MECHANICS

Prof. Anjani Kumar Tiwari

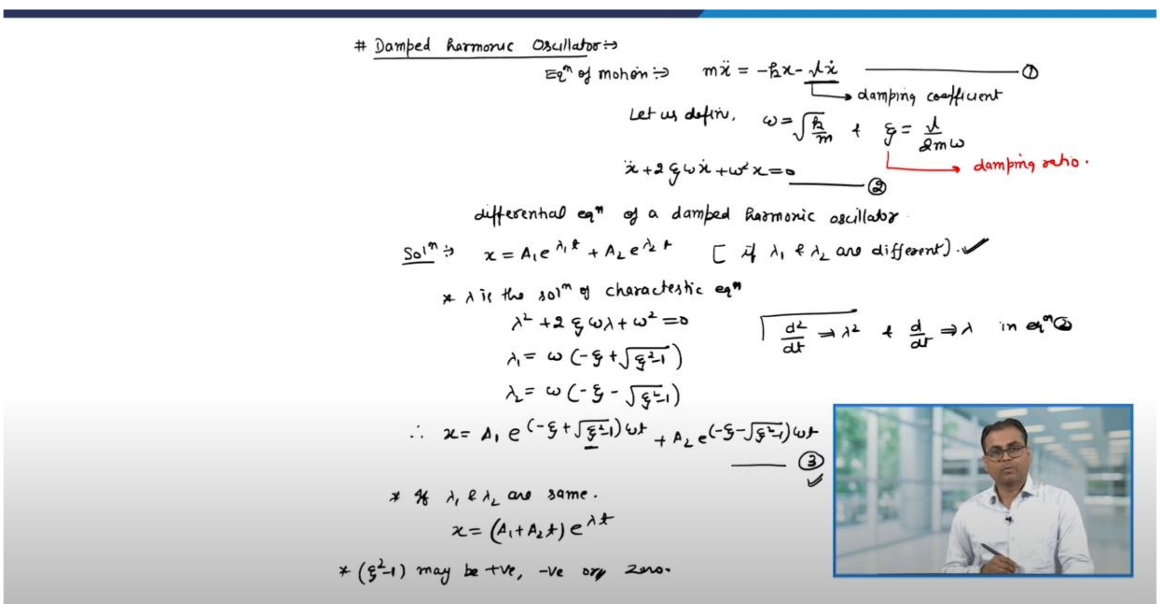
Department of Physics

Indian Institute of Technology, Roorkee

Lecture: 51

Damped harmonic oscillator

Hello everyone, welcome to the lecture again. In the last two classes, we discussed about the simple harmonic oscillator, for example, a spring mass system without any damping. But in real systems, the damping are always present.



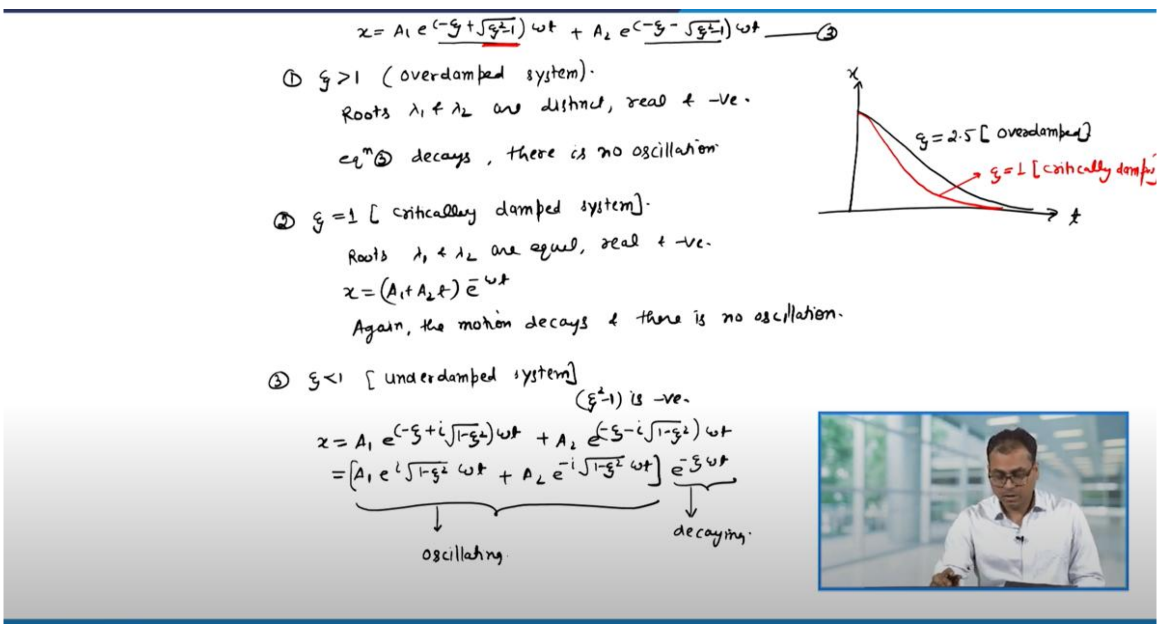
Therefore, let us discuss the damped harmonic oscillator. So, the equation of motion is $F = ma$. So, $m\ddot{x}$ equal to the forces. So, there will be the spring force $-kx$ minus let us say a velocity dependent damping. So, let us say some $\zeta\,\dot{x}$. Here, this ζ is the damping coefficient.

