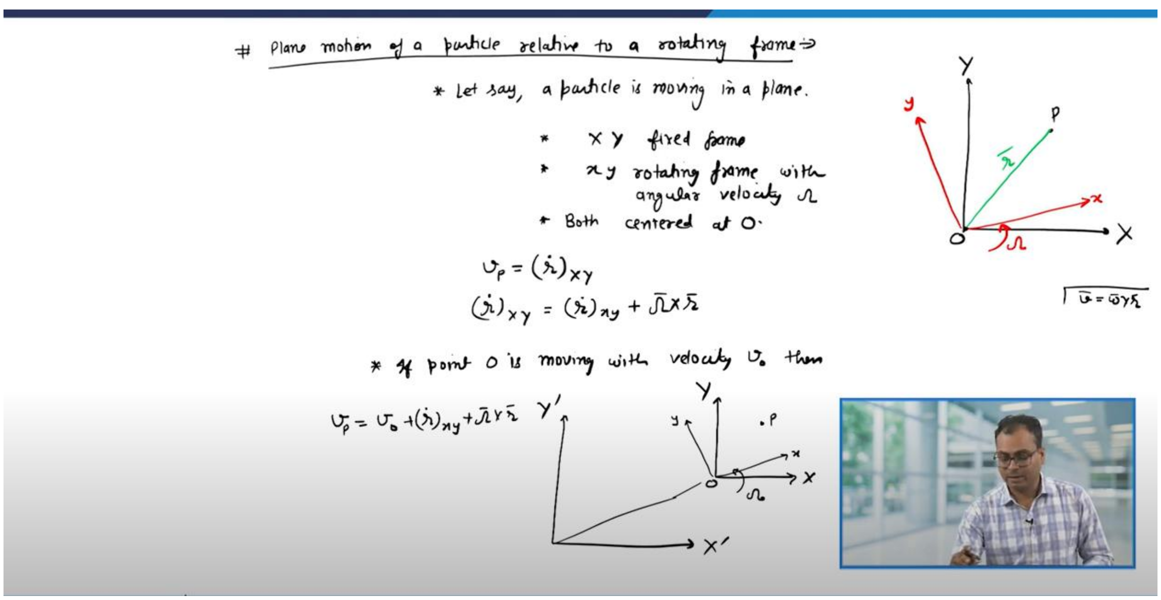
Lecture: 58

Euler's equations of motion

Hello everyone, welcome to the lecture again. Today, we are going to discuss the Euler's equation of motion. These equations are very important to study the dynamics of the rigid bodies. Now, to understand the Euler's equation of motion, we need the concept of rotating frame and to understand that, let us consider a plane motion of a particle relative to a rotating frame.

