**MECHANICS**

**Prof. Anjani Kumar Tiwari**

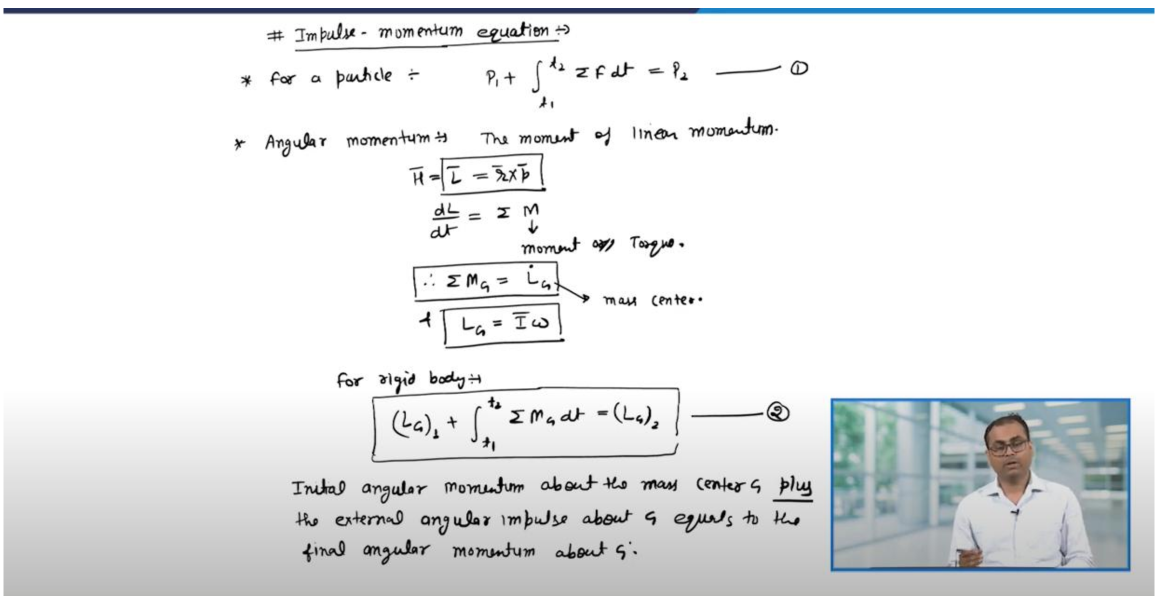
**Department of Physics**

**Indian Institute of Technology, Roorkee**

**Lecture: 56**

**Plane motion of a rigid body: impulse-momentum equation**

Hello everyone, welcome to the lecture again. In the last lecture, we look at the work energy equation of a rigid body performing a plane motion.

