MECHANICS

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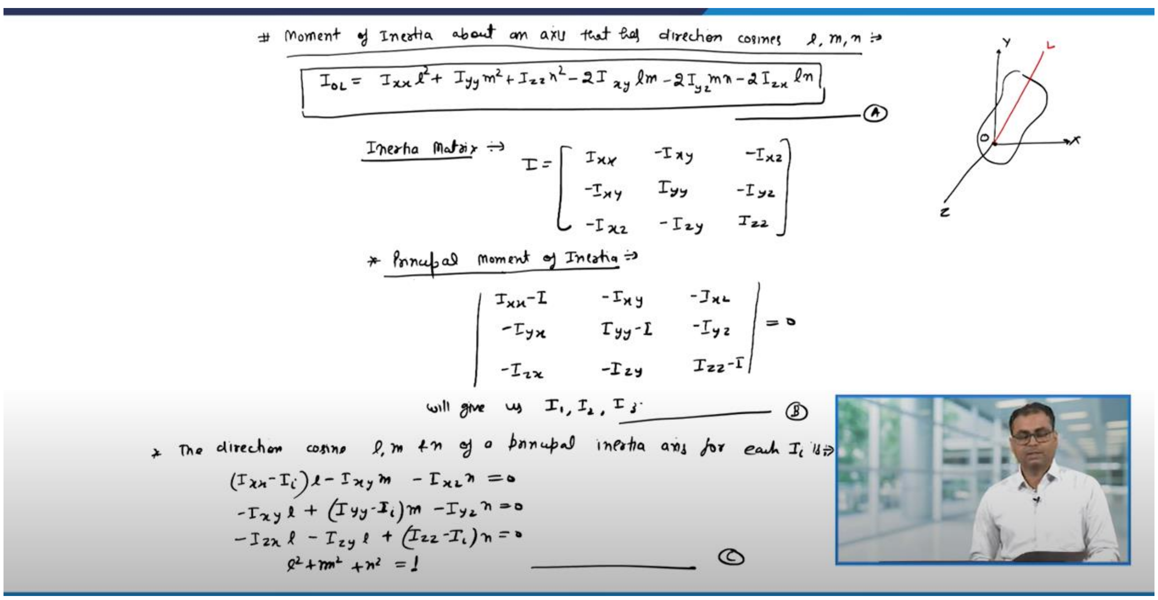
Lecture: 47

Principal axes of inertia: examples-I

Hello everyone, welcome to the lecture again. In the last class, we learned about how to find out the moment of inertia about an axis which has a direction cosine $l,\ m\ and\ n$. We learned about the inertia matrix and the principal moment of inertia, and how to find out the direction cosine of the principal axis of inertia.