So, this is the simplest type of damping for example, drag because it depends only on the velocity and this $\zeta\,\dot{x}$ is basically the damping force. Let us call this as equation number (1). Now, to solve this equation, let us define as earlier, $\omega = \sqrt{\frac{k}{m}}$ and let us say some $\xi = \frac{\zeta}{2m\omega}$. In that case, I can write down equation number (1) as $\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = 0$.

Here, this ξ is the damping ratio. Now, let me call this as equation number (2) and this is the differential equation of a damped harmonic oscillator. Now, equation number (2) can be solved mathematically, and its general solution is $x = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$ if λ_1 *and* λ_2 are different. Here, this λ is the solution of the characteristic equation $\lambda^2 + 2\xi\omega\lambda + \omega^2 = 0$.

And this equation we get by replacing $\frac{d^2}{dt^2} = \lambda^2$ and $\frac{d}{dt} = \lambda$ in equation number (2). So, we have to solve this equation to find out what are the lambdas. Its solution will be $\lambda_1 = \omega\left(-\xi + \sqrt{\xi^2 - 1}\right)$ and $\lambda_2 = \omega\left(-\xi - \sqrt{\xi^2 - 1}\right)$. So, therefore $x = A_1 e^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega t} + A_2 e^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega t}$. Let me call this equation number (3).

So, this solution we wrote when λ_1 *and* λ_2 are distinct, that is they are different. If λ_1 *and* λ_2 are same, in that case, the general solution is $x = (A_1 + A_2 t)e^{\lambda t}$. Now, this quantity ξ can be more than 1, it can be smaller than 1 or it can be 1. Therefore, $\xi^2 - 1$ may be positive, negative or 0, okay.



