MECHANICS
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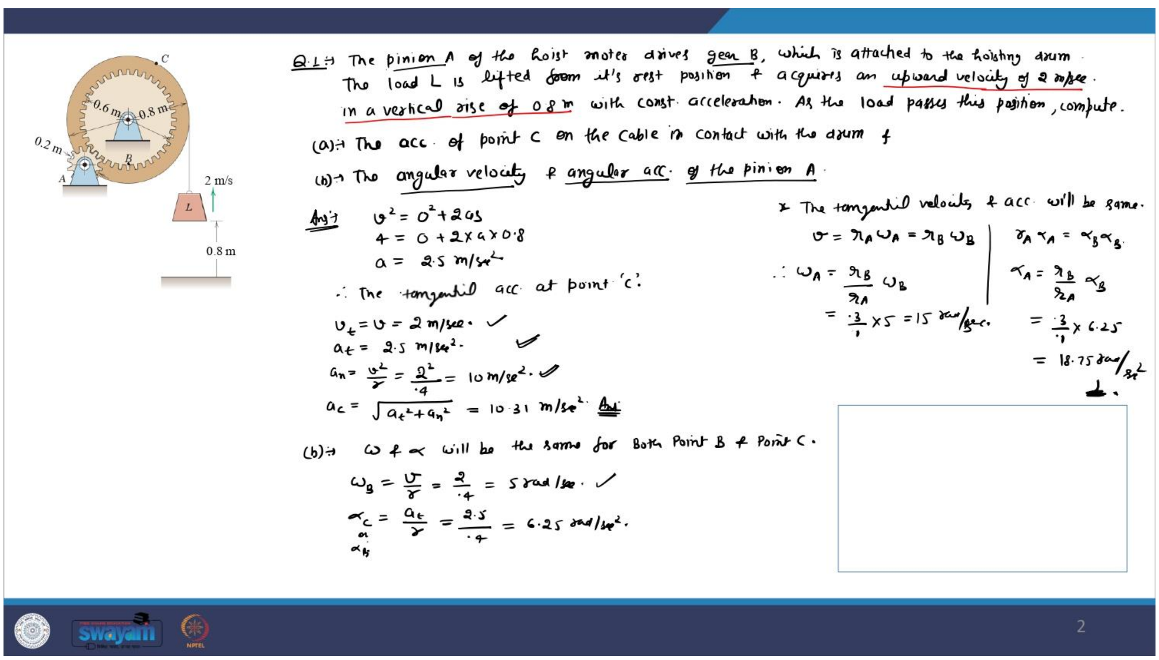
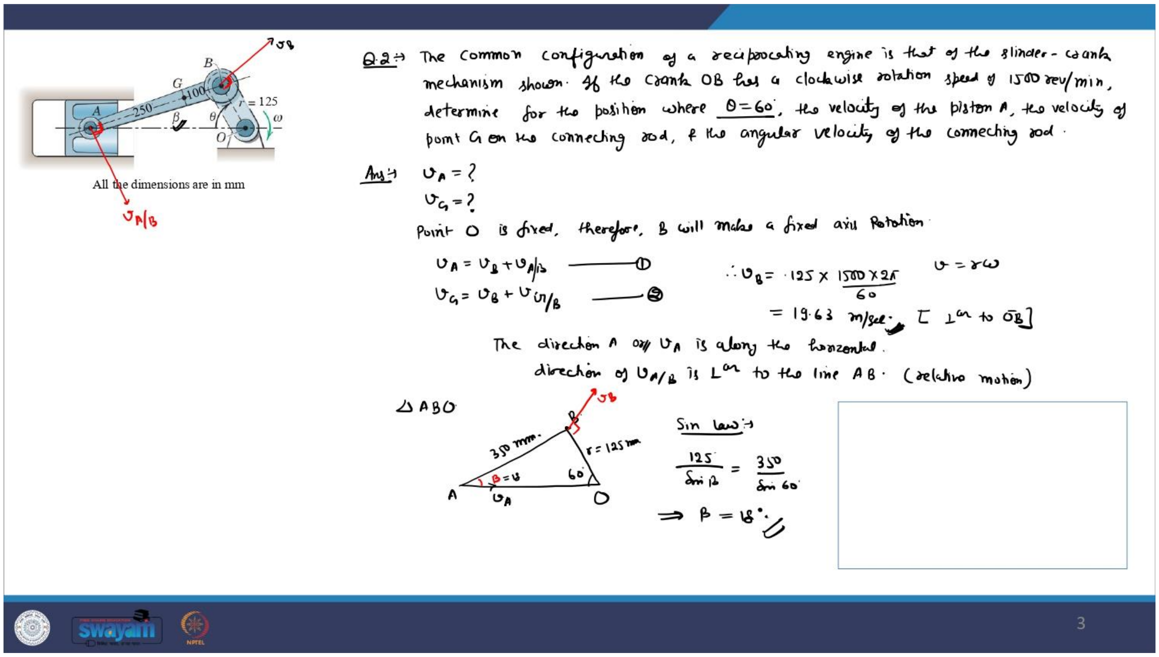
Lecture 53
Translation and rotation of rigid bodies: examples