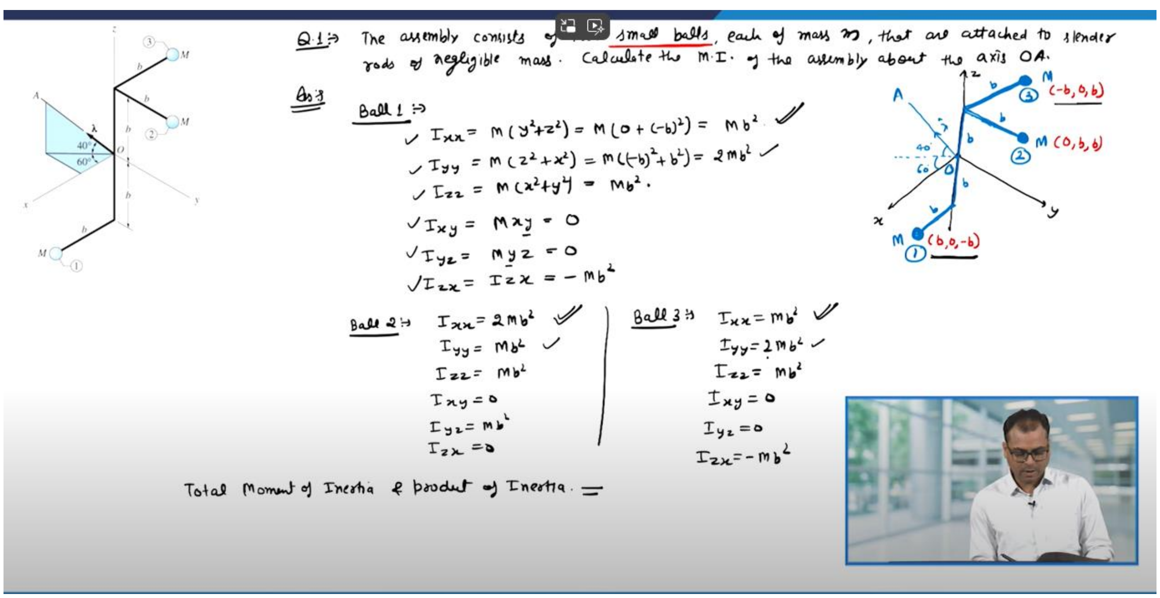


Let me first summarize these results and then we will look at couple of examples. So, the moment of inertia about an axis that has direction cosines let us say $l,\ m\ and\ n$. So, remember I have a rigid body and let us say a point O is given and this is the $x-axis,\ y-axis,\ and\ z-axis$. Then I want to find out the moment of inertia of this rigid body about an axis OL whose direction cosines are $l,\ m,\ and\ n$. Then the moment of inertia I about line OL will be $I_{xx}\,l^2 + I_{yy}\,m^2 + I_{zz}\,n^2 - 2I_{xy}\,lm - 2I_{yz}\,mn -$

$2\, I_{zx}$ln. Let us call this equation number(A). Then, the inertia matrix I was I_{xx}, I_{yy}, I_{zz} and then we have $-I_{xy}$, $-I_{xz}$, $-I_{xy}$, $-I_{yz}$, $-I_{xz}$ and $-I_{zy}$, okay.

Then the principal moment of inertia can be find out by using the following equation $I_{xx} - I$, $-I_{xy}$, $-I_{xz}$, $-I_{yx}$, $I_{yy} - I$, $-I_{yz}$, $-I_{zx}$, $-I_{zy}$, $I_{zz} - I$ and put this determinant equal to 0. Now, this equation will give you three solutions for the I. So, let us call them I_1, I_2 and I_3, call this equation number (B). So, now, we have the principal moment of inertia, then the direction cosine l, m and n of a principal inertia axis for let us say each I_i is, okay. So, that will be $I_{xx} - I_i$.

So, let us say I choose the first I, so that is $I_1 l - I_{xy} m - I_{xz} n = 0$ and I have $-I_{xy} l + (I_{yy} - I_i) m - I_{yz} n = 0$ and then we have $-I_{zx} l - I_{zy} l + (I_{zz} - I_i) n = 0$ along with that $l^2 + m^2 + n^2 = 1$. So, let us call this equation number (3), and the solution of this equation will give you the l, m, and n corresponding to the first I_1. If you use I_2, then the l, m, n corresponding to I_2 and similarly, for the I_3. So, that will be the direction cosine of the principal inertia axis.



Now, let us look at one problem statement. And the problem statement is following. Question number one, the assembly consists of three small balls, each of mass M. that are attached to slender rod of negligible mass calculate the moment of inertia of the assembly about the axis OA.

So, we have this slender and let us say it is given that this one is the x-axis, this is the y-axis and we have the z-axis and we have this slender rod and from here I have two masses. So, this rod is parallel to the x-axis, then we have this axis, which is parallel to the y-axis, and then we have this slender rod, which is parallel to the x-axis, and this is, so all of them have mass M. And let us say this one is 1, it is given there, this one is 2 and this is mass number 3, okay.

The length is also given, so this is b, this is b, this is b, this is b, and this is also b. Then, we have the axis OA, about which we have to find out the moment of inertia. Let us say the unit vector along OA is lambda or $\hat{\lambda}$ and it is given that this angle is $60°$ and this angle is $40°$. Now, first let me write down the coordinate of mass 1, 2 and 3. Note that this is a three-dimensional problem.

