MECHANICS

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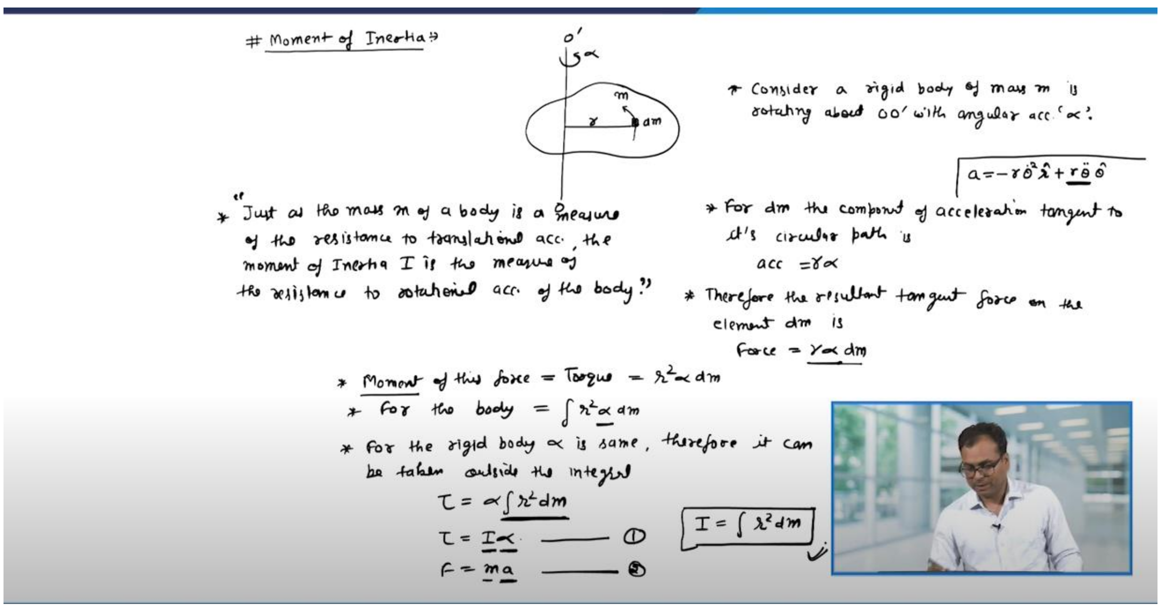
Lecture: 43

Moment of inertia

Hello everyone, welcome to the lecture again. Today, we are going to start the discussion on moment of inertia.



In this lecture, we will discuss the moment of inertia of standard bodies and the parallel axis theorem, the perpendicular axis theorem, and the radius of gyration. So, let us see what is moment of inertia. Let us say I have a rigid body and that rigid body is rotating about an axis with an angular acceleration of alpha.

So, let us say this is OO' and the mass of this rigid body is m. Then, as I said, this body or a rigid body of mass m is rotating about OO' with angular acceleration α. Now, let us consider a small mass dm which is at a distance let us say r away. So, the tangential acceleration because it is rotating. So, this mass is going to have a tangential acceleration, and we know in planar polar coordinates, the acceleration a is $-r\dot{\theta}^2\hat{r} + r\ddot{\theta}\hat{\theta}$.

This theta double dot is the angular acceleration, which is alpha here. So, therefore, for mass d m, the component of tangential acceleration or the acceleration which is tangent to its circular path because this mass d m is going in a circular path. So, circle path is.

So, the acceleration or the tangential acceleration is $r\ddot{\theta}$, which is α here. Now, let us calculate the force on this mass dm. So, therefore, the resultant tangential force or the tangent force on the element dm is a force equal to mass time acceleration. So $r\alpha dm$.