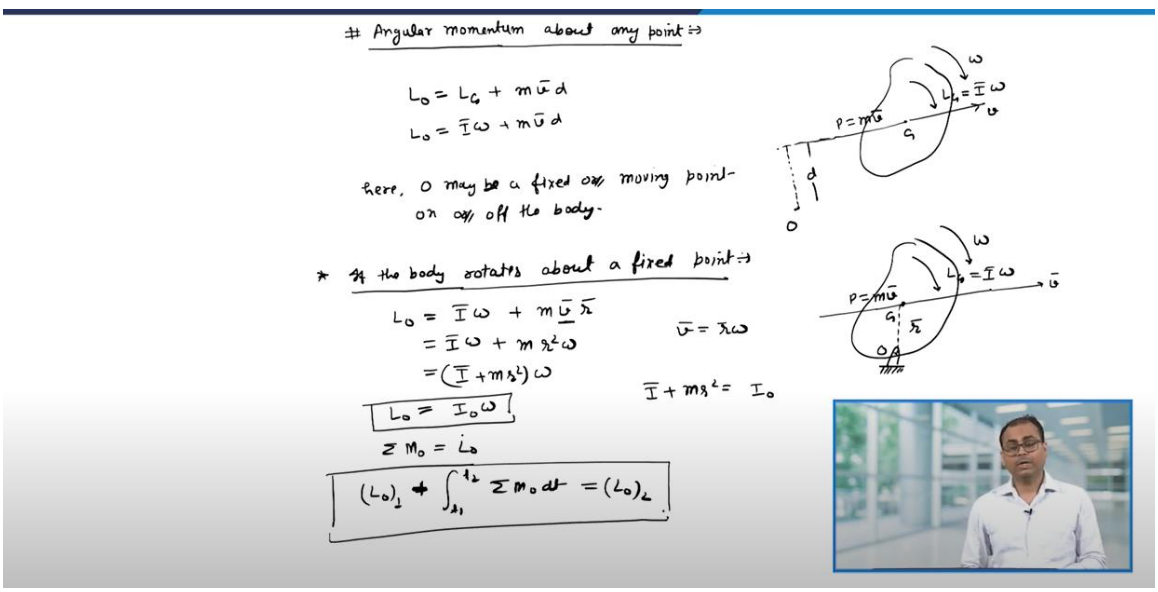



Today, we are going to look at the impulse momentum equation. So, we already know that for a particle, the impulse momentum equation is $P_1 + \int_{t_1}^{t_2} \Sigma F dt \ = P_2$. Now, we also know that the angular momentum of a rigid body is defined as the moment of linear momentum. So, it is denoted by either $H \, or \, L$ , I will prefer $L = \vec{r} \times \vec{P}$ where $P$ is the linear momentum. Now, we also know that rate of change of angular momentum $\frac{dL}{dt} = \Sigma M$. So, this I can rewrite as $\Sigma M_G = \dot{L_G}$ where dot represents the rate of change of that quantity.

So, here this G represents mass center and also the angular momentum about the mass center is $L_G = \overline{I}\,\omega$ where $\overline{I}$ is the moment of inertia of the rigid body about the center of mass. Let us call this as equation number (1). Then similar to the impulse momentum equation for the particle, we have the impulse momentum equation for the rigid body and it is given by $(L_G)_1 + \int_{t_1}^{t_2} \Sigma M_G \, dt = (L_G)_2$. Here, $(L_G)_1$ is the initial angular momentum of the rigid body about the mass center.

$\int_{t_1}^{t_2} \Sigma M_G \, dt$ Is the external angular impulse and $(L_G)_2$ is the final angular momentum again about G. So, this equation tells us that the initial angular momentum about the mass center G plus the external angular impulse about again G equals to the final angular momentum about G. Now, let us say we want to calculate the angular momentum of a rigid body about any point which may be or may not be on the rigid body. So, angular momentum about any point.

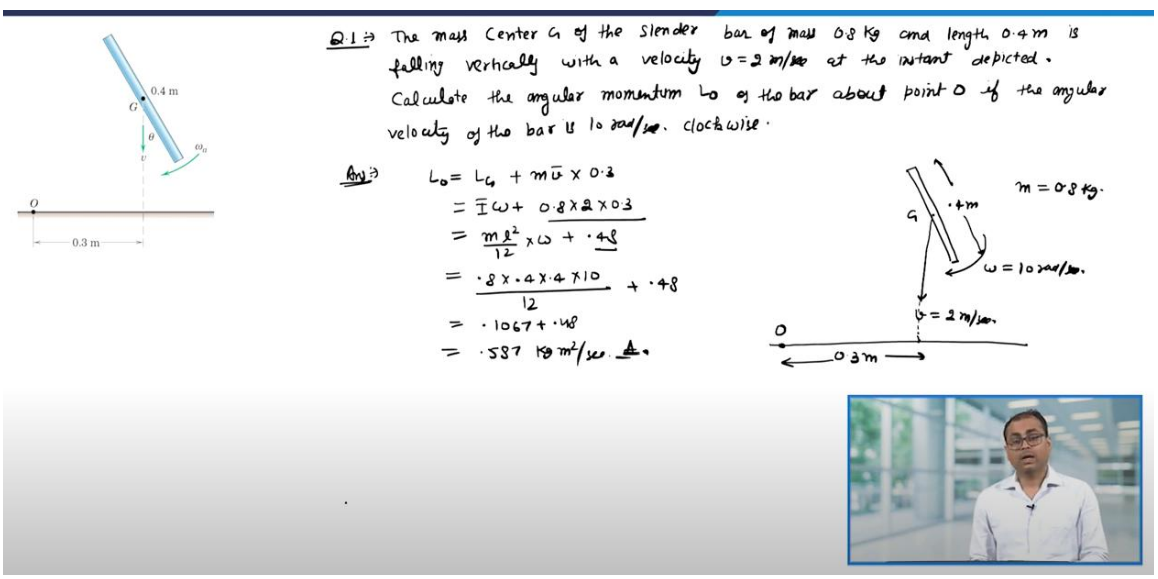
So, let us say we have a rigid body and this rigid body has a linear velocity of $v$. So, therefore, let us say the momentum that is acting on the body is $P = m\overline{v}$. And let us say this body is also rotating with an angular velocity $\omega$. Therefore, the angular momentum about G will be $L_G = \overline{I}\,\omega$ where $\overline{I}$ is the moment of inertia of the rigid body about the mass center G. Let us say I want to calculate the angular momentum about a point $O$ which is at a distance of $d$ from the line of action of P. Then the angular momentum about O will be $L_O = L_G + m\overline{v}d$. The moment is $mv$ and its momentum is you have to multiply by the perpendicular distance which is $d$. So, therefore, $L_O = \overline{I}\omega + m\overline{v}d$.




