MECHANICS

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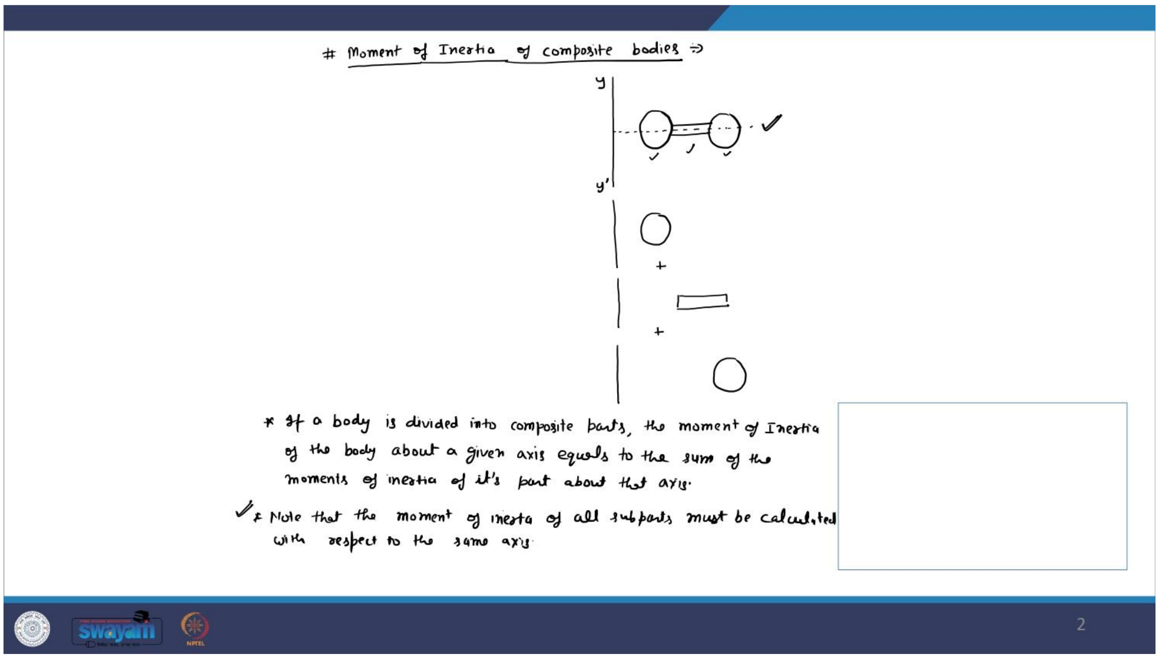
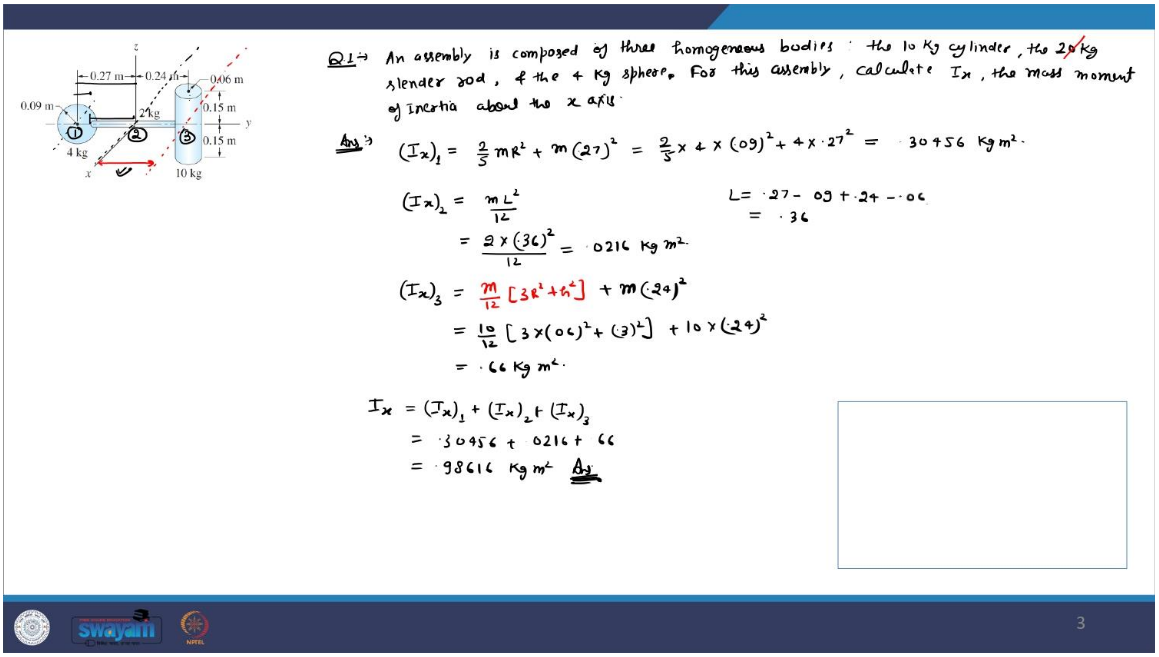
Lecture 45
Moment of inertia of composite bodies