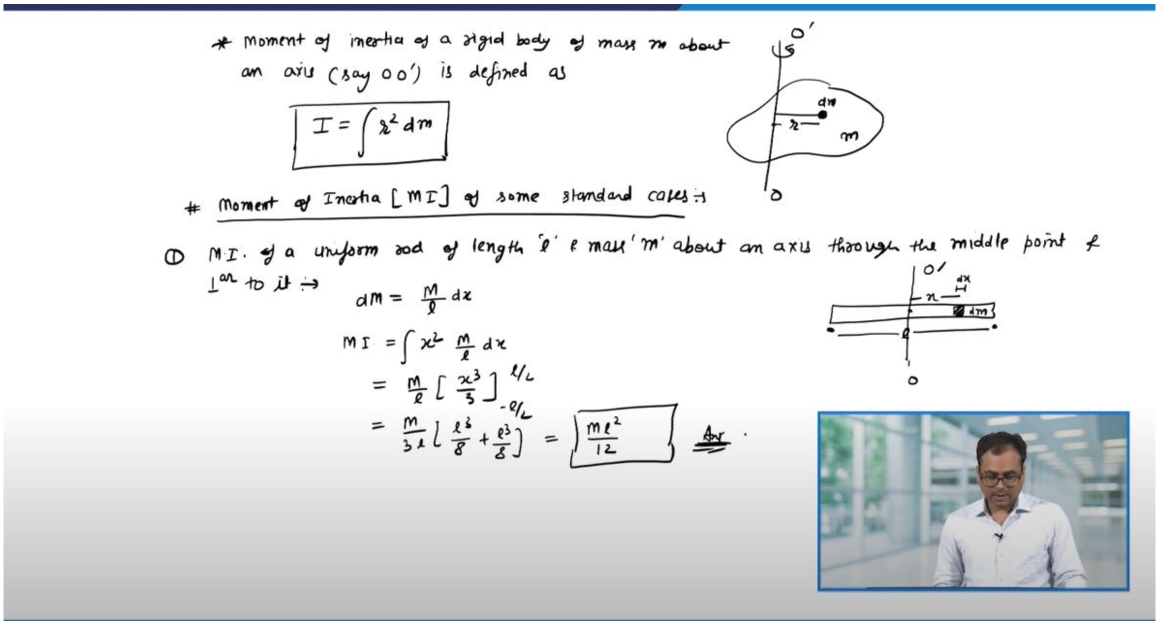
Now, let us calculate the torque that is acting on the mass dm or the moment of this force. So, the moment of this force which is also called torque is equal to. So, we have to take the moment of this. So, we have to multiply this by r. So, it becomes $r^2\alpha dm$. Now, this moment or the torque is for the small element dm for the entire body, for the body, the rigid body that we have, it will be integral $r^2\alpha dm$.

Now, note that because it is the rigid body, so every point on the rigid body is moving with the same angular acceleration α. Therefore, this α is independent of the integral, and I can take it outside. So, let me mention this for the rigid body, this α is same Therefore, it can be taken outside the integral and we can write down the moment for the body.

Let me denote it by $\tau = \alpha\int r^2dm$. Let me define this quantity r^2dm as moment of inertia I. So, τ becomes $I\alpha$. Let me call it equation number 1. Here note that the I is $\int r^2dm$. Let me compare it with $F = ma$.

So, this is famous Newton's law. Let us compare 1 and 2, and just by comparison, we can make the following statement. Just as the mass m of a body is a measure of the resistance to the translational acceleration, the moment of inertia I is the measure of the resistance to rotational acceleration of the body.

From equation number 2, you can see that the mass m is the measure of the resistance to the translational acceleration a. Similarly, the moment of inertia I is the resistance of the rotational acceleration alpha of the body. So, this is the physical significance.