So, you have to note that here this O may be a fixed point or a moving point which may be on or off the body. So, this point may be on the body as you have seen, or it may be off the body. Now, let us say this body rotates about a fixed point. So, if the body rotates about a fixed point. So, again we have this rigid body. Let us fix some point on it. So, let us say this body is rotating about this fixed point.

So, we have the linear momentum $P = m\overline{v}$ and since this body is rotating with an angular velocity $\omega$ . Therefore, this has an angular acceleration $L_G = \overline{I}\,\omega$ and let us say the perpendicular distance is r of the mass center G and let us say this point is O. So, therefore, the angular momentum $L_O$ will be $\overline{I}\,\omega$ which is the angular momentum about the mass center $G$ plus the moment of momentum. So, we have $m\overline{v}\,\overline{r}$. Now, we know that because this is a fixed axis rotation. So, therefore, $\overline{v} = \overline{r}\omega$. So, therefore, $L_O = \overline{I}\omega + mr^2\omega$. So, $mr^2\omega$ or $\omega$ is common.

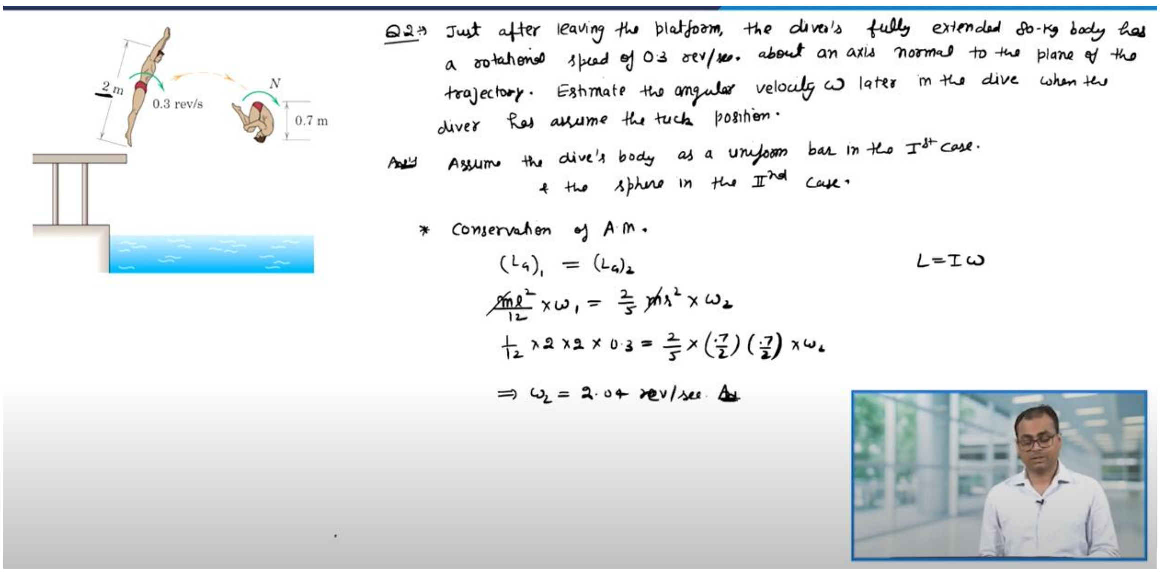
So, I can take $L_O = \left(\overline{I} + mr^2\right)\omega$. Now, note that what is $\overline{I} + mr^2$? $\overline{I} + mr^2 = I_O$, okay. So, you can use the parallel axis theorem. Therefore, I can write down $L_O = I_O\omega$, okay. We have $\Sigma M_O = \dot{L_O}$. Therefore, the moment impulse equation becomes for point O, it is the $(L_O)_1 + \int_{t_1}^{t_2} \Sigma M_O\, dt = (L_O)_2$. So, this is the equation of impulse momentum about the point O.