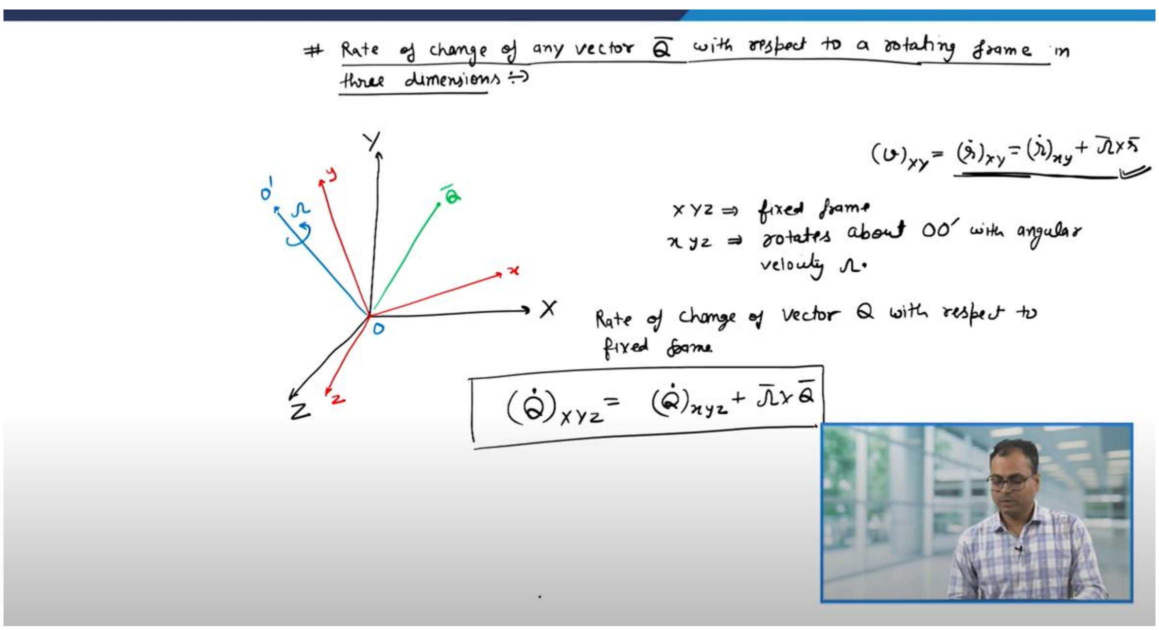


Let us say I have a particle and this particle is moving in a plane. So, we have let us say this particle P and its position vector is let us say $\vec{r}$ and let us say this X and Y, these are fixed frame. So, we have $X \; and \; Y - coordinate$ and we have another coordinate, let us say it is denoted by x and y . And this frame is rotating with an angular velocity Ω .

So, we have xy ; these are rotating frames with angular velocity Ω . And both these reference frames, they are centered at the same point, let us say O. then the angular velocity of the particle, so let me say it is v_p; this will be the $v_p = (\dot{r})_{XY}$, okay? And if I want to write down the velocity of particle p in the xy frame, then $(\dot{r})_{XY} = (\dot{r})_{xy} + \vec{\Omega} \times \vec{r}$. Now, this is basically the vector sum of the velocities.

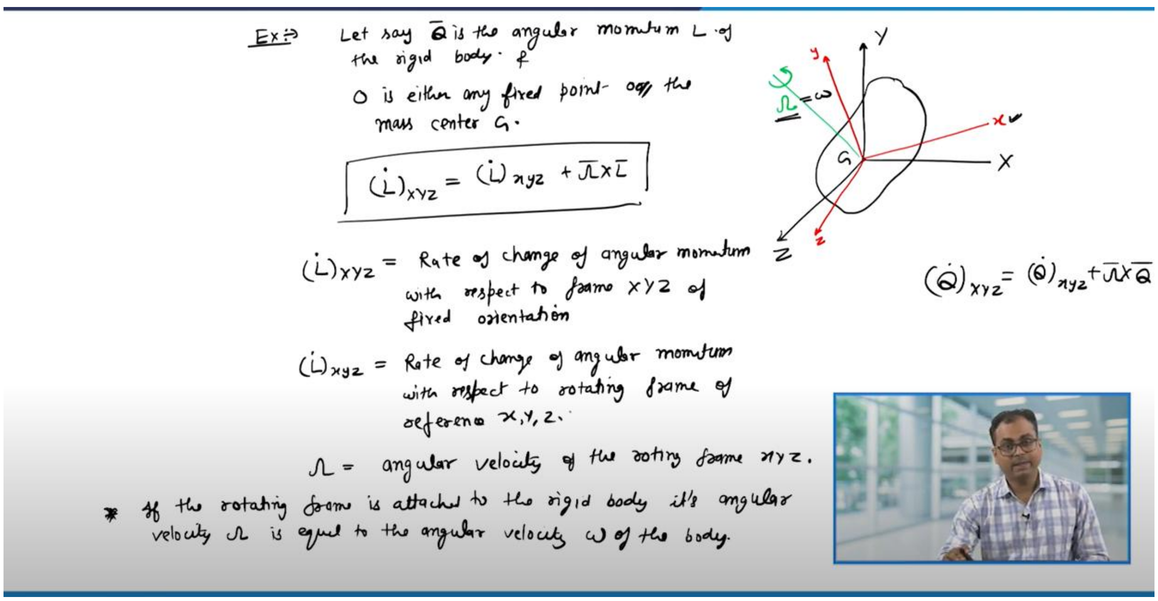
The second term comes from $\vec{v} = \vec{\omega} \times \vec{r}$. And now, let us say we want to write down the velocity in the frame for which this point O is also moving. So, if point O is moving with velocity, let us say v_o. In that case, so the situation is following: we have some coordinate system, let us call it X', Y', and then we have the fixed frame X, Y , and we have a frame which is rotating with angular velocity Ω and the velocity of point O is v_o. So, in that case, the velocity of point P can be written as $v_p = v_o + (\dot{r})_{xy} + \vec{\Omega} \times \vec{r}$.