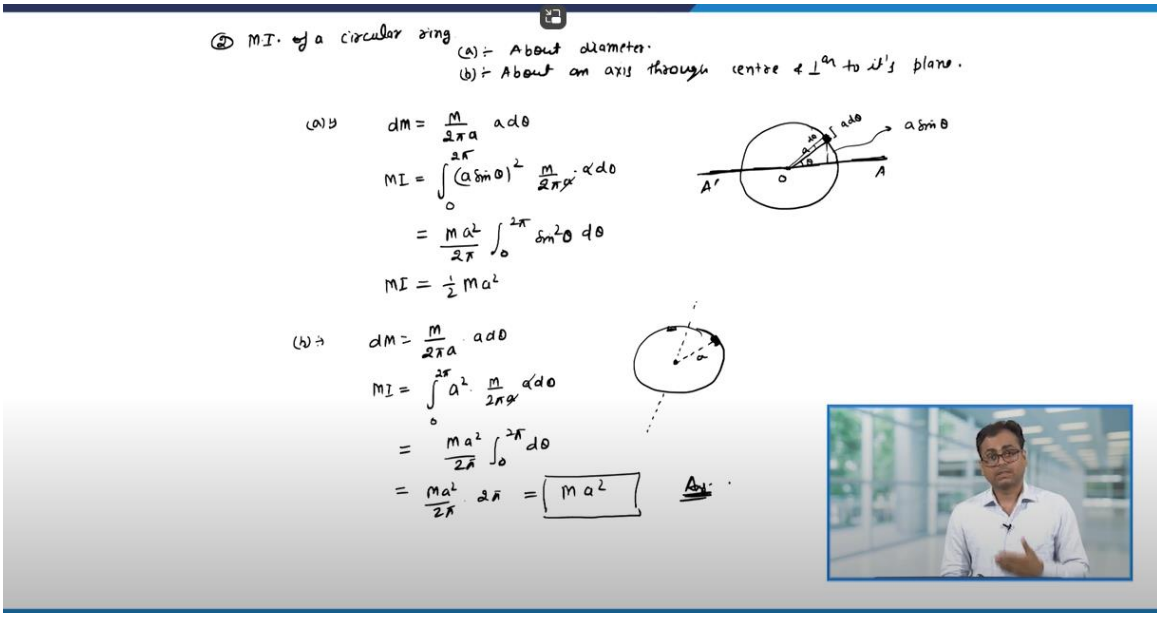


Let us look at moment of inertia I. Let us say I have a rigid body, and this rigid body has a mass m and it is rotating about OO'. Then the moment of inertia of a rigid body of mass m about an axis say OO' is defined as $I = \int r^2 dm$. So, we take a small mass dm at a distance r and then you find out this integral. Now, based on this let us look at the moment of inertia of some standard object ok. So, first let us look at the moment of inertia of a rod.

Moment of inertia of a, let us say, a uniform rod of length l and mass m about an axis through the middle point and perpendicular to it. So, we have the following situation, we have this rod, and the mass of the rod is m, the length is l, and let us say we want to find out the moment of inertia about an axis which passes through the middle point and that is perpendicular to it. So, this is our OO'. So, to find out the moment of inertia, let us consider a small mass dm at a distance of x. So, the mass is dm and this length is dx. So dm, the mass will be the, you know, mass per unit length multiplied by dx.

So, mass per unit length is $\frac{m}{l}dx$. This is my dm. Now, the moment of inertia will be $\int x^2$ and dm.

So, dm is $\frac{m}{l}dx$. So, here, m/l, I can take outside, and the integral of x^2 is $x^3/3$, and the limit are from $-l/2$ to $l/2$. So, this point is $-l/2$ and that point is $l/2$. So, it is $-l/2$ to $l/2$. So, therefore, it is $\frac{m}{3l}[\frac{l^3}{8}+\frac{l^3}{8}]$ and that comes out to be $ml^2/12$. This is the moment of inertia of a rod of length l mass m about an axis, which is perpendicular to the rod and it passes through the middle point.



Now, let us look at the moment of inertia of a circular ring. In the first case, let us say about diameter and then let us consider about an axis through the center and perpendicular to its plane. So, the first case is about diameter. So, we have a ring, and this is the diameter passes through the center O. Let us say this point is A' and A. So, we want to calculate the moment of inertia about AA'. Now, to find out the moment of inertia, I need dm. dm Will be the mass per unit length. So $m/2\pi a$, where a is the radius of this ring and multiplied by the infinitesimal length.