Now, based on this, let us look at some problems. And the first problem statement is following. The mass center G of the slender bar of mass 0.8 kg and length $0.4\ m$ is falling vertically with a velocity $v = 2\ \frac{m}{sec}$ at the instant depicted. Calculate the angular momentum $L_O$ of the bar about point O if the angular velocity of the bar is $10\ \frac{rad}{sec}$ in the clockwise direction. So, we have this rod.

It is falling with a velocity $v = 2\ \frac{m}{sec}$ and it has a initial angular velocity $\omega = 10\ \frac{rad}{sec}$. Its mass $m = 0.8\ kg$ and the length is $0.4\ m$ , okay. And we have to calculate its angular momentum about point O. And point O is at a distance of $0.3\ m$ from the line of action of $v$ . $L_O = L_G + m\bar{v} \times 0.3$. Now $L_O = \bar{I}\omega + 0.8 \times 2 \times 0.3$.

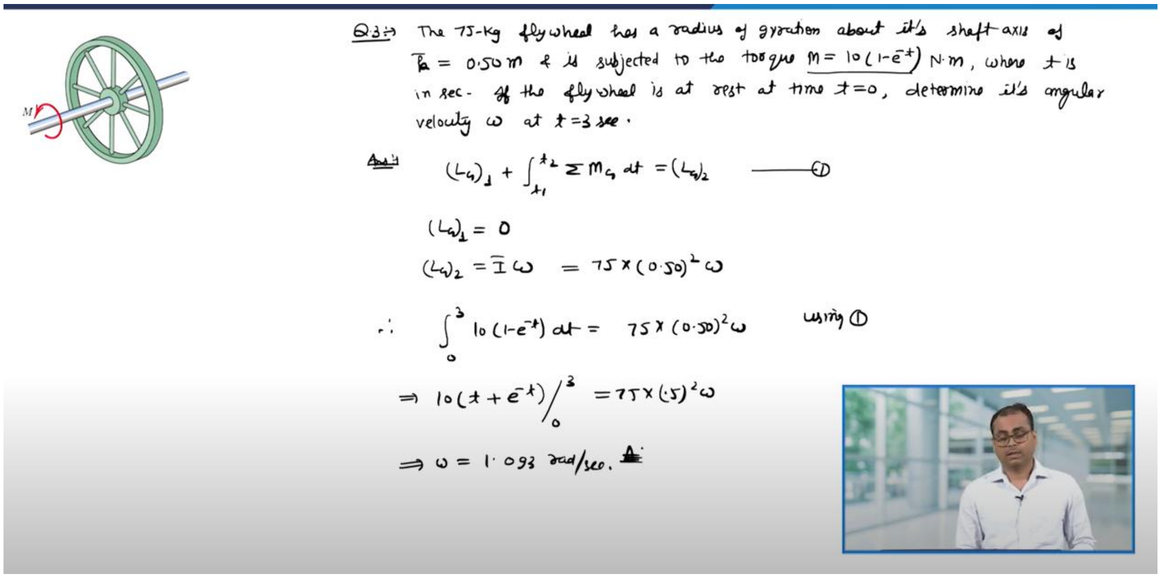
Now, $\bar{I}$, $\bar{I}$ is the moment of inertia of the rod about its center of mass and we know that it is $\frac{ml^2}{12}$. So, $\frac{ml^2}{12} \times \omega + 0.48$. So, let us put the value $m = 0.8$ , $length = 0.4$ , $\omega = 10$ . So, this comes out $L_O = \frac{0.8 \times 0.4 \times 0.4 \times 10}{12} + 0.48 = 0.1067 + 0.48$ and this is $L_O = 0.587\ \frac{kg-m^2}{sec}$.




Now, let us look at the another problem statement. Just after leaving the platform. The diverse fully extended 80 kg body has a rotational speed of $0.3\ \frac{rev}{sec}$ about an axis normal to the plane of the trajectory. Estimate the angular velocity $\omega$ later in the dive when the diver has assumed the tuck position. So, here we can assume that the diver's body as a uniform bar in the first case and the sphere in the second case.