So, here, the coordinate of mass 1 will be $(b,\ 0,\ -b)$. Because you have to move b towards the x-axis. About the y-axis, you do not need to move. So therefore, this is 0 and then minus b along the z direction to reach the mass 1.

Similarly, for mass 2, the coordinates are $(0,\ b,\ b)$, and for the third mass, the find out the moment of inertia and the product of inertia for ball 1. So, for ball 1, the moment of inertia about the x axis, which is equal to Mr^2. So, here r will be $M(y^2\ +\ z^2)$.

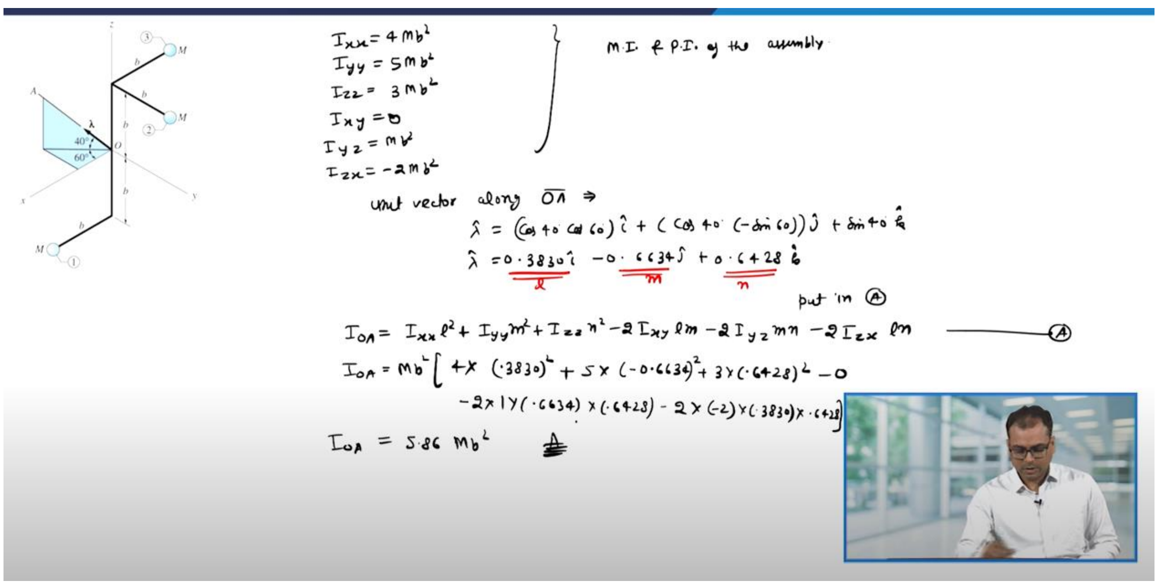
Now, we have to put the value of y and z for mass 1 from here. So, let us put that it will be M y is 0 plus z is minus b square and that will give you Mb^2. This is the moment of inertia of mass 1 about the x axis. Now, let us look at I_{yy} moment of inertia about the y axis. It will be $M(z^2\ +\ x^2)$ equal to m z is minus b. So therefore $(-b)^2\ +\ b^2$. And that is equal to $2Mb^2$.

Similarly, I_{zz}. so, that is $I_{zz} = M(x^2\ +\ y^2)\ =\ Mb^2$. Now, the product of inertia I_{xy} so, I_{xy}, note that here $y\ =\ 0$ because the definition is Mxy , but since $y\ = 0$, therefore, the product of inertia will be 0. I_{yz}, again it will be Myz , but then $y\ =\ 0$ therefore it will be 0. Now, let us calculate I_{zx} , that is equal to $I_z\, x$. So, this will be equal to $z\ =\ -\,b\ and\ x\ =\ b$. So, therefore, it will be $-Mb^2$. So, we have the moment of inertia and the product of inertia for ball 1. Now, let us look at ball 2. For ball 2, the moment of inertia about the x-axis similarly, I am not writing down the formula, it will be $2Mb^2,\ I_{yy} = Mb^2,\ I_{zz} =\ Mb^2$.