Hello everyone, welcome to the lecture again. In the last lecture, we discussed about the translation and rotation of rigid bodies. We saw that the general plane motion can be considered as the combination of the translation and rotation about a fixed point. We also discussed about the relative velocity. Today, we will discuss few examples of the relative velocity and also discuss the relative acceleration.



So, let me write down the first problem statement is following. The pinion A of the hoist motor drives gear B which is attached to the hosting drum. The load L is lifted from its rest position and acquires an upward velocity of $2\ m/sec$ in a vertical rise of $0.8\ m$ with constant acceleration. As the load passes this position, compute a) the acceleration of point C on the cable in contact with the drum and b) the angular velocity and angular acceleration of the pinion A. So, first of all, it is given the load L moves in the upward direction with a velocity of $2\ m/sec$ in a vertical rise of $0.8\ m$. From here, we can find out what is the velocity of the load at that instant. So, let us do that. We know that $v^2 = u^2 + 2as$. Now, $4 = 0 + 2 \times a \times 0.8$ and from here we get the acceleration. So, this is the acceleration. It comes out to be $a = 2.8\ m/sec^2$. This is the acceleration of the load. Now, this load is connected by a cable. Therefore, the acceleration of the cable or rather the tangential acceleration of the cable will be $a_t = 2.5\, m/sec^2$. So, this is the tangential acceleration at point C and the same is true for the velocity also. The tangential velocity at C will be $2\, m/sec$. So, let me write down the tangential velocity let me call it v because there is no radial velocity will be $v_t = v = 2\, m/sec$ and the tangential acceleration at point C will be $a_t = 2.5\, m/sec^2$. Now, from this velocity, I can find out what is the normal acceleration. Now, note that the normal acceleration a_n still at point C will be $a_n = \frac{v^2}{r}$. Now, v is given, it is 2. So, therefore, this will be $\frac{2^2}{r}$. r is the radius of the gear. It is given that the diameter is $0.8\, m$. So, this is the diameter. Therefore, the radius will be $r = 0.4$ and that comes out to be $a_n = \frac{2^2}{0.4} = 10\, m/\text{sec}^2$. Therefore, the acceleration of point C will be $a_C = \sqrt{a_t^2 + a_n^2}$ and a_t and a_n is known to us. So, that comes out to be $a_C = 10.31\, m/\text{sec}^2$. Now, in part b, we have been asked to find out the angular velocity and angular acceleration of the pinion A. Note that the pinion A is connected inside the gear. So, therefore, to find out the velocity and acceleration, first we have to study what is happening at point B, which is inside the gear. So, note that the ω, the angular velocity and the angular acceleration α will be the same for both point B and point C. So, therefore, the ω_B or the $\omega_C = \frac{v}{r}$. Now, the velocity at point C is known, it is 2 and the radius at C is 0.4. So, that comes out to be $\omega_B = \frac{v}{r} = \frac{2}{0.4} = 5\, rad/sec$. Similarly, the angular velocity at point B or C will be $\alpha_C = \frac{a_t}{r} = \frac{2.5}{0.4} = 6.25\, rad/\text{sec}^2$. Now, note that the gear B drives the pinion A. Therefore, the tangential velocity and the acceleration will be the same. So, velocity $v = r_A\omega_A = r_B\omega_B$ and the acceleration $r_A\alpha_A = r_B\alpha_B$. So, therefore, $\omega_A = \frac{r_B}{r_A}\omega_B$. And ω_B is known. r_B and r_A is also given in the figure. So, that comes out to be $\omega_A = \frac{0.3}{0.1} \times 5 = 15\, rad/sec$. Now, $\alpha_A = \frac{r_B}{r_A}\alpha_B = \frac{0.3}{0.1} \times 6.25 = 18.75\, rad/\text{sec}^2$.