Now, this result we wrote for the velocity, but actually they can be generalized for any vector. So, let us say we want to find out the rate of change of any vector, let us say some $\vec{Q}$ with respect to a rotating frame in three dimensions. So, remember earlier, we have the velocity v in XY frame.



So, this was nothing but $(\dot{r})_{XY}$. This was $(\dot{r})_{xy}$, the frame which is rotating plus $\vec{\Omega} \times \vec{r}$. Here, in three dimensions, we have a fixed frame $X, Y \; and \; Z$. Then we have the rotating frame $x, y, z,$ and let us say I have some $\vec{Q}$, this is some $\vec{Q}$, and this x, y, z frame they are rotating with angular velocity Ω . So, let us say this point is O and about OO', it is rotating with an angular velocity Ω .

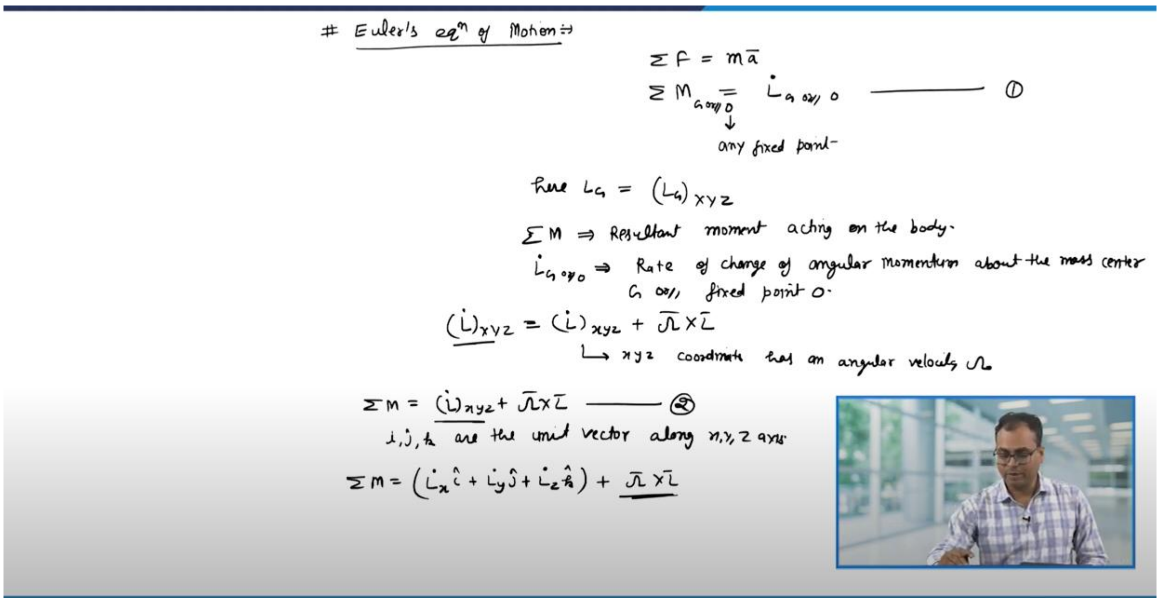
Let me again mention that there are two frames of references, and both of them are centered at the same point O. XYZ is the fixed frame, and xyz is the frame which rotates about OO' with angular velocity Ω. So, I can write down from this equation the rate of change of $\vec{Q}$ with respect to the fixed frame. So, a fixed frame means XYZ . So, we have $\dot{Q}$. So, remember this equation: X, Y, Z will be $\left(\dot{Q}\right)_{xyz} + \vec{\Omega} \times \vec{Q}$ which is $\vec{Q}$ here. Now, let us look at it by an example.



So, let us say $\overline{Q}$ here is the angular momentum. So, again the same situation. We have a rigid body and let us say at its Center of mass G, I have the XYZ frame and then I have a $x,\ y\ and\ z - axis$ and this axis is rotating with an angular velocity Ω So, let us say it is rotating about this axis and the angular velocity is Ω . Then the angular momentum, so we are interested in angular momentum.