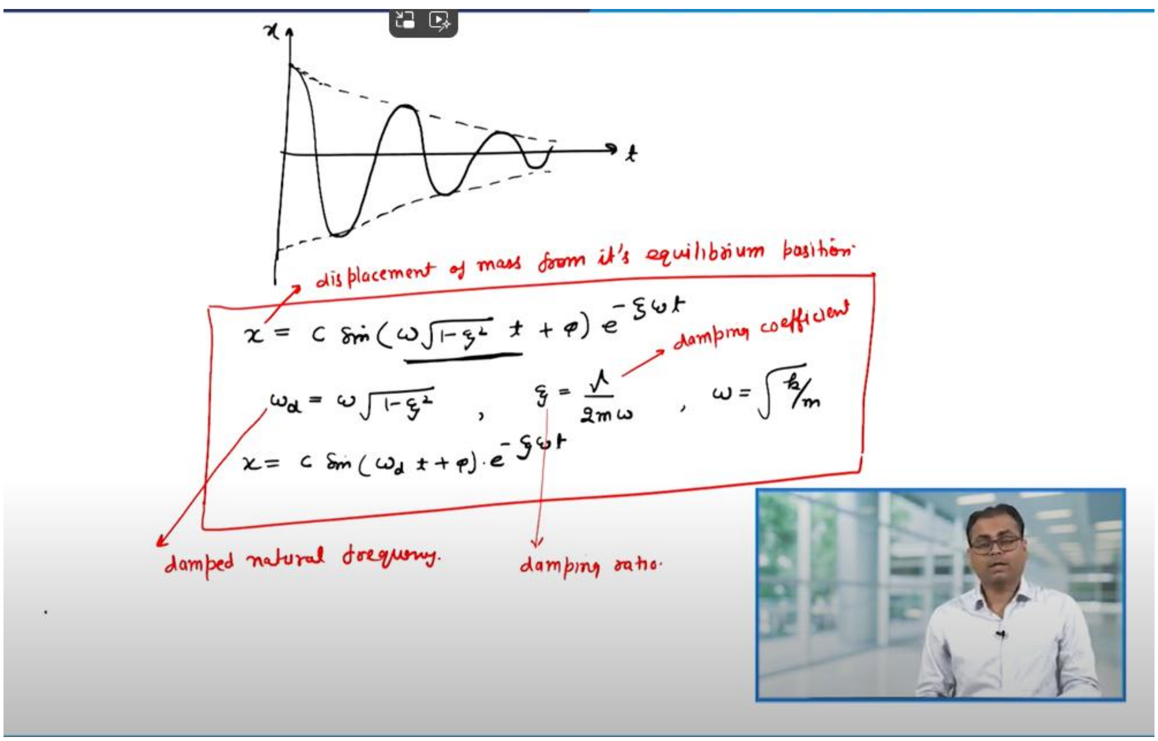
Let me write down equation number (3) once again. We had $x = A_1 e^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega t} + A_2 e^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega t}$. Now, let us look at the case when $\xi > 1$. This is called the overdamped system. In this case, the roots λ_1 *and* λ_2, that is this one and that one, they are distinct, they are different, and they are real and they are negative.

Therefore, equation number (3) suggests that the solution or the value of the x is going to decay exponentially. So, therefore, equation number (3), it decays and there is no oscillation. Or there is no periodic motion. So, let me just plot for some representative value of x_i, equation number (3).

So, let us say this is $t-axis$ and this one is x . So, in this case, this is the plot of x . So, let us say this is for x_i equal to more than 1, let us say 2.5. And of course, this is over-damped system. Now, let us look at the second case when x_i is equal to 1, this is called the condition for critical damping, critically damped system. So, in this case, the roots λ_1 and λ_2, they are equal, they are real, and they are negative. Therefore, its solution will be $x = (A_1 + A_2 t)e^{-\omega t}$. So, again it is an exponentially decaying function. So, the motion decays and again there is no oscillation, okay. So, let me again plot it, okay.

So, this is the case when $\xi = 1$, this is called critically damped system. Now, let us look at the case when $\xi < 1$ that is called the underdamped system. So, in this case because $\xi < 1$ therefore, $\xi^2 - 1$ will be negative and therefore, its square root is going to be imaginary. Therefore, λ_1 and λ_2, they become complex.

So, here as I said, $\xi^2 - 1$ is negative. Therefore, roots are complex. In this case $x = A_1 e^{\left(-\xi + i\sqrt{1-\xi^2}\right)\omega t} + A_2 e^{\left(-\xi - i\sqrt{1-\xi^2}\right)\omega t}$. And this equation I can rewrite as $x = \left(A_1 e^{i\sqrt{1-\xi^2}\omega t} + A_2 e^{-i\sqrt{1-\xi^2}\omega t}\right) e^{-\xi\omega t}$. Now, the quantity inside the bracket is in the form of $e^{i\omega t}$ or $e^{-i\omega t}$ which is an oscillating term and the quantity outside the bracket is in the



