**MECHANICS**
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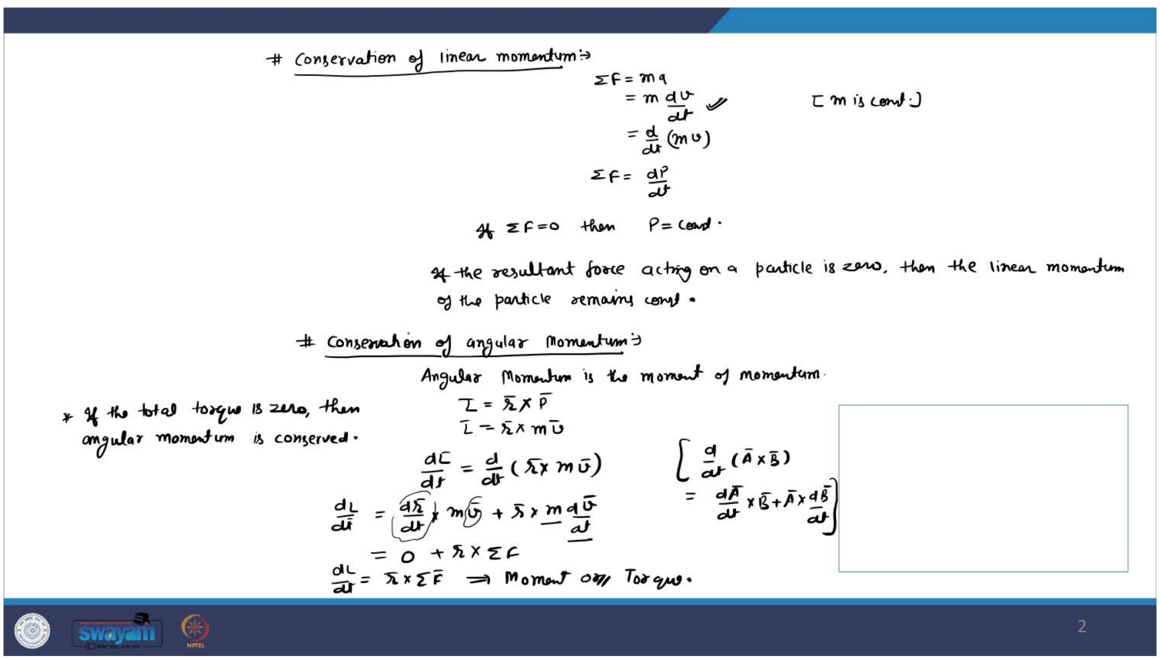
**Lecture 37**
# Work energy method

Hello everyone, welcome to the lecture again. In the last few lectures, we integrated Newton's law to find out the velocity and position of a particle as a function of time. Now, we are going to look at an alternate procedure that is based on energy. Using this method, certain classes of problems can be handled more easily, and we do not need to integrate the differential equation. This is known as work energy method.