So, let us say $\overline{Q}$ is the angular momentum, okay, L of the rigid body. And point O, so earlier we have this point O, this point O is here either any fixed point or the centre of mass point G. O is either any fixed point or the mass centre. So, in that case, note that this equation we had $\left(\dot{Q}\right)_{XYZ} = \left(\dot{Q}\right)_{xyz} + \vec{\Omega} \times \vec{Q}$. Here Q is the angular momentum, so therefore, we have $\left(\dot{L}\right)_{XYZ} = \left(\dot{L}\right)_{xyz} + \vec{\Omega} \times \vec{L}$. Let me again emphasize that here $\left(\dot{L}\right)_{XYZ}$ is the rate of change of angular momentum with respect to the frame XYZ which has fixed orientation.

So, this XYZ is not rotating it is fixed. Similarly, $(\dot{L})_{xyz}$ is the rate of change of angular momentum with respect to rotating frame of reference xyz and what is Ω ? Ω Is the angular velocity of the rotating frame xyz . Now, this is very important if the rotating frame is attached to the rigid body, its angular velocity Ω will be equal to the angular velocity ω of the body. So, if this is xyz earlier it has an angular velocity of Ω . Now, I attach this xyz with the rigid body. So, that when the rigid body rotates xyz also rotate in that case this angular velocity Ω . It will be equal to the angular velocity of the rigid body which is equal to ω .

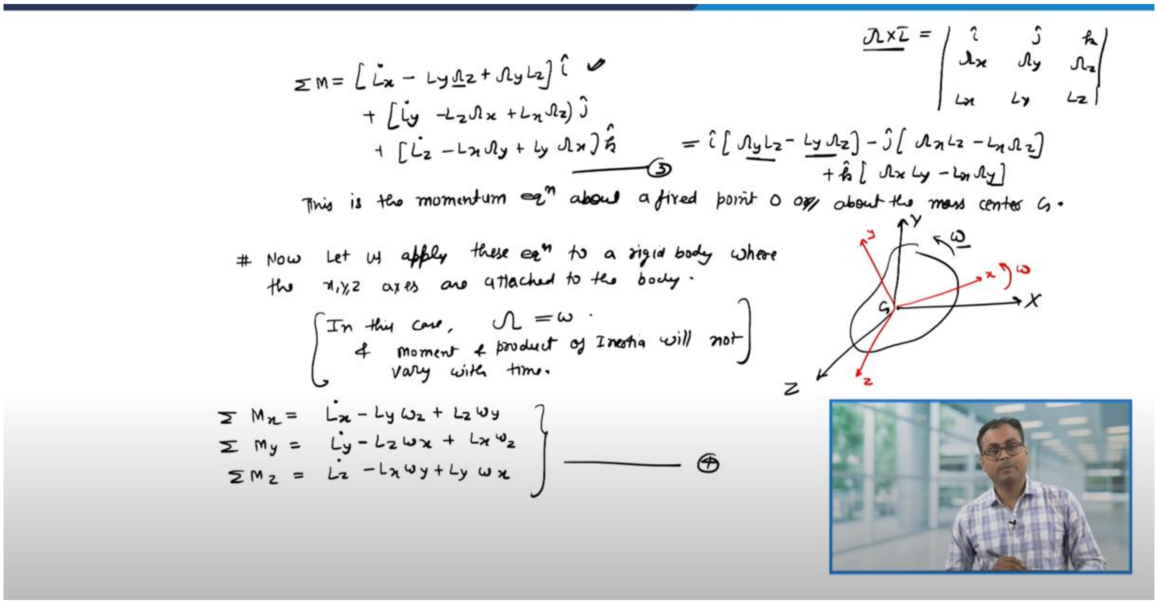


Now, keeping these concepts in mind, let us look at the Euler's equation of motion and let us consider the equation of motion of the rigid body in three dimensions. So, we know the governing equation. The governing equation are $\Sigma F = m\overline{a}$, where this equation tells you the motion of the center of mass of the body. And we have $\Sigma M = \dot{L}$.

And this moment can be calculated either about the center of mass G or any fixed-point O. So, this is any fixed point. Similarly, this $\dot{L}$ can be the rate of change of angular momentum about the mass center G or any fixed-point O. Here, this L_G, so let me call this as equation number (1). Here, this $L_G \; or \; L_O$, this is the angular momentum about the fixed frame XYZ .