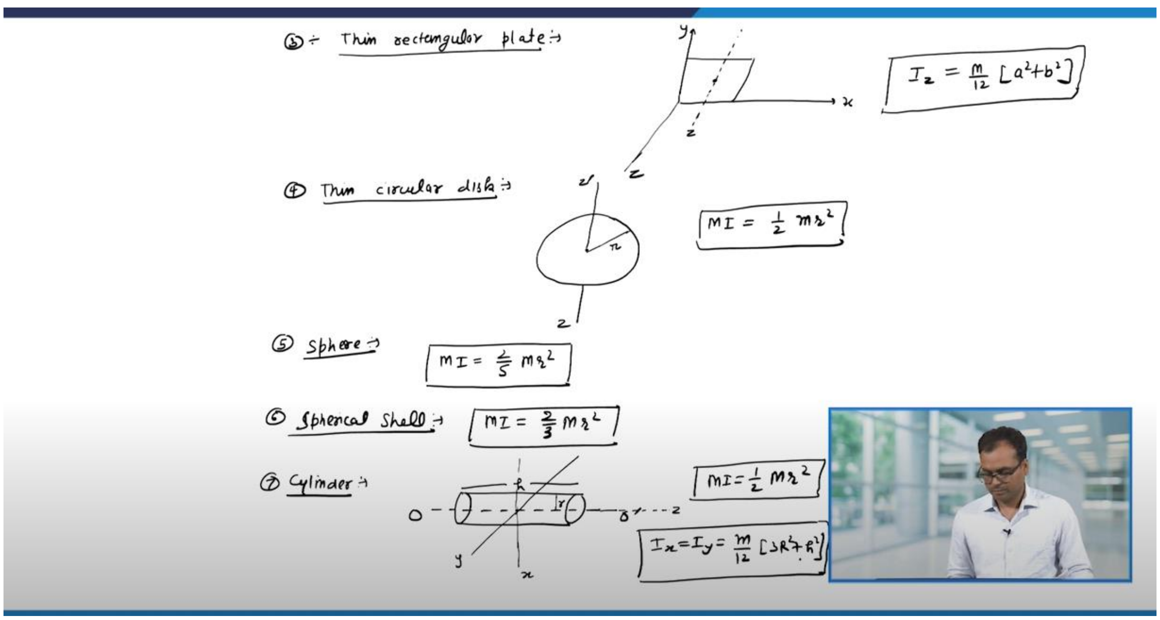
So, let us say this is a, this angle is θ. So, therefore, this is $d\theta$ and that length will be $ad\theta$. So, multiplied by $ad\theta$. This is the mass of this small part dm.

Now, the moment of inertia will be $\int r^2 dm$. So, we want to calculate the moment of inertia about this axis. So, we have to see how far this mass is from this axis. Now, from the geometry, you can see that since this length is a, this angle is θ.

Therefore, this length will be $asin\theta$. So, we will have $(asin\theta)^2 dm$ is $\frac{M}{2\pi a} ad\theta$. Now, the limit of θ, you can clearly see that theta varies from 0 to 2π. So, this comes out to be M and a^2, this a will get cancelled with that a. So $\frac{Ma^2}{2\pi}$, 0 to $2\pi \sin^2\theta d\theta$ and this comes out to be $\frac{1}{2}Ma^2$. So, this is the moment of inertia of the circular ring about the diameter.

Now, let us consider the second case, the moment of inertia about the axis which is passing through the center and perpendicular to its plane. So, we have this ring and the moment of inertia about let us say this axis, okay. So, in this case again, the d m will be the same. So, dm will be $\frac{M}{2\pi a} ad\theta$. But the distance r from this is going to be fixed and it is going to be a only.

So, it does not matter whether you are here or there. From this axis, the distance is always a. So, therefore, the moment of inertia will be $a^2 dm = \frac{M}{2\pi a} ad\theta$, and the limit is from 0 to 2π. So, let me take the constant outside this a will get cancelled with that a. So, we have $\frac{Ma^2}{2\pi}$, 0 to $2\pi d\theta$ which is $\frac{Ma^2}{2\pi} 2\pi$ that is equal to Ma^2. Similarly, the moment of inertia of other standard objects can be calculated.