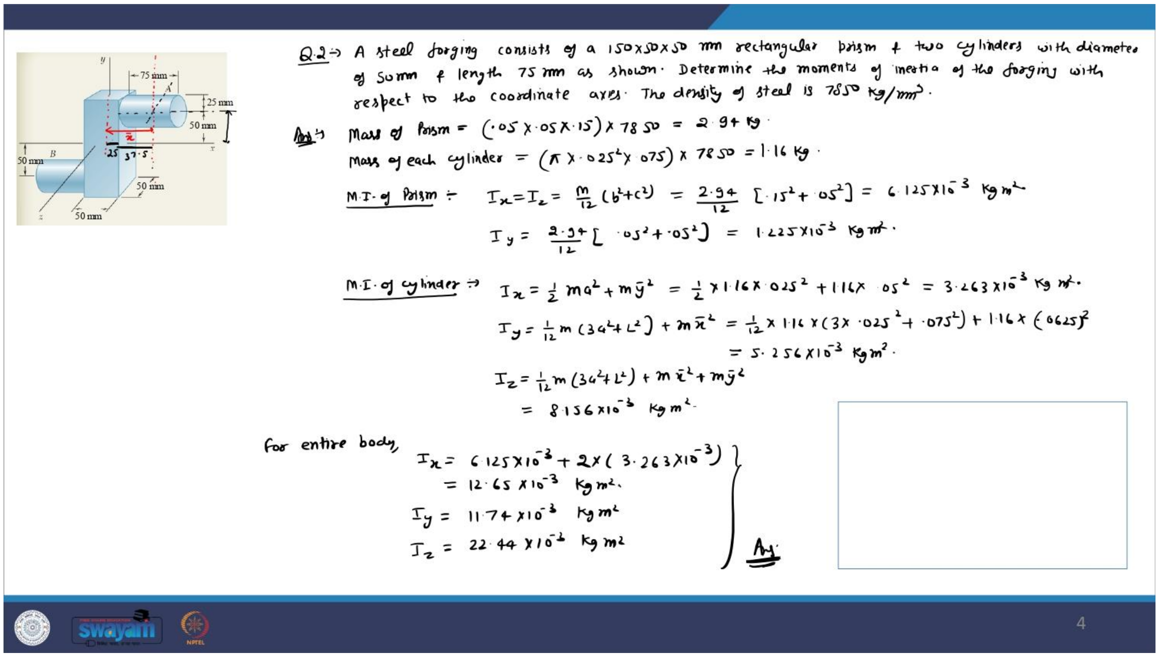
Hello everyone, welcome to the lecture again. In the last class, we saw how can we find out the moment of inertia of a body.