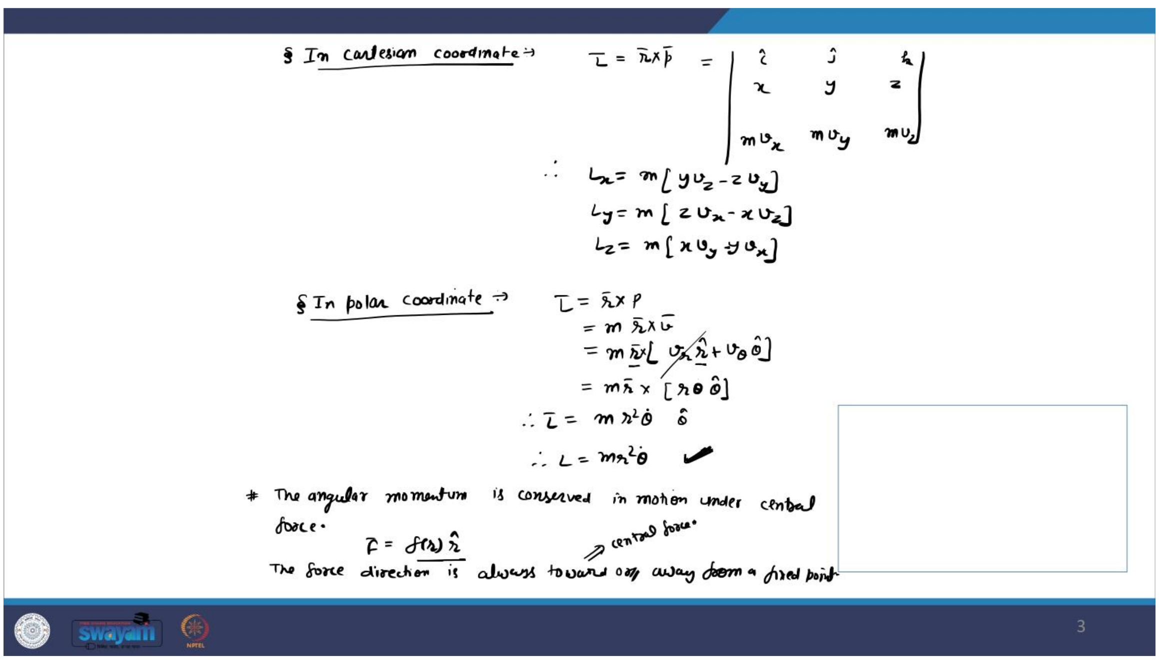

Before we go deep into the work energy method, let us look at the conservation laws. So, let us first look at the conservation of linear momentum. So, we have the Newton equation of motion,

$$\sum F = ma$$
$$= m\frac{dv}{dt}$$

Suppose $m$ is constant, that is the mass is constant and does not vary with time.

$\sum F = \frac{d(mv)}{dt}$.

We know that $mv$ is the momentum. Therefore, $\sum F = \frac{dp}{dt}$. Now, from this equation, it is clear that if $\sum F = 0$, then rate of change of momentum is $0$. That means $p$ is constant. So, what does this mean? This implies that if the resultant force acting on a particle is $0$, then the linear momentum of the particle is constant or remains constant. Now, let us look at conservation of angular momentum. The angular momentum is defined as moment of momentum. So, angular momentum is the moment of momentum. So, when we take moment, we multiply by $\bar{r}$ and then momentum is $m\bar{v}$. Let us denote it by $\bar{L}$. Therefore, $\bar{L}$ is moment of momentum $\bar{p}$. So, I can write down as, $\bar{L} = \bar{r} \times \bar{p} = \bar{r} \times m\bar{v}$. Now, let us look at $\frac{d\bar{L}}{dt} = \frac{d}{dt}\,(\bar{r} \times m\bar{v})$ and we know $\frac{d}{dt}\,(\bar{A} \times \bar{B}) = \frac{d\bar{A}}{dt} \times \bar{B} + \bar{A} \times \frac{d\bar{B}}{dt}$. Let us use this identity. So, we have $\frac{dL}{dt} = \frac{d\bar{r}}{dt} \times m\bar{v} + \bar{r} \times m\frac{d\bar{v}}{dt} = 0 + \bar{r} \times \sum F$. We know that $r \times F$ is the moment or torque. So, we have $\frac{d\bar{L}}{dt} = \bar{r} \times \sum F$ which is nothing but moment or torque. This implies that if the moment or torque is $0$, then $\frac{d\bar{L}}{dt} = 0$. Therefore, $L$ becomes the constant of motion. So, let me write it down in the text. If the total torque acting on the particle is $0$, then angular momentum is conserved.




Now, let us look at the angular momentum in Cartesian coordinate. So, as we saw the angular momentum $\bar{L} = \bar{r} \times \bar{p}$. Therefore, let us write down in the matrix form,

$$\bar{L} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x & y & z \\ mv_x & mv_y & mv_z \end{vmatrix}$$