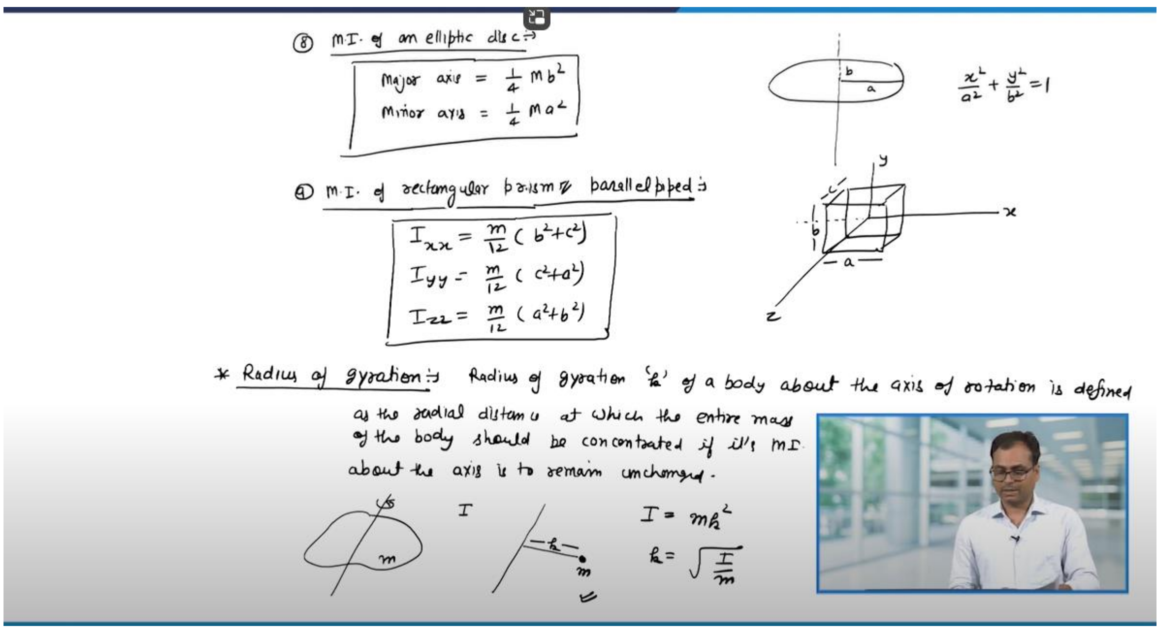


Now, I am not calculating it, but I am writing down the end results. So, let us say a thin rectangular plate. So, let us say I have the x axis, the y axis, and the z axis and we want to calculate the moment of inertia about a point that is passes through the center and which is perpendicular to the plane of the rectangular plate.

So, in this case, I_z. So, let us say this is z axis will be $\frac{M}{12}(a^2 + b^2)$. If you have a thin circular disc, So, this is a disc, and the moment of inertia again about the axis which is passing through the center and perpendicular to the plane will be $\frac{1}{2} Mr^2$ where r is the radius of it and this moment of inertia I have written let us say about z'. Now, if you have a sphere, then the moment of inertia about the center is $\frac{2}{5} Mr^2$.

For spherical cell, the moment of inertia is $\frac{2}{3}Mr^2$. So, all of them you can calculate using this relation $I = \int r^2 dm$. Now, for the cylinder, so let us say I have a cylinder of let us say height is h and the radius is r, and let us say this is OO', then the moment of inertia about OO'.

So, let me also call it z axis will be $\frac{1}{2}Mr^2$. And if you calculate the moment of inertia about the x or y-axis, so let us say this is your x-axis and this is your y-axis, then it is, you know, about x and y-axis. It will be the same and $I_x = I_y = \frac{M}{12}(3R^2 + h^2)$. So, r or R, they are the same here.