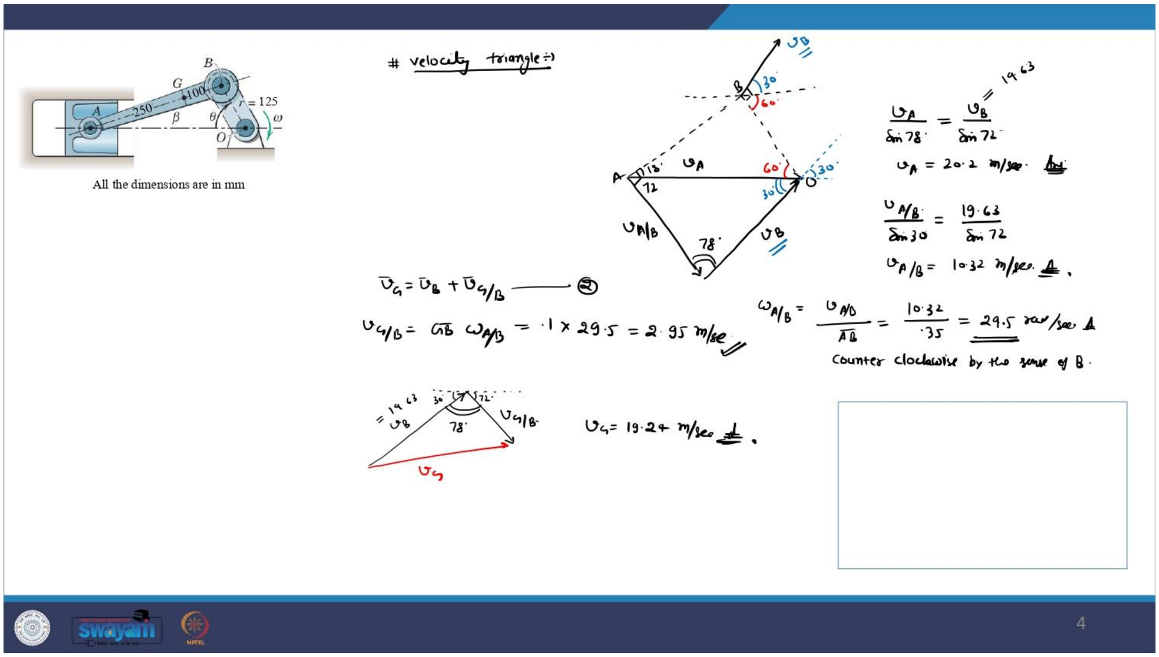
Now, let us look at another problem statement. The common configuration of a reciprocating engine is that of the cylinder crank mechanism shown. If the crank OB has a clockwise rotation, speed of $1500\, rev/min$, determine for the position where $\theta = 60°$, the velocity of the piston A, the velocity of point G on the connecting rod and the angular velocity of the connecting rod. So, in this question, we have been asked to find out what is v_A and what is v_G. So, first of all, we have to note that the point O here is fixed and therefore, point B is making a fixed axis rotation. So, if it is making a fixed axis rotation that at any instant, the velocity of point B will be perpendicular to OB. So, therefore, this angle has to be 90°. Let me write this down. Point O is fixed. Therefore, B will make a fixed axis rotation. And therefore, the velocity of point B will be perpendicular to OB. Now, if an observer is fixed at point B, then he will see as if point A is making a fixed axis rotation. So, from the prospect of B, point A is making a fixed axis rotation.