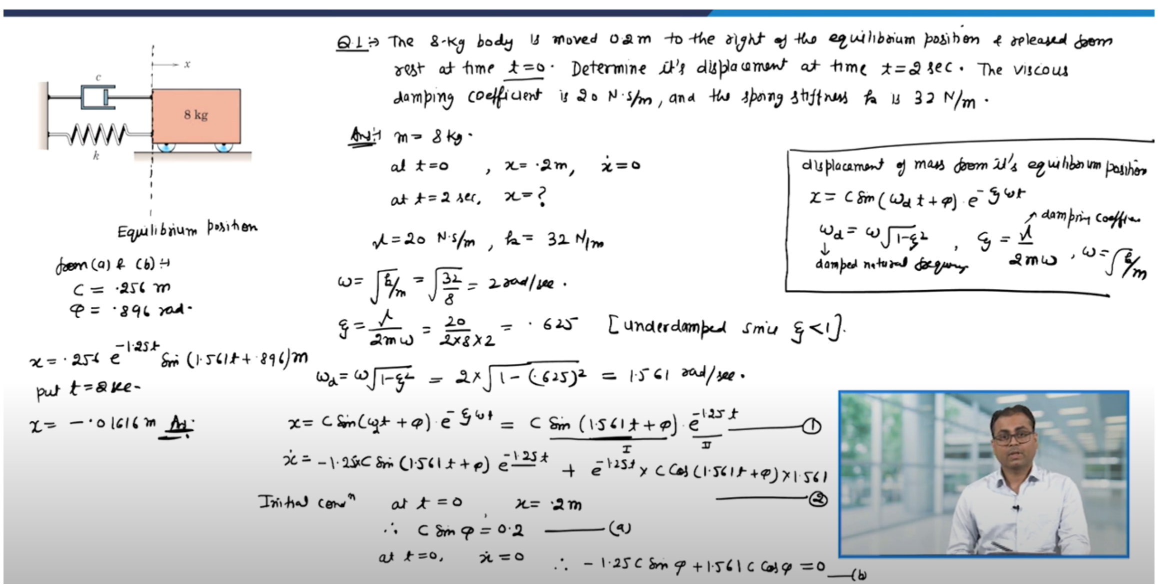
form of $e^{-\omega t}$ which is a decaying term.

Therefore, this time the x is going to decay exponentially and simultaneously it is going to oscillate. So, the solution will look like this. So, this is $t - axis$, this one is x. It will oscillate and it will decay. Now, this same solution I can rewrite in different form.

So, the $x = C\,Sin\left(\omega\sqrt{1-\xi^2}\,t + \varphi\right)e^{-\xi\omega t}$. So, this quantity that you see here is the same as this quantity. Now, let us define $\omega_d = \omega\sqrt{1-\xi^2}$. Then, the above solution can be written as $x = C\,Sin(\omega_d\,t + \varphi)e^{-\xi\omega t}$. And note that what is ξ ?

$\xi = \frac{\zeta}{2m\omega}$, the damping ratio. So, $\xi = \frac{\zeta}{2m\omega}$ and our $\omega = \sqrt{\frac{k}{m}}$. So, just let me write down x is the displacement of the mass from its equilibrium position. ω_d Is the damped natural frequency.

ξ Is the damping ratio. ζ Is the damping coefficient and k is of course the spring constant. Now, let us look at couple of problems based on these concepts.