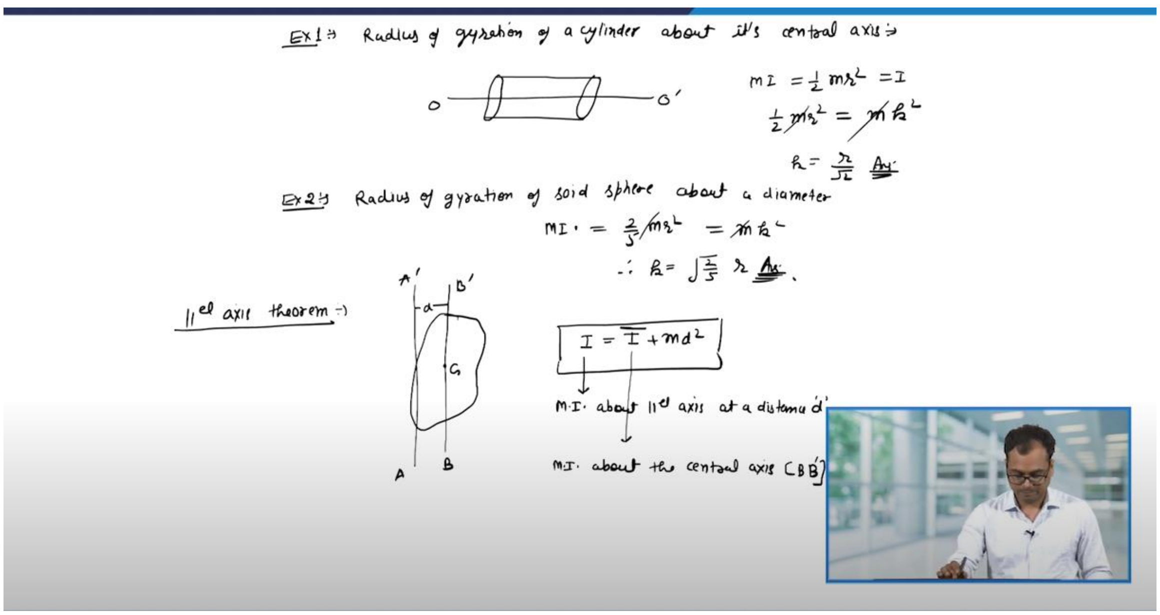


Now, let us look at the moment of inertia of an elliptic disc. So, let us say we have a elliptic disc. Its semi-major axis is A and semi-minor axis is B, then the moment of inertia about major axis will be $\frac{1}{4}Mb^2$, and about the minor axis, it will be $\frac{1}{4}Ma^2$. And note that the equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Now, the moment of inertia of a rectangular prism or sometimes it is also called parallel piped okay. So, we have this parallel pipette, it is let us say length is a, b and c, okay.

Let us say this is the x-axis, y-axis and z-axis, then the moment of inertia I about x x-axis, so about this x x let us say, it will be $\frac{m}{12}(b^2 + c^2)$. I_{yy} Will be $\frac{m}{12}(c^2 +^2)$, and I_{zz} will be $\frac{m}{12}(a^2 + b^2)$. Now, let me define the radius of gyration. So, first let me write down the definition the radius of gyration it is denoted by k of

A body about the axis of rotation is defined as the radial distance at which the entire of the body should be concentrated if its moment of inertia about the axis is to remain unchanged. So, for example, let us say I have this rigid body, and it has some, you know, axis of rotation, and let us say the mass is m. Now, you find out what is the moment of inertia. Let us say the moment of inertia is, you know, whatever it is, it is I. Then what you do is you take this mass and keep it at a distance, let us say k, which is the radius of gyration.

So, the entire mass is kept at this distance, okay. Then, we put I equal to the moment of inertia of this system is mk^2, okay. So, therefore, this k becomes a $\sqrt{\frac{I}{m}}$. So, you take this body, it has a mass, it has a moment of inertia, take this mass at a distance of k, calculate the moment of inertia and then put it back equal to I. So, whatever you know distance you get that is the radius of gyration.