You can check it by putting the coordinates, and the formula is already given here. Similarly, $I_{xy} = 0$, $I_{yz} = Mb^2$ and $I_{zx} = 0$. Now, let us look at ball 3. So, here $I_{xx} = Mb^2$, $I_{yy} = 2Mb^2$, $I_{zz} = Mb^2$ and $I_{xy} = 0$. Why?

Because for ball 3, we are here, $I_{xy} = Mxy$, but then $y = 0$. So, therefore, it will be 0. $I_{yz} = 0$, and $I_{zx} = -Mb^2$. Note that while calculating this, we have used the fact that the ball is small. So, it is not a sphere, it is a point mass.

Therefore, the moment of inertia was, you know, Mr^2 and so on. Now, the total moment of inertia and the product of inertia will be the sum of ball 1, 2 and 3. So, total moment of inertia and product of inertia will be. So, for example, I_{xx} will be the sum of this plus this plus that.



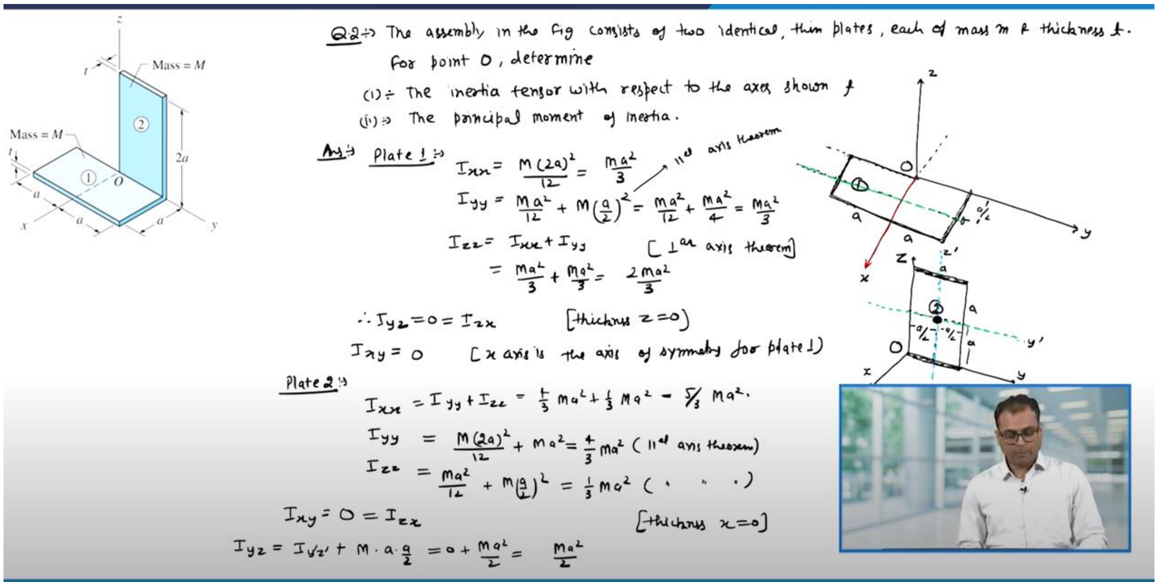
So, we have 1 plus 2 plus 1 equal to $4Mb^2$. Let me write it down on the next page. So, $I_{xx} = 4Mb^2$. Similarly, $I_{yy} = 2Mb^2 + Mb^2 + 2Mb^2$. So, that is $I_{yy} = 5Mb^2$.

$I_{yy} = 5Mb^2$. This is the moment of inertia and product of inertia of the assembly of the entire structure that we have. Similarly, I_{zz} you can sum them up. It will be $3Mb^2$. $I_{xy} = 0$.