Let us look at component by component. Therefore, $L_x$ is the coefficient of $i$. So,
$L_x = m(yv_z - zv_y)$
$L_y = m(zv_x - xv_z)$
$L_z = m(mv_y - yv_x)$
Now, let us look at the angular momentum in polar coordinates. So, again angular momentum $\bar{L}$ is moment of momentum and $\bar{p} = m\bar{v}$. So, I can write down as $m\bar{r} \times \bar{v}$. Now, $v$ in polar coordinate can be written as $\bar{v} = v_r\hat{r} + v_\theta\hat{\theta}$, where $v_r$ is the coefficient or the component along the $\hat{r}$ direction and $v_\theta$ is the component of the velocity along the $\hat{\theta}$ direction. Now, the first term here is the cross product of $\bar{r}$ with $\hat{r}$ and they are in the same direction. Therefore, it will be 0. So, this term will be 0. Therefore, we have $m\bar{r} \times (r\dot{\theta}\hat{\theta})$. Now, we know the velocity in the $\hat{\theta}$ direction is $r\dot{\theta}\hat{\theta}$. So, therefore, $\bar{L} = mr^2\dot{\theta}\hat{\theta}$ and its direction will be in the $\hat{\theta}$ direction. So, therefore, $L = mr^2\dot{\theta}$. Now let me show you that the angular momentum is conserved in the motion under central force. So we are going to see that the angular momentum is conserved in motion under central force. So, the central force is defined by a force which is always directed along the $\hat{r}$ direction. So, the force $F$ in the form of some function of $f(r)$ and then its direction along the $\hat{r}$ is a central force. So, let me mention that the force direction is always towards or away from a fixed point. This is the definition of the central force.

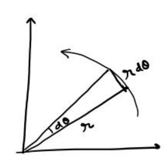
$$\frac{d\bar{L}}{dt} = \bar{r} \times \Sigma\bar{F}$$
$$= \bar{r} \times [f(r)\,\hat{r}]$$
$$= f(r)\ \bar{r} \times \hat{r}$$
$$= 0$$

$\therefore$ L is conserved.

* For the motion under central force, the rate at which the area is swept out by the radius vector is const or areal velocity $\left(\frac{dA}{dt}\right)$ is const.

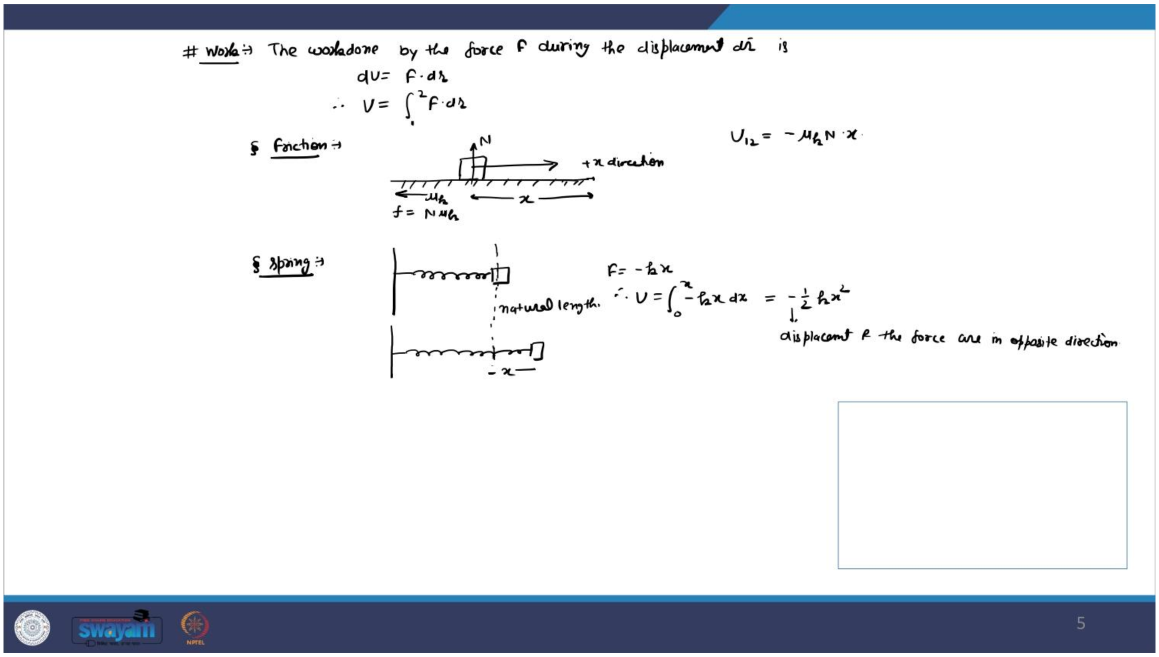
Proof ⇒ Conservation of angular Moment.

$$\frac{d}{dt}(L) = 0$$
$$\frac{d}{dt}(mr^2\dot{\theta}) = 0$$
$$\cancel{m}\frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}\right) = 0$$
$$\frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}\right) = 0$$
$$\frac{d}{dt}\left[\frac{dA}{dt}\right] = 0$$
$$\frac{dA}{dt} = \text{Const.}$$