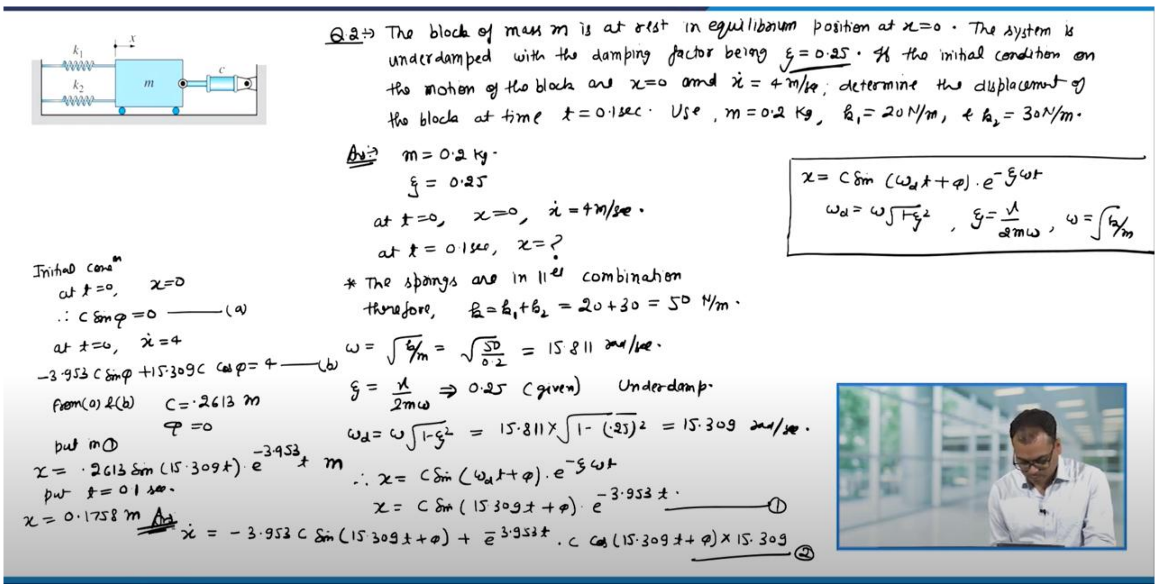


The first problem statement is following. The 8 kg body is moved 0.2 meter to the right of the equilibrium position and released from rest at time $t = 0$. Determine its displacement at time $t = 2\,sec$ and it is given that the viscous damping coefficient is 20 $\frac{Ns}{m}$ and the spring stiffness k is 32 $\frac{N}{m}$, okay. So, here this is the equilibrium position. And it is given that $m = 8\,kg$ and at initial time $t = 0$, the body is released from rest.

So, therefore, $\dot{x} = 0$ and it is moved by 0.2 meter. So, $x = 0.2\,m$ and $\dot{x} = 0$. In the question, we have been asked to find out at $t = 2$ sec, what is x? And it is given that damping coefficient is 20.

So, $\zeta = 20\ \frac{Ns}{m}$ and $k = 32\ \frac{N}{m}$. We know that the x which is the displacement of the mass from its equilibrium position is $x = C\,Sin(\omega_d\, t + \varphi)e^{-\xi\omega t}$, where this $\omega_d = \omega\sqrt{1-\xi^2}$ and ω_d is the damped natural frequency and $\xi = \frac{\zeta}{2m\omega}$ where this ζ is the damping coefficient and of course $\omega = \sqrt{\frac{k}{m}}$. So, let us first calculate the ω.

So $\omega = \sqrt{\frac{k}{m}}$. k is given, m is also given. So, $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{32}{8}} = 2\ \frac{rad}{sec}$. Now, let us find out ξ . $\xi = \frac{\zeta}{2m\omega}$. Gamma is given m is also given and omega we have calculated, it is 2. So, therefore $\xi = \frac{\zeta}{2m\omega} = \frac{20}{2\times8\times2} = 0.625$, okay. Since $\xi < 1$, therefore, the system is underdamped, okay. And ω_d, which is the damped natural frequency, that will be $\omega_d = \omega\sqrt{1-\xi^2}$ and that will be $\omega_d = \omega\sqrt{1-\xi^2} = 2\times\sqrt{1-(0.625)^2} = 1.561\ \frac{rad}{sec}$.



Now, let us look at x . $x = C\,Sin(\omega_d\, t + \varphi)e^{-\xi\omega t}$. Let us put the values. So, that will be $x = C\,Sin(1.561t + \varphi)e^{-1.25t}$.

This is the position at time t. Let us look at the velocity $\dot{x}$. $\dot{x}$ Will be the derivative of this. So, let us differentiate equation number 1 with respect to t. So, we get, so let us say this is the (I) function and this one is the (II) function. So, we get first function as it is $C\,Sin(1.561t + \varphi)$ and then the derivative of second function.

So $e^x\ is\ e^x$. So, $-1.25t$ and then differentiation of that, that will be -1.25 , okay, plus second function as it is $e^{-1.25t}$ into the derivative of the first function. So, which is $C\,Cos(1.561t + \varphi)$ and then the derivative of that which is 1.561. So, let me call this equation number(2). Now, initial conditions are given.

So, initial condition. So, it is at $t = 0$. $x = 0.2\ m$. So, let me put this in equation number (1) at $t = 0$. So, we will have $C\ Sin\varphi = 0.2$.

Let me call this equation number (a). Now, let us use the another condition. At $t = 0$, it is given that $\dot{x} = 0$. Therefore, we will have $-1.25C\ Sin\varphi + \ 1.561C\ Cos\varphi = 0$. Let me call this equation number (b). Now, there are two equations (a) and (b) and there are two unknown $C\ and\ \varphi$.