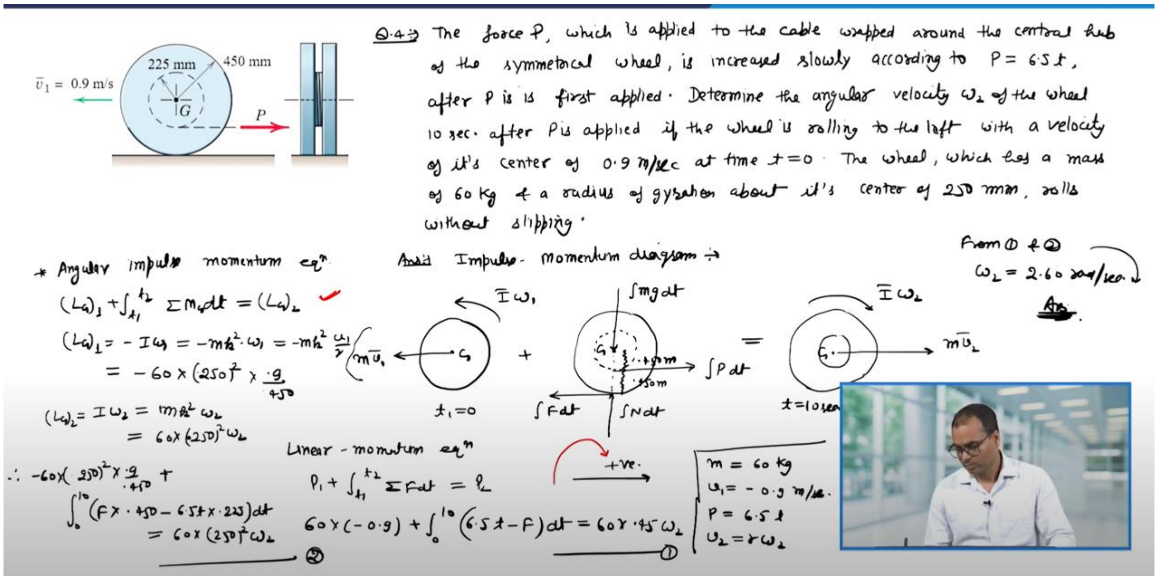
So, for this assumption, we can apply the conservation of angular momentum because no external angular impulse is applied. So, let us use the conservation of angular momentum. So, we have $(L_G)_1 = (L_G)_2$. So, $L$ is the angular momentum and that is $L = I\omega$ . So, we have in the first case $\frac{ml^2}{12} \times \omega_1 = \frac{2}{5} mr^2 \times \omega_2$ because $L = I\omega$ .

So, let us put the values $m$ will get cancelled because the mass is the same in both the cases. So, we have $\frac{1}{12} \times 2 \times 2 \times 0.3 = \frac{2}{5} \times \left(\frac{0.7}{2}\right)\left(\frac{0.7}{2}\right) \times \omega_2$. r is the radius the diameter is given it is $d = 0.7m$ . So, therefore, the radius will be $r = \frac{0.7}{2}$. And this gives $\omega_2 = 2.04\ \frac{rev}{sec}$. So, you can see that the omega has increased.




Now, let us look at another problem statement. The 75 kg flywheel has a radius of gyration about its shaft axis of $k = 0.50\ m$ and is subjected to the torque $M = 10(1 - e^{-t})\ N.m$, where t is in second. If the flywheel is at rest. At time $t = 0$ , determine its angular velocity $\omega$ at $t = 3\ sec$.

So, again for this we can use the impulse momentum equation. So, we have $(L_G)_1 + \int_{t_1}^{t_2} \Sigma M_G\, dt = (L_G)_2$. Here, it is given that initially, the flywheel is at rest. So, therefore, $(L_G)_1 = 0$. Now $(L_G)_2 = \bar{I}\omega$, where $\bar{I}$ is the moment of inertia and it is given that the radius of gyration $k = 0.50\ m$ , okay. So, therefore $I = mk^2$. So $(L_G)_2 = 75 \times (0.50)^2\omega$, okay. So, let us use equation number (1). So, therefore, we have $\int_0^3 10(1-e^{-t})dt = 75 \times (0.50)^2\omega$. This is using equation number (1). So, that gives you $[10(t+e^{-t})]_0^3 = 75 \times (0.50)^2\omega$ and that gives you $\omega = 1.093\ \frac{rad}{sec}$.




Now, let us look at one more question. And the problem statement is following. The force P which is applied to the cable wrapped around the central hub of the symmetrical wheel is increased slowly according to $P = 6.5t$ , where t is in second.