Therefore, the angle of v_A with respect to B will be perpendicular to the connecting rod AB. Now, let us look at the velocity of point A. $v_A = v_B + v_{A/B}$ $-----$ (1). Similarly, $v_G = v_B + v_{G/B}$ $-----$ (2). Velocity of B can be very easily find out $v = r\omega$. So, therefore, $v_B = 0.125 \times 1500 \times \frac{2\pi}{60}$=19.63 m/sec. And as I said, the direction of v_B will be perpendicular to OB. Now, what is the direction of A? The direction of A or let me call it v_A is along the horizontal. Because this piston A is moving in the horizontal direction and the direction of $v_{A/B}$, so this is again I said that the direction of $v_{A/B}$ is perpendicular to the line AB. And this is because of the relative motion. This is how point B will perceive point A. Now, let us look at the triangle ABO. So, we have the velocity of A and we have this point O and then we have point B. It is given that this length is 350 mm. This angle is 60°, and the arc is 125 mm. Note that the velocity will be perpendicular to OB. So, this is the velocity of B. Now, from here, I can find out what is this angle β. This angle β. So, for that, I can use the sin law which is $\frac{125}{sin\beta} = \frac{350}{sin60}$ and from here I get $\beta = 15°$.

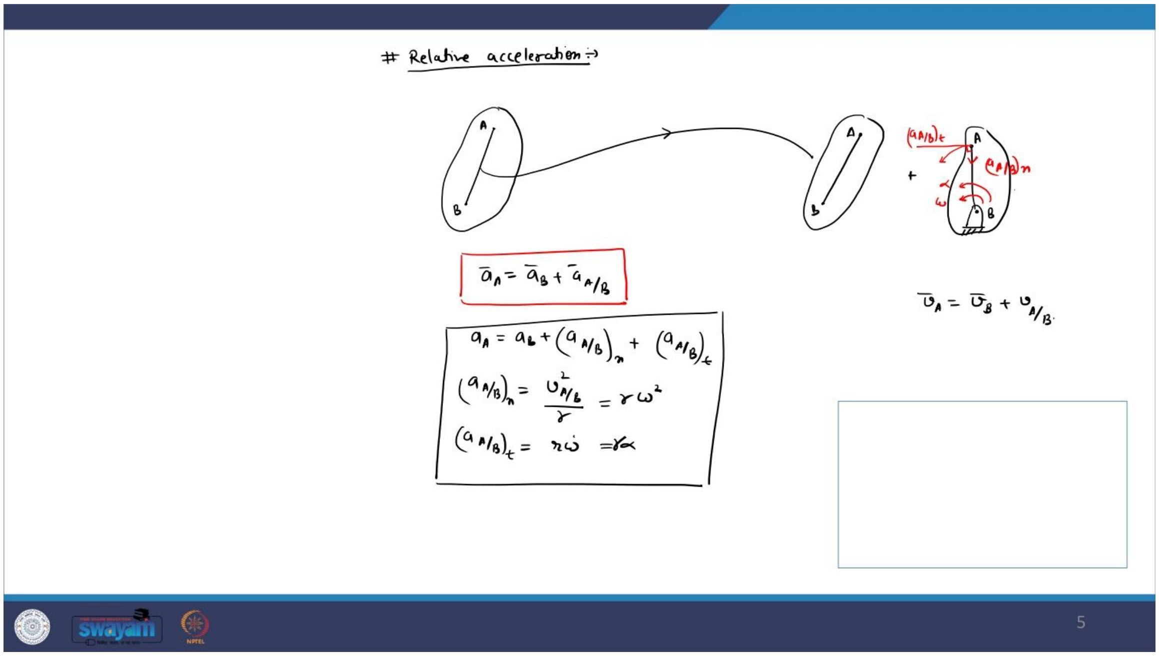
Now, since one of the velocity v_B is known and also we know the angle β, let us look at the velocity triangle to find out what is v_A and what is $v_{A/B}$. So, we are looking at the velocity triangle. So, some of the geometry is involved. So, we have the velocity of point A in the horizontal direction. AB rod is making an angle of 18°. So, this is something that we find out and $v_{A/B}$ is perpendicular to AB. So, therefore, that angle will be 72°. So, we have v_A, $v_{A/B}$. Now, we have v_B. Now, note that this v_B is making an angle of 90° with respect to OB and this angle is given it is 60°. So, if this angle is 60° therefore, that angle will be 60° and therefore, this angle will be 30°. And since this is also v_B, that is also v_B.