So, from (a) and (b) they can be find out and it comes out that$C = 0.256\ m$ and $\varphi = 0.896\ rad$. Let us put this value of $C\ and\ \varphi$ to find out how x varies as a function of t. So $x = 0.256\ e^{-1.25t}\ Sin(1.561t\ + 0.896)\ m$, okay. And we have been asked to find out x at time $t = 2$ sec. So, let us put $t = 2$ sec, then x comes out to be $-0.0161\ m$.

Now, let us look at another problem and the problem statement is following. The block of mass m is at rest in equilibrium position at x equal to 0.

The system is underdamped with the damping factor being $\xi = 0.25$, if the initial condition on the motion of the block are $x = 0\ and\ \dot{x} = 4\ \frac{m}{sec}$. Determine the displacement of the block at time $t = 0.1$ sec and it is given that use$m = 0.2\ kg, k_1 = 20\ \frac{N}{m}\ and\ k_2 = 30\ \frac{N}{m}$. So, let me again write down that $x = C\ Sin(\omega_d\ t\ +\ \varphi)e^{-\xi\omega t}$. And $\omega_d = \omega\sqrt{1-\xi^2}$.

$\xi = \ \frac{\zeta}{2m\omega}$ And $\omega = \sqrt{\frac{k}{m}}$. So, here $m = 0.2\ kg$. $\xi = 0.25$ And the initial conditions are at time $t = 0,\ x = 0\ and\ \dot{x} = 4\ \frac{m}{sec}$. And again, we have been asked at $t = 0.1\ sec$, what is x ? So, previously we have one spring, in this problem we have two spring and they are connected in parallel, okay. So, we know that when the springs are connected in parallel, then the spring constant $k = k_1 + k_2$, okay $k = k_1 + k_2$. Now, $k_1 = 20$ and $k_2 = 20$. So, therefore, $k = 50\ \frac{N}{m}$. Now, let us find out ω . $\omega = \sqrt{\frac{k}{m}}$. So, for the combined system, it will be $\sqrt{\frac{50}{0.2}}$.

So, this comes out to be $\omega = 15.811\ \frac{rad}{sec}$. Now, ξ is given $\xi = 0.25$. So $\xi = \ \frac{\zeta}{2m\omega} = 0.25$. So, therefore, it is obvious that the system will be underdamped. Now, let us find out ω_d. And $\omega_d = \omega\sqrt{1-\xi^2}$.

So $\omega_d = 15.811\sqrt{1-(0.25)^2} = 15.309\ \frac{rad}{sec}$. Now, x will be $x = C\ Sin(\omega_d\ t\ +\ \varphi)e^{-\xi\omega t}$. Let us put the values $x = C\ Sin(15.309t\ +\ \varphi)e^{-3.953t}$. Okay. So, let us call this as equation number (1). Now, $\dot{x}$ also I can find out by differentiating it.

So $\dot{x} = -3.953C\ Sin(15.309t + \varphi)\ + e^{-3.953t} C\ Cos(15.309t + \varphi) \times 15.309$. Okay. Let us say this is equation number (2). Now, the initial conditions are given.

Okay. And what are the initial condition? At $t = 0$, $x = 0$. So, let us put $t = 0$ and $x = 0$ in equation number (1). So, we have $C\ Sin\varphi = 0$.

Let me call this equation number (a) and at $t = 0$, $\dot{x} = 4$. Let us put that in equation number (2). So, we have $-3.953C\ Sin\varphi + 15.309C\ Cos\varphi = 4$. Let us call this equation number (b). Now, again in $(a)\ and\ (b)$, there are two equations and there are two unknowns.

So, that can be found out. So, from $(a)\ and\ (b)$, you get $C = 0.2613\, m$ and $\varphi = 0$. So, let us put this value in equation number (1) to find out how x varies as a function of t. So, let us put in equation number (1). So, you get $x = 0.2613\ Sin(15.309t)e^{-3.953t}\ m$. And we have been asked to find out x at $t = 0.1sec$. So, let us put $t = 0.1sec$ so, you get $x = 0.1758\, m$.

So, with this, let me stop here. See you in the next class. Thank you.