After P is first applied, determine the angular velocity $\omega_2$ of the wheel 10 $sec$ after P is applied if the wheel is rolling to the left with a velocity of its center of 0.9 $\frac{m}{sec}$ at time $t = 0$ . It is given that the wheel which has a mass of 60 kg and a radius of gyration about its center of $250mm$ rolls without slipping. To solve this question, let us look at the impulse momentum diagram of the wheel. That is what is the linear momentum, angular momentum, linear impulse and angular impulse which are acting on the wheel. So, initially we have this wheel which is moving with a linear momentum of $m\bar{v}_1$ and this wheel is also rotating.

So, therefore, there will be angular momentum which is equal to $\bar{I}\omega_1$. This is the situation at time $t_1 = 0$. Then let us look at the linear and angular impulse which are acting on it. So, we have this wheel and this is the hub.

So, first of all its weight is going to act. So, because of the weight the linear impulse will be $\int mgdt$ which is $\int Fdt$. And to balance this, there will be a normal force. So, that will be $\int Ndt$ and we are applying an external force.

So, therefore, the impulse will be $\int Pdt$ and also, there will be a frictional force, okay, the rolling resistance and that is equal to let us say $\int Fdt$, okay. And because of that, we have the final situation, okay. So, we have this hub, center of mass is G, and this moves with a velocity of $v_2$.

So, therefore, linear momentum will be $m\overline{v_2}$ and it rolls with angular velocity of $\omega_2$. So, therefore, angular momentum will be $\bar{I}\omega_2$. This is the situation at $t = 10sec$. So, we can use both the linear impulse momentum equation and the angular impulse momentum equation. So, first let me apply the linear momentum equation which is $P_1 + \int_{t_1}^{t_2} \Sigma F dt = P_2$. Let me take this direction as positive. Now, all the parameters are given. For example $m = 60\ kg$ . $v_1 = -0.9\ \frac{m}{sec}$, $P = 6.5t$ and $v_2 = r\omega_2$.

So, let us put this over here. So, we have $60 \times (-0.9) + \int_0^{10} (6.5t - F)dt = 60 \times 0.45 \times \omega_2$. Let me call this equation number (1). Similarly, we have angular impulse momentum equation. Okay. So, that is $(L_G)_1 + \int_{t_1}^{t_2} \Sigma M_G\, dt = (L_G)_2$. Now, $(L_G)_1 = I\omega_1$. Okay. But its direction has changed.

So, let us write down $(L_G)_1 = -I\omega_1$ that is $(L_G)_1 = -mk^2\omega_1$ and $\omega = \frac{v_1}{r}$, okay. So, therefore, this becomes $-mk^2\frac{v_1}{r}$, okay. And we can put the values. So, this comes out to be $(L_G)_1 = -60 \times (0.250)^2 \times \frac{0.9}{0.450}$. And $(L_G)_2 = I\omega_2 = mk^2\omega_2$ and again we have the values. So, $(L_G)_2 = 60(0.250)^2\omega_2$. Now, for the rotation, let me take this direction as positive direction and put these values over here. So, we have $-60 \times (0.250)^2 \times \frac{0.9}{0.450} + \int_{t_1}^{t_2} \Sigma M_G\, dt$. So, $M_G$ is the moment.

So, that will be equal to integral 0 to 10. We have the force F which is acting at a distance of $0.450m$ . So, $F \times 0.450$ , this is the moment minus we have force P, which is acting at a distance of $0.225m$ . So $-60 \times (0.250)^2 \times \frac{0.9}{0.450} + \int_0^{10}(F \times 0.450 - 6.5t \times 0.225)dt = (L_G)_2$., And $(L_G)_2 = 60 \times (0.250)^2 \times \omega_2$. So, we have $-60 \times (0.250)^2 \times \frac{0.9}{0.450} + \int_0^{10}(F \times 0.450 - 6.5t \times 0.225)dt = 60 \times (0.250)^2 \times \omega_2$ . Let us call this equation number $(2)$. Now, we have equation number $(1)$ and equation number $(2)$ and this equation has two parameters and there are two equations. Therefore, from $(1)\ and\ (2)$, we can find the value of $\omega_2$ and $\omega_2 = 2.60\ \frac{rad}{sec}$, which is in the clockwise direction. With this, let me stop here. See you in the next class. Thank you.