$I_{yz} = Mb^2$ And $I_{zx} = -2Mb^2$. Now, in the question statement, we have been asked to calculate the moment of inertia of the assembly about OA. So, what we have to do is we have to put this value in equation number (A), but for that, I should know what is the$l,\ m,\ and\ n$, which is, the direction cosine of the axis OL. So, for that, let us look at the unit vector along OA. So, the unit vector, let us call it $\hat{\lambda}$.

So, $\hat{\lambda}$ will be so; let us look at the projection along the x-axis. So, it will be $(Cos40°Cos60°)\hat{\imath}\ + \big(Cos40°\ (-Sin60°)\big)\hat{\jmath}\ +\ Sin40°\ \hat{k}$. So, this is the unit vector which I have wrote in terms of $i,\ j\ and\ k$. This can be, you know, find out because the value of cos 40 and cos 60 we know. So, it will be $0.3830\ \hat{\imath}\ -\ 0.6634\ \hat{\jmath}\ +\ 0.6428\ \hat{k}$. So, this is your l, that is your m, and this is your n. The direction cosine is the value of $Cos\alpha,\ Cos\beta,\ and\ Cos\gamma$, where $\alpha,\ \beta,\ \gamma$ is the angle that it makes from the $x,\ y,\ and\ z-axis$. So, let us put this in equation number (A). Remember the equation number (A) was this. So, I_{OL} or here $I_{OA}\ = I_{xx}\ l^2\ +\ I_{yy}\ m^2\ +\ I_{zz}\ n^2\ -\ 2I_{xy}\ lm\ -\ 2I_{yz}\ mn\ -\ 2\ I_{zx}ln$. This was our equation number (A). So, we have I_{OA} Equal to, now the value of $I_{xx} = 4Mb^2$. Let me take Mb^2 outside. So, we have 4 into $l^2\ =\ (0.3830)^2$ plus $I_{yy}\ =\ 5Mb^2$. Mb^2 I have taken as a common factor. So, that is 5 into $m^2\ =\ (-\ 0.6634)^2$ plus 3 into $(0.6428)^2$ minus$2I_{xy}$. So, that is equal to 0 and then minus $2I_{yz}$. So, 2 into I_{yz} is just Mb^2, Mb^2 I have taken outside. So, Imn. So, m is $0.6634n$ is $0.6428 - 2I_{zx}$ is $-2Mb^2$.

So, -2 and l and n. So, the value of l is $0.3830n$ is 0.6428, and this can be calculated. So, that comes out to be $5.86\ Mb^2$. This is the moment of inertia of this assembly about the axis OA. So, this is how we can find out the moment of inertia about any axis if we know the direction cosines.



Now, let us look at another problem statement. The assembly in the figure consists of two identical thin plates each of mass M and thickness t. For point O, determine number 1, the inertia tensor with respect to the axis shown. And the principal moment of inertia. Here, this assembly consists of two plates to find out the inertia tensor of the assembly.

We have to find out the moment of inertia and the product of inertia of both the plates. So, let us first look at the plate 1. So, plate 1 is like this. We have the y-axis, we have the x-axis and we have the z-axis. Then plate 1 is like that and it is given that this length is a, this is also a, that is also a.

So, this is our plate 1 and the second plate is like this. So, this is also a and this is also a. That is also a. Now, let us find out the moment of inertia and product of inertia for plate 1. It is given that the thickness of the plate is very small. So, I_{xx} , the moment of inertia of this plate about the x-axis.

So, which is this axis. So, the moment of inertia of the plate of length L, as we already know it is $\frac{ML^2}{12}$. So, here it will be $\frac{M(2a)^2}{12}$ and that is equal to $\frac{Ma^2}{3}$. The moment of inertia of the first plate about the y-axis, which is this that will be equal to the moment of inertia, let us say about this axis plus m into h square. So, I am just using the parallel axis theorem.

So, that will be equal to $\frac{Ma^2}{12}$. This is the moment of inertia of plate one about the axis, which I have drawn by the green dashed line plus Mh^2 . $h = \frac{a}{2}$ Because we want to calculate about the y-axis. So, this is $\frac{a}{2}$.