So, they are supposed to be parallel. Therefore, this angle will also be 30° and so is this angle. And from the triangle, total sum should be 180°.



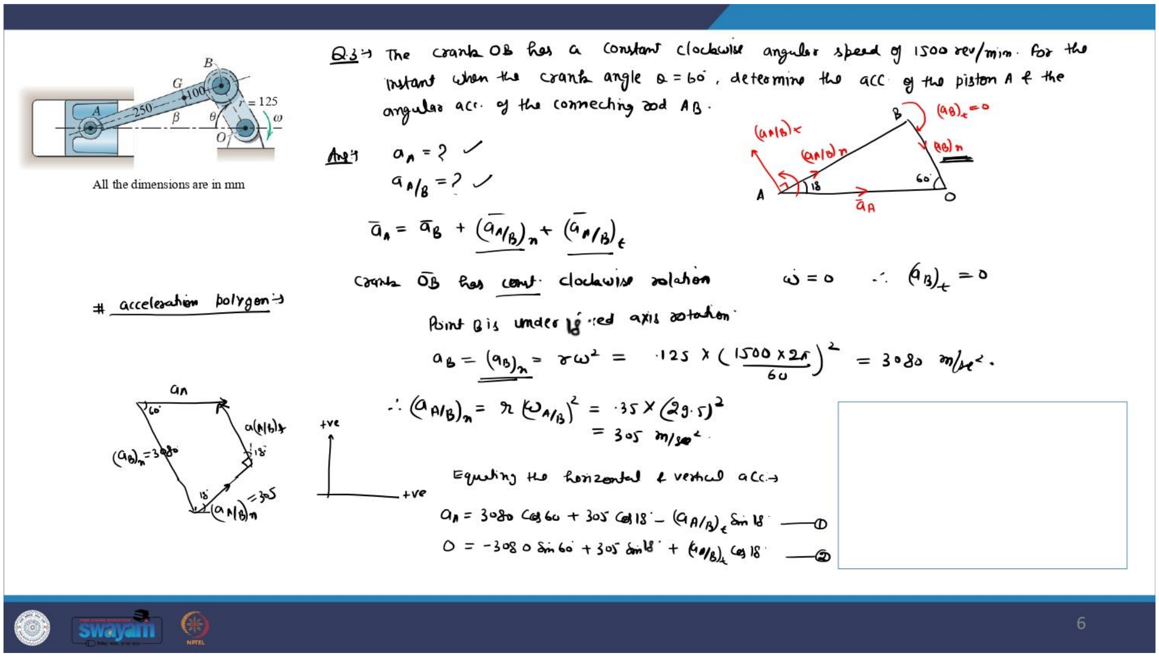
Therefore, this comes out to be 78°. Now, we can use the sin rule to find out, you know, all the velocities. So, we have $\frac{v_A}{sin78} = \frac{v_B}{sin72}$. And the velocity of B is known, this is $v_A = 20.2\ m/sec$. Now, we can also use $\frac{v_{A/B}}{sin30} = \frac{19.63}{sin72}$ and that gives me $v_{A/B} = 10.32\ m/sec$. Therefore, we can find out what is $\omega_{A/B}$, the angular velocity of A with respect to B. It will be $\omega_{A/B} = \frac{v_{A/B}}{\overline{AB}} = \frac{10.32}{0.35} = 29.5\ rad/sec$. And note that since B is rotating in the clockwise direction, therefore, B will see as if A is rotating in the counterclockwise direction. So, this will be in the counterclockwise by the sense of B. So, this is how B will sense. We have been asked to find out what is v_G. So, $v_G = v_B + v_{G/B}$. $v_{G/B} = \overline{GB}\omega_{A/B} = 0.1 \times 29.5 = 2.95\ m/sec$. So, now we know $v_{G/B}$, we also know v_B. So, in principle we can calculate what is v_G, but note that this is a vector sum. So, you have to take care of the direction also. So, we have v_B which is equal to 19.63 and this makes a 30° angle from the horizontal and we have $v_{G/B}$. So, this is my v_B. We have $v_{G/B}$ and that makes an angle of 72° from the horizontal. So, therefore, this angle will be 78° and from here I can find out what is v_G. So, that will be square root this square plus that square plus $2v_Bv_G$ and the angle between them. So, that gives you $v_G = 19.24\ m/sec$.