Today, we are going to discuss how to find out the moment of inertia of composite bodies. So, the moment of inertia of composite bodies. Let's say I have a body like this. So, a sphere, a cylinder and then a sphere and I want to calculate what is the moment of inertia of this about let's say the y-axis. So, yy'. The moment of inertia will be the moment of inertia of the first sphere plus the moment of inertia of this cylinder plus the moment of inertia of the second sphere. Let me write it down in the text. If a body is divided into composite parts, the moment of inertia of the body about a given axis equals to the sum of the moment of inertia of its part about that axis. Now, note that the moment of inertia of all subpart must be calculated with respect to the same axis. So, this is what I said when you want to calculate the moment of inertia, let's say for this composite structure, then the moment of inertia will be equal to this plus that plus that and we have to calculate the moment of inertia of all the subpart about the same axis which is yy' here. Now, based on this concept, let us look at a few questions.



So, this is the first problem statement. An assembly is composed of three homogeneous bodies, the 10 kg cylinder, the 2 kg slender rod and the 4 kg sphere. For this assembly, calculate I_x, which is the mass moment of inertia about the x-axis. So, here we have three bodies. Let us label them 1, 2 and 3. We have to calculate the moment of inertia about the x-axis. So, let us first look at the moment of inertia of the first subpart which is the sphere. The moment of inertia of the first part about the x-axis will be, so our x-axis is this. We have to calculate what is the moment of inertia about this axis. So, we can use the parallel axis theorem. It will be the moment of inertia of this 4 kg sphere about its central axis plus mh^2. So, it will be $(I_x)_1 = \frac{2}{5}mR^2 + m(0.27)^2$and now we can put the values. So, it will be $\frac{2}{5} \times 4 \times (0.09)^2 + 4 \times (0.27)^2$and this comes out to be $0.30456\ kg\ m^2$. Now, let us look at the moment of inertia of the second object which is the rod about the x-axis. So, we know that the moment of inertia of a rod about this x-axis is $\frac{mL^2}{12}$. Now, let us see what is L. So, the $L = 0.27 - 0.09 + 0.24 - 0.06 = 0.36$. So, let us put the value $m = 2\ kg$. So, it is $(I_x)_2 = \frac{mL^2}{12} = 2 \times \frac{(0.36)^2}{12}$. So, this is $0.0216\ kg\ m^2$. Now, the moment of inertia of the third subpart which is the cylinder about the x-axis will be the moment of inertia about this axis plus m into this distance square. So, the moment of inertia about that axis will be $(I_x)_3 = \frac{m}{12}(3R^2 + h^2) + m(0.24)^2$. Now, let us put the values. So, it will be $\frac{10}{12}(3 \times (0.06)^2 + (0.3)^2) + 10 \times (0.24)^2$ and that comes out to be $0.66\,kg\,m^2$. Therefore, the moment of inertia of the whole assembly about the x-axis will be $I_x = (I_x)_1 + (I_x)_2 + (I_x)_3$ which is equal to $0.30456 + 0.0216 + 0.66$ that is equal to $0.98616\,kg\,m^2$.



Now, let us look at another problem. Question number 2, a steel forging consists of a $150 \times 50 \times 50\,mm$ rectangular prism and two cylinders with diameter of $50\,mm$ and length $75\,mm$ as shown. Determine the moments of inertia of the forging with respect to the coordinate axis. The density of steel is $7850\,kg/mm^3$. This system consists of two cylinder and prism. We have to find out the moment of inertia about the x, y and the z-axis. Let us first look at the mass of this prism. So, the mass of the prism will be the density multiplied by the volume. So, the volume is $0.05 \times 0.05 \times 0.15$ because the length is $150\,mm$ and multiplied by the density 7850 and that comes out to be $2.94\,kg$.

$mass\ of\ prism = (0.05 \times 0.05 \times 0.15) \times\ 7850 = 2.94\,kg.$

Now, let us look at the mass of each cylinder. So, the mass of each cylinder is again the volume. So, the volume is $\pi \times r^2$. So, $mass\ of\ each\ cylinder = (\pi \times (0.025)^2 \times 0.075) \times 7850 = 1.16\,kg$.

Now, let us first look at the moment of inertia of the prism. And because of the symmetry, the moment of inertia about the x and the z-axis, it will be the same. So, $I_x = I_z$ and we know the formula.