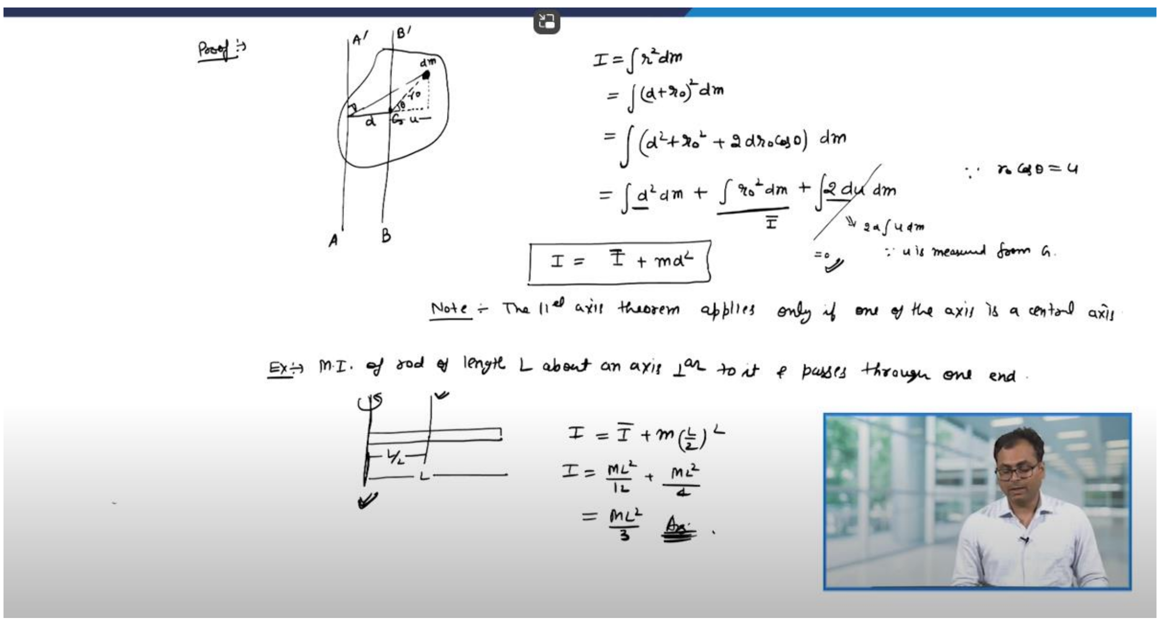


Let us understand it by an example so that it will be clear. So, radius of gyration of a cylinder. Let us say about its central axis. So, we have a cylinder, and this is the central axis, let us say OO', then we know the moment of inertia about OO' is $\frac{1}{2}$. So, this is your I, this you put equal to mk^2. So, let me put your I, which is $\frac{1}{2}mr^2 = mk^2$.

Then k is the radius of gyration. Let us find out what is k. So, k is $r/\sqrt{2}$. Let us look at another example. Let us say radius of gyration of solid sphere about a diameter. So, the moment of inertia of a solid sphere is $\frac{2}{5}mr^2$.

Let us put this equal to mk^2. So, m will get cancelled. So, from here, k comes out to be $\sqrt{\frac{2}{5}}r$. Now, let us look at the parallel axis theorem. Let us say I have a rigid body, and we have an axis that is passing through the center of mass. Let me call it BB', and you want to calculate what is the moment of inertia of this body about an axis which is parallel to it.

So, let us say it is situated at a distance of d. So, this is $AB. A'$, then the moment of inertia about AA', let me denote it by I equal to the moment of inertia about BB', let me denote it by $\bar{I} + md^2$, okay. So, here this I is the moment of inertia about parallel axis. At a distance d and $\bar{I}$ is the moment of inertia about the central axis. So, here it is BB'.



Let us prove this statement. So, we have this rigid body, and we have the axis which is passing through the centre of mass, which is G here, and we want to calculate. So, this was our BB', and we want to calculate the moment of inertia about AA'. So, the moment of inertia about AA' will be, let us take a small mass dm, okay?

And let us say this mass d m is at a distance of r, okay? So, this is the perpendicular distance, okay? Because this object is in 3D. So $I = \int r^2 dm$, okay? Now, this A' axis is situated at a distance of d. So, this distance is d, and let us say from g, the mass dm is at a distance of r_0.

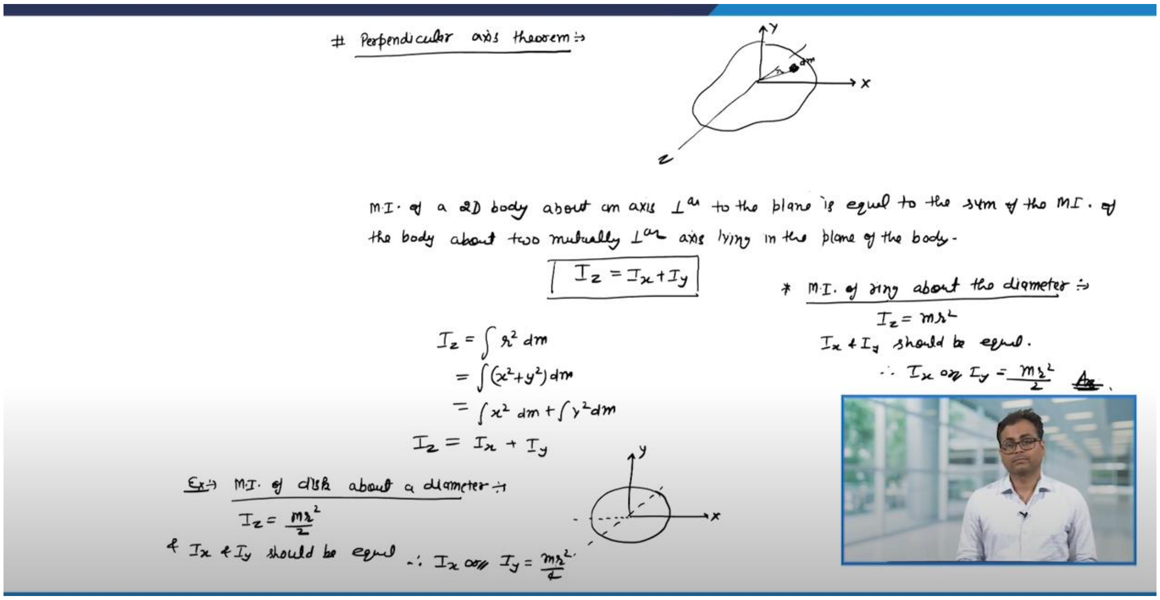
Then I can write this down as $\int d + r_0^2 dm$ because r is $d + r_0$, and let us say this r_0 is making an angle θ from the horizontal line, and this distance is let us say u. In that case, so for this geometry, I can write down this as $d^2 + r_0^2 + 2dr_0 cos\theta dm$, or I can write this down as $d^2 dm + \int r_0^2 dm + 2dr_0 cos\theta$ is nothing but u. So, $2udm$ because from the geometry $r_0 cos\theta = u$. So, this $\int r_0^2 dm$ is the moment of inertia of this rigid body about BB'.