Now, let us understand the concept of relative acceleration. We saw that if a rigid body is making a general plane motion then the motion can be analyzed as a combination of a translation and a fixed axis rotation about a point. So, let us say that point is B. So, it is the combination of the translation and the fixed axis rotation about B. And we also saw that the velocity of point A will be the velocity of point B plus velocity of A with respect to B.



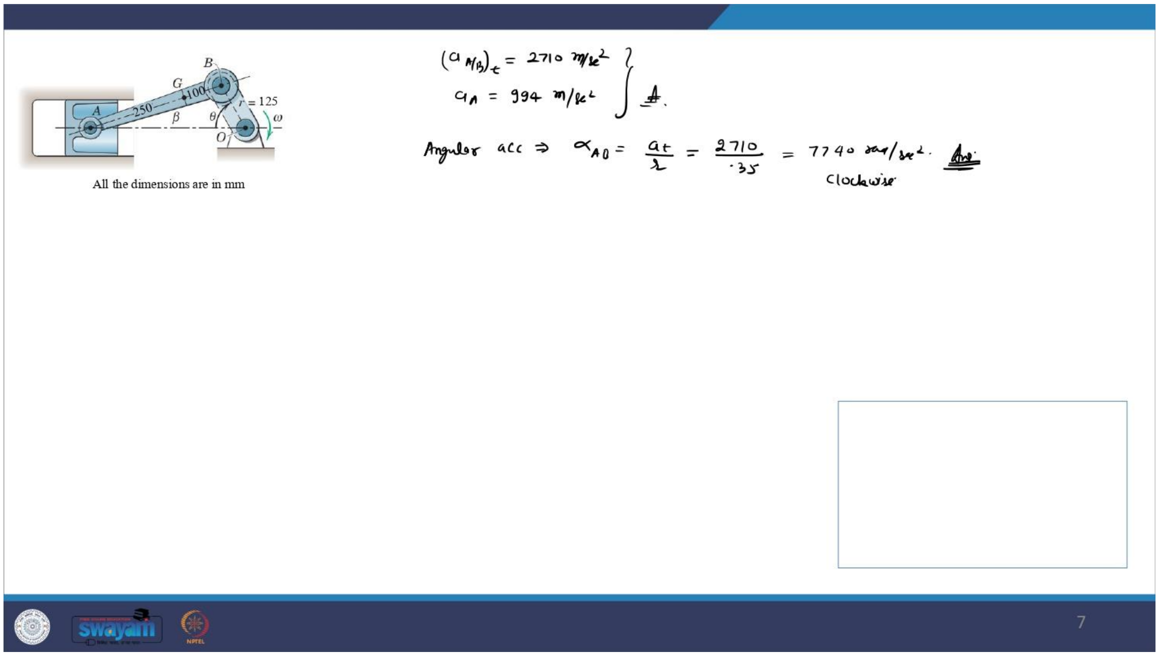
Now, if we differentiate it, then we get the acceleration of point A equal to the acceleration of point B plus the acceleration of point A with respect to B and all of them are vector quantities. Now, note that this acceleration of point A with respect to B will have two components. It will have the normal component and the tangential component. So, $a_A = a_B + a_{A/B} = a_B + \left(a_{A/B}\right)_n + \left(a_{A/B}\right)_t$. And what is the value of the normal component? $\left(a_{A/B}\right)_n = \frac{v_{A/B}^2}{r} = r\omega^2$ and $\left(a_{A/B}\right)_t = r\dot{\omega} = r\alpha$. Let us look at the direction. So, let us say ω is like this, α is also like that, then $\left(a_{A/B}\right)_n$ will be towards AB and $\left(a_{A/B}\right)_t$ will be perpendicular to AB.