So, therefore, $M\left(\frac{a}{2}\right)^2$ and that is equal to $\frac{Ma^2}{12} + \frac{Ma^2}{4} = \frac{Ma^2}{3}$. Now, note that this plate is in the xy plane . Therefore, we can use the perpendicular axis theorem to find out the moment of inertia about the z-axis and $I_{zz} = I_{xx} + I_{yy}$. So, here in I am using the perpendicular axis theorem because the plate is in xy plane and herein I have used the parallel axis theorem. So, this is equal to $\frac{Ma^2}{3} + \frac{Ma^2}{3} = \frac{2Ma^2}{3}$. Now, let us look at the product of inertia. So, note that this thickness of the plate is very small.

It is given that it is small. This is a thin plate. So, therefore, in the z direction, it will be 0 because the thickness is small. So, I have thickness $z = 0$. Therefore, I_{yz} . $I_{yz} = Myz,\ but\ then\ z = 0$. Therefore, this will be equal to 0 and also $I_{zx} = 0\ because\ again\ z = 0$. Now, what is left is I_{xy}. Now, $I_{xy} = 0$ because the x axis here is the symmetric axis for this plate. So, $I_{xy} = 0$ since the x-axis is the axis of symmetry for plate 1.

So, either you calculate or from the symmetry you can say because the limit has to be from $-a\ to\ +a$ and therefore, it will be 0. Now, let us consider plate 2. So, for plate 2 also, we have to find out what is its moment of inertia about the x-axis, y-axis and the z-axis. Note that this plate 2, it lies in the y-z plane. Therefore, moment of inertia about the x-axis I can find out using the perpendicular axis theorem and for that I should know what is $I_{yy}\ and\ I_{zz}$.

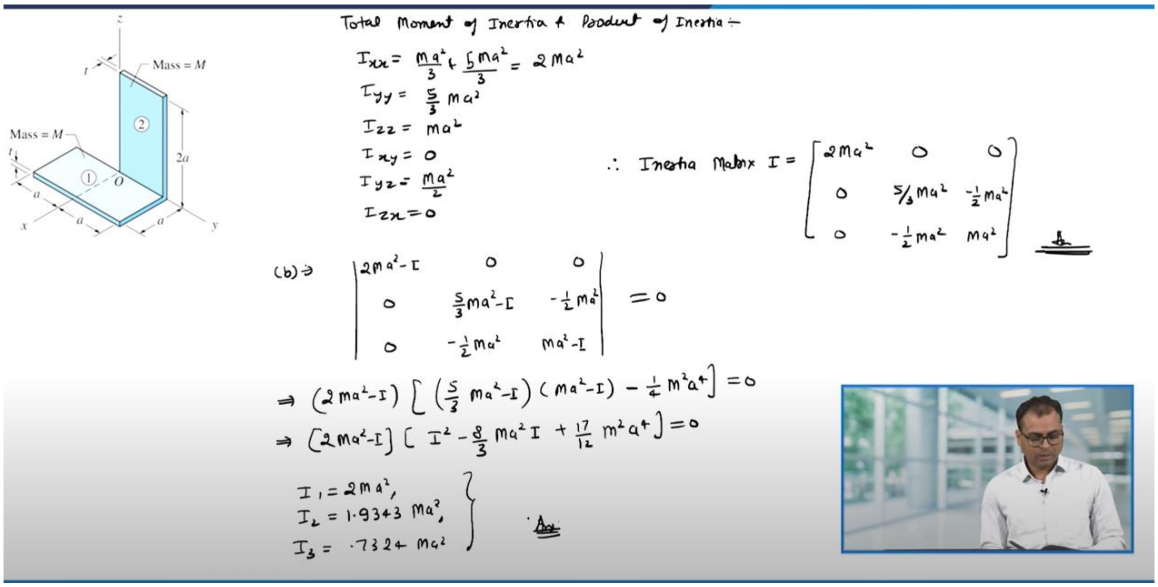
So, let us first calculate what is I_{yy}. So, that is the moment of inertia about the y-axis and for this I can use the parallel axis theorem. So, the moment of inertia of this plate about this green axis which is passing through the center will be $\frac{ML^2}{12}$, $L = 2a$. So, this divided by 12 plus from the parallel axis theorem, it will be Ma^2.