And ΣM is the resultant moment or the torque of the external force acting on the rigid body and $\dot{L}_{\,G \, or \, O}$ is the rate of change of angular momentum about the mass center G. or any fixed-point O. Now, from here, I can write down what is $(\dot{L})_{XYZ}$ in terms of xyz or the rate of change of angular momentum about the rotating frame. So, $(\dot{L})_{XYZ} = (\dot{L})_{xyz} + \vec{\Omega} \times \vec{L}$, where this Ω is the angular velocity of a xyz frame.

So, here in we say that this xyz coordinate has an angular velocity Ω . Now, I can plug this in equation number (1). So, then I have $\Sigma M = (\dot{L})_{xyz} + \vec{\Omega} \times \vec{L}$. Now, I can write down this equation component by component. For that, let us say that $\hat{\imath}, \hat{\jmath}, \hat{k}$, these are the unit vector along $x, y, z - axis$. So, in that case M , let us look at this term, it will be $\Sigma M = (\dot{L_x}\hat{\imath} + \dot{L_y}\hat{\jmath} + \dot{L_z}\hat{k}) + \vec{\Omega} \times \vec{L}$. Now, let us find out the value of $\vec{\Omega} \times \vec{L}$.



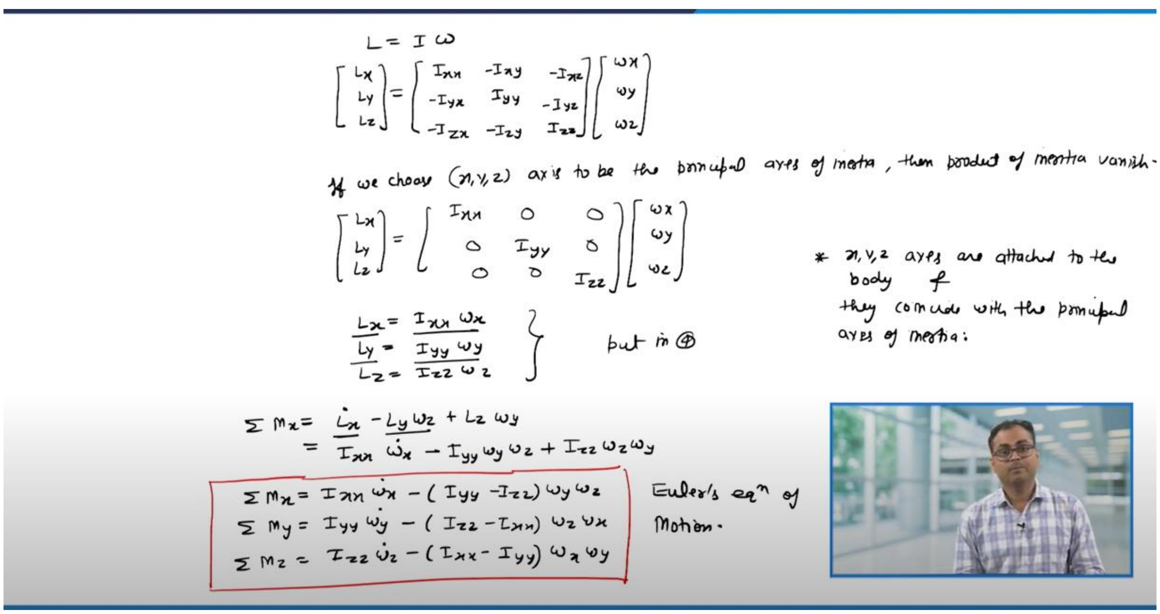
So, $\vec{\Omega} \times \vec{L}$, it is a vector product. So, it will be $\hat{\imath}, \hat{\jmath}, \hat{k}, \Omega_x, \Omega_y, \Omega_z, L_x, L_y, L_z$, and this will be $\hat{\imath}[\Omega_y L_z - L_y \Omega_z] - \hat{\jmath}[\Omega_x L_z - L_x \Omega_z] + \hat{k}[\Omega_x L_y - L_x \Omega_y]$. And we can put this over here. So, we have a ΣM equal to let us collect the component of $\hat{\imath}$ or the coefficient of $\hat{\imath}$. So, we have $\Sigma M = [\dot{L_x} - L_y\Omega_z + \Omega_y L_z]\hat{\imath} + [\dot{L_y} - L_z\Omega_x + L_x\Omega_z]\hat{\jmath} + [\dot{L_z} - L_x\Omega_y + L_y\Omega_x]\hat{k}$. Let us say this is equation number (3), and this is the momentum equation about the fixed-point O or about the mass center G. So, let me mention it that this is the momentum equation about a fixed-point O or about the mass center G. Now, let me mention it again that we have this rigid body, on this rigid body, we have the mass center G and XYZ are the fixed axis, and we have $x, y, and\ z$, which are rotating with an angular velocity Ω .