Now, based on this concept, let us look at a problem statement and the previous question is repeated here. So, the crank OB has a constant clockwise angular speed of $1500\ rev/min$ for the instant when the crank angle $\theta = 60°$, determine the acceleration of the piston A and the angular acceleration of the connecting rod AB. So, we have been asked what is the acceleration of piston A and what is a with respect to B. Now, just now we saw that $a_A = a_B + \left(a_{A/B}\right)_n + \left(a_{A/B}\right)_t$, and all of them are vector quantities. Here, the crank OB has a constant clockwise rotation. So, since it is constant, therefore, $\dot{\omega} = 0$, and therefore, $(a_B)_t = 0$. So, there is no tangential acceleration of point B. So, therefore, point B is only under fixed axis rotation. So, we have a_B only the n component because this a_B can also have a_t and a_n, but $a_t = 0$. So, $a_B = (a_B)_n = r\omega^2 = 0.125 \times \left(1500 \times \frac{2\pi}{60}\right)^2 = 3080\, m/\mathrm{sec}^2$. So, therefore, $\left(a_{A/B}\right)_n$ can be find out now. So, $\left(a_{A/B}\right)_n = r\left(\omega_{A/B}\right)^2 = 0.35 \times (29.5)^2 = 305\, m/\mathrm{sec}^2$.



So, now we know $(a_B)_n$ and if you find out $(a_B)_t$, then we know what is a_A and what is $a_{A/B}$. For that, let us look at the geometry of the acceleration. So, we have O here, A here and we have B. The acceleration of A will be along AO. So, this is the direction of the acceleration. Point B is making a fixed axis rotation like that. So, therefore, the direction of $(a_B)_n$ will be towards BO and there is no $(a_B)_t$. To point B, point A is rotating in the anti-clockwise direction and the component of $\left(a_{A/B}\right)_n$ will be along AB and $\left(a_{A/B}\right)_t$ will be perpendicular to AB. And also note that we have already determined that this angle is 18° and this angle was given that it is 60°. So, now let us look at the acceleration polygon because we know some of the accelerations and the angle between them. So, we have the acceleration A which is in the horizontal direction, then at 60°, we have $(a_B)_n$. So, this is 60° and this one is $(a_B)_n$. Its value, we have find out, the value is 3080. And at 18°, we have $\left(a_{A/B}\right)_n$ and its value also we have find out it is 305. This is 18° and then at 90° to it we have $(a_B)_t$. So, this is $\left(a_{A/B}\right)_t$ and this angle will also be 18° from the geometry. So, now let us equate the horizontal and vertical component so that we can find out you know all the values. So, let me take this as positive direction and that as positive direction and I am equating the horizontal and vertical accelerations. So, $a_A = 3080\, cos60 + 305\, cos18 - \left(a_{A/B}\right)_t sin18 - - - - - (1)$ and the vertical component is $0 =$

$-3080\ sin60 + 305\ sin18 + \left(a_{A/B}\right)_t cos18 - - - - - (2)$. So, now we have two equations and there are two unknown.



From here, we can find out that $\left(a_{A/B}\right)_t = 2710\ m/\sec^2$ and $a_A = 994\ m/\sec^2$. And the angular acceleration of AB, which is $\alpha_{AB} = \frac{a_t}{r} = \frac{2710}{0.35} = 7790\ rad/\sec^2$ in the clockwise direction.

With this, let me stop here. See you in the next class. Thank you.