So, here I have used the parallel axis theorem and this comes out to be $\frac{4}{3}Ma^2$. Now, the moment of inertia about the z axis, again I can find out if I calculate the moment of inertia about this blue axis and then use the parallel axis theorem. So, here it will be $\frac{Ma^2}{12} + M\left(\frac{a}{2}\right)^2$ because this is $\frac{a}{2}$. So, that comes out to be $\frac{1}{3}Ma^2$.

This is again the parallel axis theorem and $I_{xx} = I_{yy} + I_{zz}$ because the plate lies in the $yz\ plane$. So, that is equal to $\frac{4}{3}Ma^2 + \frac{1}{3}Ma^2 = \frac{5}{3}Ma^2$. Now, this plate again has a thickness of almost 0 because it is very thin. So, therefore, I can say that $x = 0$ because thickness is small. So therefore, $x = 0$.

Therefore, I_{xy} which is $Mxy,$ but $x = 0$. So therefore, it will be 0, and also, I_{zx}, okay. Now, let us calculate I_{yz} , and for that, I can use the parallel axis theorem. So, I_{yz} will be equal to let us say this axis is y' and this axis is z'. So, that will be equal to $I_{y'z'}$ plus from the parallel axis theorem I can say $Ma\left(\frac{a}{2}\right)$ because this is a and this is $\frac{a}{2}$.

Now, $I_{y'z'} = 0$ because the plate is symmetric about the y' and z' $axis$. So, that is equal to $0 + \frac{Ma^2}{2} = \frac{Ma^2}{2}$. So, now we have the moment of inertia and the product of inertia of plate 1 and plate 2.



Now, the total moment of inertia and the product of inertia will be let me write down total moment of inertia and product of inertia will be I_{xx} equal to I_{xx} will be equal to this plus that.

Similarly, I_{yy} will be equal to this plus that and I_{zz} will be equal to this plus this. So, let me write it down. It will be equal to $\frac{Ma^2}{3} + \frac{5Ma^2}{3} = 2Ma^2$. I_{yy} Similarly, it will be $\frac{5}{3}Ma^2$, and $I_{zz} = Ma^2$. Now, let us look at I_{xy}.

I_{xy} Was 0 and here I_{xy} was also 0. So, therefore, $I_{xy} = 0 + 0 = 0$. I_{yz} Will be equal to I_{yz}. So, here it was 0 and here I_{yz} was $\frac{Ma^2}{2}$. So, that is $\frac{Ma^2}{2}$ and $I_{zx} = 0$.

Therefore, the inertia matrix I will be equal to I_{xx} , so which is $2Ma^2$, $\frac{5}{3}Ma^2$, $and\ I_{zz} = Ma^2$, and then we have 0, 0, 0, $-\frac{Ma^2}{2}$, $0\ and\ -\frac{Ma^2}{2}$. Now, let us look at the second part. In part b, we have been asked to find out the principal moment of inertia. So, that will be the solution of the following determinant: $2Ma^2 - I$, $0\ and$ 0, 0, $\frac{5}{3}Ma^2 - I\ and\ -\frac{Ma^2}{2}$, 0 , $-\frac{Ma^2}{2}\ and\ Ma^2 - I$ equal to 0. So, let us solve this equation. So, we have $(2Ma^2 - I)\left[\left(\frac{5}{3}Ma^2 - I\right)(Ma^2 - I) - \frac{1}{4}M^2a^4\right] = 0$. So, this I can rewrite as $(2Ma^2 - I)\left[I^2 - \frac{8}{3}Ma^2 I + \frac{17}{12}M^2a^4\right] = 0$

And this equation gives you the three values of the I. So, here, $I_1 = 2Ma^2$, $I_2 = 1.9343Ma^2$, $and\, I_3 = 0.7324Ma^2$. These are your principal moment of inertia. With this let me stop here. See you in the next class. Thank you.