Then these are the momentum equation. Now, let us say we apply these equations to the rigid body wherein this xyz −axis , they are fixed, or they are attached to the rigid body. In that case, when the body rotates with an angular velocity ω , then the $xyz - axis$ also rotates with the same angular velocity ω . So, we have the case wherein this rigid body is rotating with angular velocity ω .

Therefore, the $x,\ y,\ and\ z-axes$ will also rotate with the same angular velocity ω because now they are attached to the rigid body. So, now let us apply this equation to a rigid body. Where the xyz −axis are attached to the body. So, in that case, what will happen? In that case, this Ω that we had earlier, this will become ω .

So, we have $\Omega = \omega$. And now, since $x,\ y,\ z-axes$ are attached to the rigid body, therefore, the moment and product of inertia will not vary with time. So, therefore, let us look at equation number (3). Under this condition, we have ΣM_x.

So, let us look at the x component. This will be $\Sigma M_x = \dot{L_x} - L_y\omega_z + L_z\omega_y$. And $\Sigma M_y = \dot{L_y} - L_z\omega_x + L_x\omega_z$. And $\Sigma M_z = \dot{L_z} - L_x\omega_y + L_y\omega_x$. Let us call it equation number (4).



Now, to rewrite these equations in the desired form, we need the relation between the angular momentum and moment of inertia, and this is something that we discussed last time that $L = I\omega$ wherein this I is the moment of inertia tensor. Therefore, component by component, we can write down $L_x,\ L_y,\ L_z$ equal to the moment of inertia tensor$\omega_x,\ \omega_y,\ \omega_z$. And this moment of inertia tensor is $I_{xx},\ I_{yy},\ I_{zz}$, then $-I_{xy},\ -I_{yx},\ -I_{xz},\ -I_{zx},\ -I_{zy},\ -I_{yz}$. Now, we also saw that if we choose the $x,\ y,$ and z −axis to be the principle of inertia axis, then in that case, the product of inertia is 0. So, in that case. $L_x,\ L_y,\ L_z$, they become $I_{xx},\ I_{yy},\ I_{zz}$ and $\omega_x,\ \omega_y,\ \omega_z$, and all the product of inertia, they are 0 or I can write down $L_x = I_{xx}\omega_x$, $L_y = I_{yy}\omega_y$ and $L_z = I_{zz}\omega_z$. Let us put this in equation number (4) . Equation number (4) was $\Sigma M_x = \dot{L_x} - L_y\omega_z + L_z\omega_y$. So, $\dot{L_x}$ because we have fixed our axis with the body. So, therefore, the product of inertia and the moment of inertia, they will not vary with time. Therefore, $\dot{L_x}$ will be

So, I_{xx} is independent of time. So, we have this, and then $\dot{\omega_x} - L_y\omega_z$, $L_y = I_{yy}\omega_y$ into ω_z plus $L_z = I_{zz}\omega_z$ into ω_y or I can write down $\Sigma M_x = I_{xx}\dot{\omega_x} - (I_{yy} - I_{zz})\omega_y\omega_z$. Similarly, $\Sigma M_y = I_{yy}\dot{\omega_y} - (I_{zz} - I_{xx})\omega_z\omega_x$ and $\Sigma M_z = I_{zz}\dot{\omega_z} - (I_{xx} - I_{yy})\omega_x\omega_y$. These are the Euler equation of motion.

And when we are using these equations, we must remember two things. Number one $x,\ y,\ z - axis$, they are attached to the body and they coincide with the principal axis of inertia at the fixed point or at the mass center G. So, with this, let me stop here. See you in the next class. Thank you.