So, let me write it as $\bar{I}$. So, I have I equal to $\bar{I}$ plus this d is constant. So, I can take d^2 outside, and integral dm is the total mass. So, it becomes md^2. Now, look at here, 2 and d is constant. So, that I can take outside. So, this I can rewrite as $2d\int udm$, okay. But since this x is $B'B$ is passing through the center of mass.

So, therefore, $\int udm$ will be 0 because it is the definition of the, you know, center of mass axis. If you take, you know, rdm and then you integrate it, it has to be 0, okay? The reason is u is measured from the axis which is passing through the center of mass, passing through g, and it is you know, the first moment. So, therefore, it has to be 0.

So, we got $I = \bar{I} + md^2$. Now, here, you have to note that the parallel axis theorem applies only if one of the axis is a central axis, because only then this term will be 0 and then we can use $I = \bar{I} + md^2$, otherwise we cannot use it. Let us understand it by an example. So, let us say we want to find out the moment of inertia of a rod of length L about an axis, which is perpendicular to it and passes through one end.

So, we have this rod, and we want to find out the moment of inertia about this axis. So, it is given that the length of this is L. Therefore, this length will be $L/2$, and the moment of inertia about this axis will be the moment of inertia about the central axis. So, this $1 + md^2$. So, d is $L/2$. So $I = \bar{I}$. Just now we saw it was $\frac{ml^2}{12} + \frac{ml^2}{4}$. So, that comes out to be $ml^2/3$.



Now, let us look at the perpendicular axis theorem. Let us say I have a body which is in the xy plane. And let us say this is the z-axis. So, the statement is following the moment of inertia of a 2D body. So, here, the body is two-dimensional about an axis perpendicular to the plane is equal to the sum of the moment of inertia of the body about two mutually perpendicular axis lying in the plane of the body. So, that means that the moment of inertia about the z-axis is equal to $I_x + I_y$. So, again, it can be proved very easily because the moment of inertia about the z-axis will be let us take a small mass dm and let us say it is at a distance r. So, I_z will be $\int r^2 dm$, and this r can be written as in terms of x and y. So, r is $\sqrt{x^2 + y^2}$. Therefore, r^2 is $x^2 + y^2 dm$, and this I can write as $\int x^2 dm + \int y^2 dm$ and $\int x^2 dm$ is the moment of inertia about the x axis and the $\int y^2 dm$ is the moment of inertia about the y axis. So, we have $I_z = I_x + I_y$ if the body lies in the x y plane. Let us look at it again by an example.

So, let us look at the moment of inertia of a disc about a diameter. So, we have a disc, and let us say this is x-axis, this one is the y-axis, and then the question is what is the moment of inertia about the diameter? So, about let us say x-axis. So, we know that the moment of inertia about I_z is $mr^2/2$, and because of the symmetry, I_x and I_y should be equal.

Therefore, I_x or I_y should be $mr^2/4$. Because $I_x + I_y$ should give you I_z. Let us look at the moment of inertia of the ring about the diameter. So, again about the axis which passes through the center and perpendicular to the plane is I_x. mr^2 In case of the ring, and again,

I_x and I_y should be equal. Therefore, I_x or I_y should be $mr^2/2$. With this, let me stop here. See you in the next class. Thank you.