$I_x = I_z = \frac{m}{12}(b^2 + c^2) = \frac{2.94}{12}((0.15)^2 + (0.05)^2) = 6.125 \times 10^{-3}\,kg\,m^2.$

And the moment of inertia about the y, similarly, it will be

$$I_y = \frac{2.94}{12}\left((0.05)^2 + (0.05)^2\right) = 1.225 \times 10^{-3}\ kg\ m^2.$$

Now, let us look at the moment of inertia of the cylinder. So, the moment of inertia of the cylinder about the x-axis is $\frac{1}{2}ma^2 + my^2$ let us say. So, first I calculate what is the moment of inertia about this axis and then m into this distance square. So, it will be

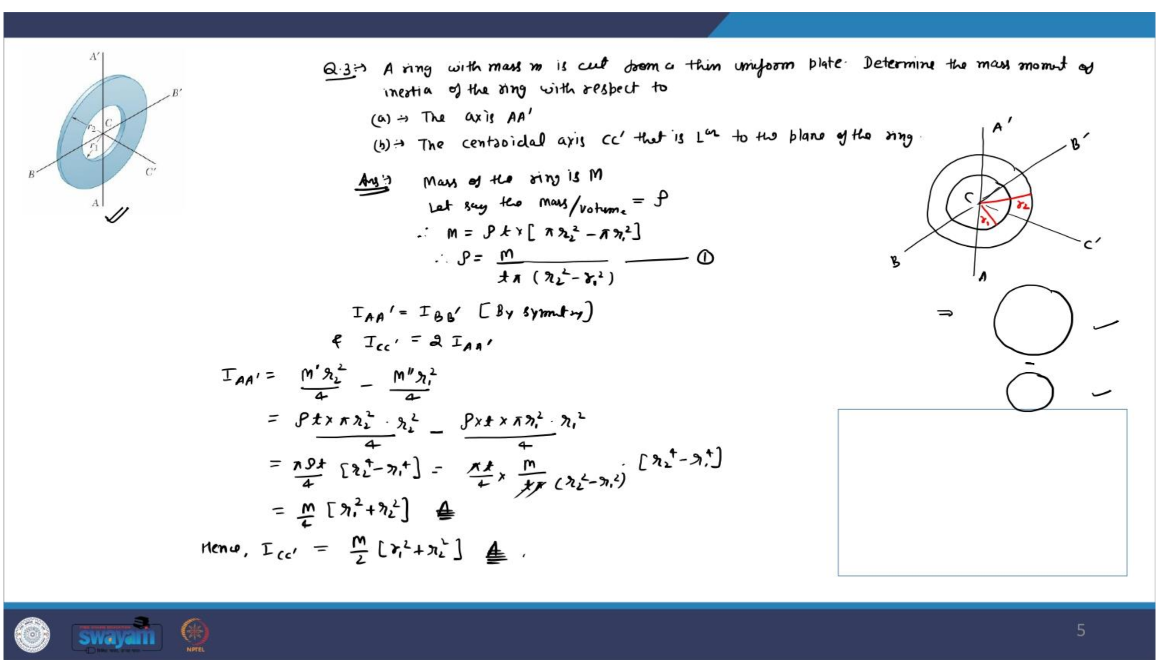
$$I_x = \frac{1}{2}ma^2 + m\bar{y}^2 = \frac{1}{2} \times 1.16 \times 0.025^2 + 1.16 \times 0.05^2 = 3.263 \times 10^{-3}\ kg\ m^2$$

Now, the moment of inertia of the cylinder about the y-axis will be equal to the moment of inertia about this axis which passes through the centre plus m into this distance square. Let us call it $\bar{x}$. So, we have used the parallel axis theorem. So, the moment of inertia about the y-axis will be $I_y = \frac{1}{12}m(3a^2 + L^2) + m\bar{y}^2 = \frac{1}{2} \times 1.16 \times (3 \times 0.025^2 + 0.075^2) + 1.16 \times 0.0625^2 = 5.256 \times 10^{-3}\ kg\ m^2$.