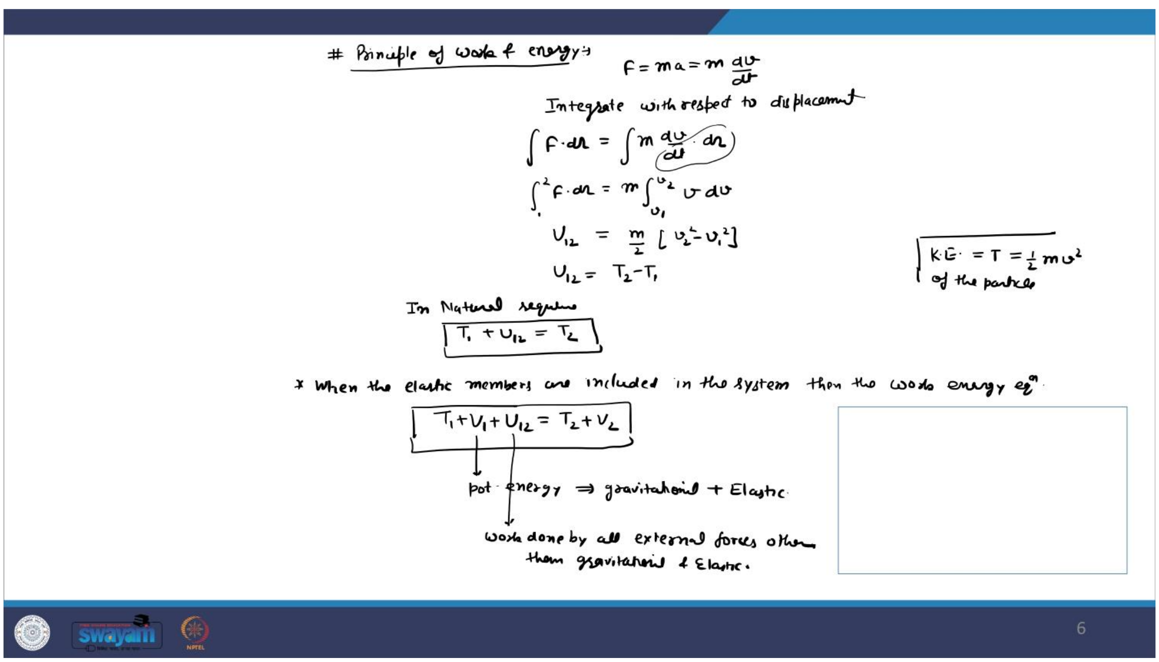



$$dA = \frac{1}{2}r\,r\,d\theta$$
$$dA = \frac{1}{2}r^2 d\theta$$
$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}$$
$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta}$$

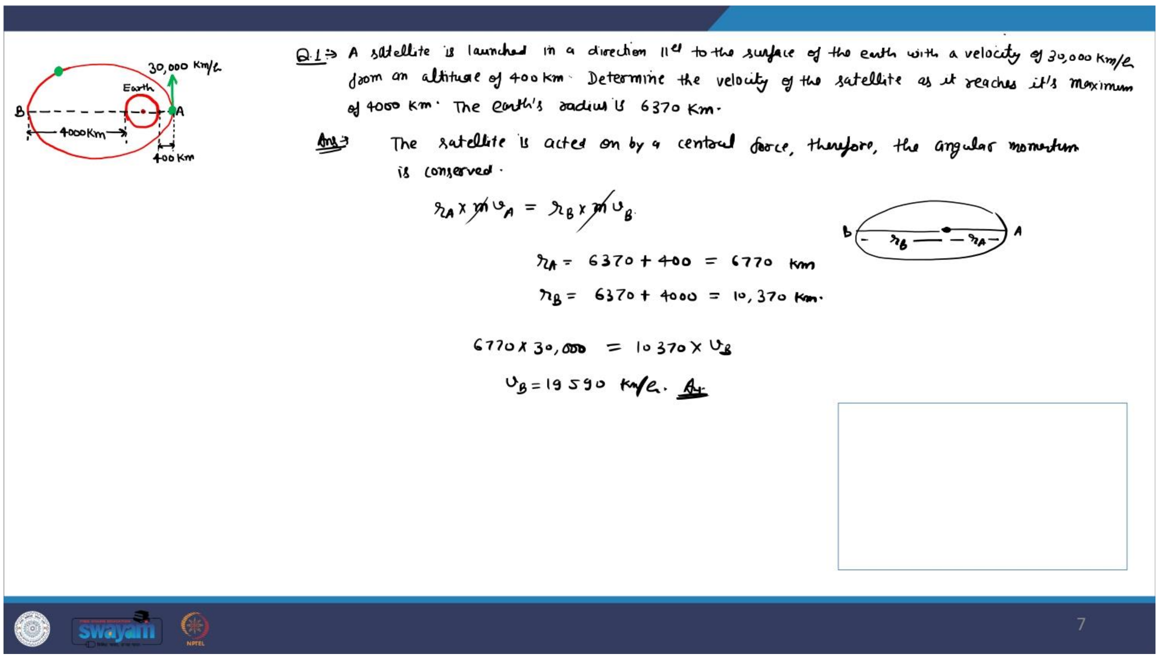
So, just now we proved that $\frac{dL}{dt} = r \times F$ and $F$ is the central force. Therefore, it is going to act in the $\hat{r}$ direction and it will have this functional form. So, let me put $F = f(r)\hat{r}$. $\frac{d\bar{L}}{dt} = \bar{r} \times f(r)\hat{r}$. Now, $\bar{r} \times \hat{r}$, they are in the same direction and there is a cross product. So, therefore, it will be $0$. So, $f(r)\, \bar{r} \times \hat{r}$. So, that means that $\frac{dL}{dt} = 0$. Therefore, $L$ is conserved or the angular momentum is conserved. Now, let us prove another identity and the statement is following. For the motion under central force the rate at which the area is swept out by the radius vector is constant or the areal velocity $\frac{dA}{dt}$ is constant. This is also known as Kepler's law. Now, let us prove this statement. So, just now we saw that under the central motion, the angular momentum remains conserved. So, conservation of angular momentum that means that $\frac{d}{dt}(L) = 0$. And we also saw that in polar coordinate system, $L = mr^2\dot{\theta}$. So, I can write down this as $\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$. Now, $m$ is a constant. So, therefore, $m\frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}\right) = 0$, but $m$ cannot be $0$. Therefore, $\frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}\right) = 0$. Now, let us see what is this quantity. So, let's say a particle is moving under the central force and at sometimes its position vector is $\bar{r}$ and after sometimes the position vector changes by $d\theta$. Therefore, this distance will be $rd\theta$ and therefore, the area under this triangle $dA = \frac{1}{2}rrd\theta$ or $dA = \frac{1}{2}r^2 d\theta$. Therefore, $\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}$ or $\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta}$. And $\frac{1}{2}r^2\dot{\theta} = \frac{dA}{dt}$, I can put it over here. So, I have $\frac{d}{dt}\left(\frac{dA}{dt}\right) = 0$. This implies that $\frac{dA}{dt}$ is a constant of motion or $\frac{dA}{dt} = constant$. And this was the statement that the areal velocity is constant under central force motion.



Now let us revisit the concept of work. So, the work done by the force let's say $F$ during the displacement $d\bar{r}$ is $dU = F.dr$. And if you want to find out total work $U$, then you have to integrate $F.dr$. Let us look at it for the case of friction. So, suppose I have a block and I apply some force because of that this block moves by a distance of let's say $x$ and because it is on a frictional surface, so therefore the frictional force will act on it. Let's say the kinetic friction is $\mu_k$. Therefore, the frictional force will develop in that direction and its value will be $N\mu_k$. So, $f = N\mu_k$ where $N$ is the normal force which is decided by the weight of this block. So, you can see here that the block is moving in the $+x$ direction and the frictional forces are acting in $-x$ direction. Therefore, because of the frictional force, the work done will be the force which is $-\mu_k N.x$. Let us look at the case of spring. Let us say I have a spring and a mass $m$ is attached to it. Let's say this is the natural length of the spring. Now, you extend this spring by an amount $x$. Then, we know that the force $F = -kx$. Therefore, $U$ will be from 0 to $x$ because here in I fix my coordinate. So, let's say this is $x = 0$. So, $U = \int_0^x -kx$ and this will give you $-\frac{1}{2}kx^2$. Here the minus sign is because the displacement and the force, they are in opposite direction. So, this implies that the displacement and the force are in opposite direction.