Now, let us look at the moment of inertia about the z-axis. So, that is equal to $I_z = \frac{1}{12}m(3a^2 + L^2) + m\bar{x}^2 + m\bar{y}^2$ .So, we calculate the moment of inertia about this axis and then we use the parallel axis theorem. So, that is equal to you can calculate $8.156 \times 10^{-3} kg\ m^2$. Now, for the entire body, the moment of inertia about the x-axis will be the moment of inertia of the prism plus the moment of inertia of two cylinders. So, that is equal to 6.125 into 10 to the power 3 plus 2 into, because they are two cylinders, the moment of inertia of each cylinder. So,

$$I_x = 6.125 \times 10^{-3} + 2 \times (3.26 \times 10^{-3}) = 12.65 \times 10^{-3}\ kg\ m^2$$

Similarly, the moment of inertia of the entire body about the y-axis is, $I_y = 11.74 \times 10^{-3}\ kg\ m^2$ and the moment of inertia about the z-axis will be, $I_z = 22.44 \times 10^{-3}\ kg\ m^2$.



Now, let us look at another problem statement. Question number 3, a ring with mass M is cut from a thin uniform plate, determine the mass moment of inertia of the ring with respect to (a) the axis AA' and (b) the centroidal axis CC' that is perpendicular to the plane of the ring. So, here the moment of inertia of this ring let's say this is your AA' and this one is BB' and then we have CC'. So, this radius is r_2 and this is r_1. Now, the moment of inertia of this assembly will be equal to the moment of inertia of the outer ring minus the moment of inertia of the inner ring. So, let us calculate. So, it is given that the mass of the ring is M. Let's say the density or the mass per volume is ρ. So, therefore, mass M will be equal to ρ times the volume and the volume is the thickness multiplied by the area. So, which is $\pi r_2^2 - \pi r_1^2$.

$M = \rho t(\pi r_2^2 - \pi r_1^2) \Rightarrow \rho = \frac{M}{t(\pi r_2^2 - \pi r_1^2)} ------(1)$.

Now, by symmetry, you can see here that the moment of inertia about AA' will be equal to the moment of inertia about BB'. So, this is from the symmetry $I'_{AA} = I'_{BB}$ and I'_{CC} from the perpendicular axis will be equal to $I'_{AA} + I'_{BB}$. So, therefore, that is equal to $2I'_{AA}$. Now, let us calculate the moment of inertia about AA'. So, that is equal to the moment of inertia of the outer ring minus the moment of inertia of the inner ring and the moment of inertia of the outer ring about AA' will be equal to $\frac{Mr^2}{4}$. So, it will be $\frac{M' r_2}{4}$ wherein M' is the mass of the outer ring minus $\frac{M'' r_1}{4}$ wherein M'' is the mass of the inner ring.

$$I'_{AA} = \frac{M' r_2}{4} - \frac{M'' r_1}{4}$$

So, let us put the value M' will be equal to the density into the volume. So,

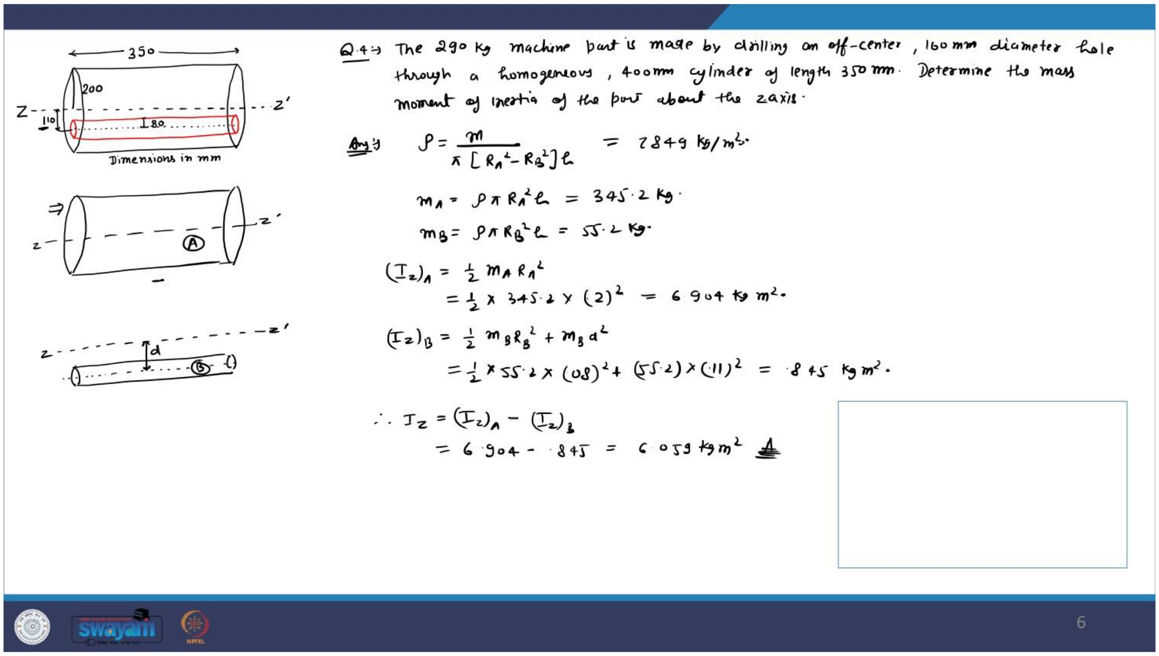
$$= \rho t \times \frac{\pi r_2^2}{4} \times r_2^2 - \rho t \times \frac{\pi r_1^2}{4} \times r_1^2.$$

$$= \frac{\rho \pi t}{4}(r_2^4 - r_1^4) = \frac{\pi t}{4} \times \frac{M}{t(\pi r_2^2 - \pi r_1^2)} \times (r_2^4 - r_1^4)$$

$$I'_{AA} = \frac{M}{4}(r_1^2 + r_2^2)$$

Hence, $I'_{CC} = \frac{M}{2}(r_1^2 + r_2^2)$.

Now, let us look at one more problem on the same concept. The $290\ kg$ machine part is made by drilling an off-center $160\ mm$ diameter hole through a homogeneous $400\ mm$ cylinder of length $350\ mm$, determine the mass moment of inertia of the part about the z-axis. So, again here the moment of inertia of this entire structure will be equal to the moment of inertia of the outer cylinder So, we calculate the moment of inertia of this about zz' minus the moment of inertia of the cylinder that is removed. So, let us find out the moment of inertia. So, first of all, the density ρ will be equal to the mass divided by the volume. So, volume is $(\pi R_A^2 - \pi R_B^2) \times h$, wherein I call this as A and this as B. So,



we can put the values because everything is given, mass is also given and R_A and R_B and h are known.

$\rho = \frac{m}{\pi(R_A^2 - R_B^2)h} = 7849\ kg/m^3$.

Now, the mass of A, $m_A = \rho\pi R_A^2 h$. Now, rho is known and all other parameters are also known. So, this you can find out it will be $345.2\ kg$. Similarly, the mass of B, $m_B = \rho\pi R_B^2 h$ and that is equal to $55.2\ kg$. Now, the moment of inertia of cylinder A about the z-axis that will be $(I_z)_A = \frac{1}{2} m_A R_A^2$. Let us put the value. So, that is equal to $\frac{1}{2} \times 345.2 \times 0.2^2$ and that comes out to be $6.904\ kg\ m^2$. Now, the moment of inertia of cylinder B about the z-axis. So, this we can find out using the parallel axis theorem that is the moment of inertia about the central axis plus m into this distance square. So, it is equal to $(I_z)_B = \frac{1}{2} m_B R_B^2 + m_B d^2$. Let us put the value $= \frac{1}{2} \times 55.2 \times (0.08)^2 + (55.2) \times (0.11)^2$. You can put the value and it comes out to be $0.845\ kg\ m^2$. Therefore, the moment of inertia about the z-axis will be the moment of inertia of A about the z-axis minus the moment of inertia of B about the z-axis. $I_z = (I_z)_A - (I_z)_B$. So, that is equal to $6.904 - 0.845$ that is equal to $6.059\ kg\ m^2$. So, this is how we can calculate the moment of inertia of the composite bodies.

With this let me stop here. See you in the next class. Thank you.