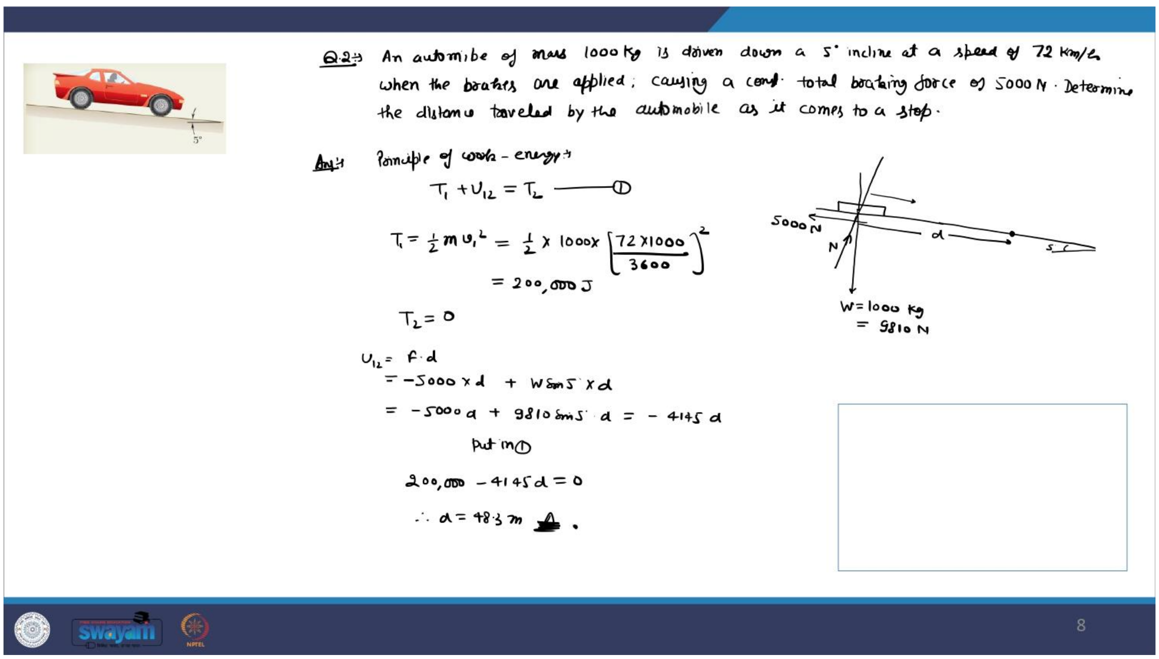



Now, let us look at the principle of work and energy. And let us start with the Newton's second law, which is $F = ma = m\frac{dv}{dt}$. To find out the principle of work and energy, we have to integrate this equation with respect to displacement. So, let us integrate with respect to displacement, that is $dr$. So, $\int F.dr = \int m\frac{dv}{dt}.dr$. Now, $\frac{dr}{dt}$ is nothing but velocity. So, we have $\int_1^2 F.dr = m\int_{v_1}^{v_2} v\,dv$. So, we have $F.dr$ is the work from 1 to 2. So, let me denote it by $U_{12} = \frac{m}{2}(v_2^2 - v_1^2)$. Let us define the kinetic energy, let us denote it by $T = \frac{1}{2}mv^2$ and this is the kinetic energy of the particle. So, we can write down $U_{12} = T_2 - T_1$ or let me write down in the natural sequence, we have a particle which is moving with a kinetic energy $T_1$, work 1 to 2 is done on it, then its kinetic energy becomes $T_2$. $T_1 + U_{12} = T_2$. Let's say there are elastic members which are included in the system. So, when the elastic members are included in the system then, the work energy equation can be written as $T_1 + V_1 + U_{12} = T_2 + V_2$ . Here $V_1$ is the potential energy which include the gravitational potential energy plus elastic potential energy. So, gravitational plus elastic and $U_{12}$ is the work done by let's say all external forces other than the gravitational and elastic. This is the statement of principle of work and energy. It is based on the conservation of energy. It states that if $T_1$ is the initial kinetic energy and work $U_{12}$ is done on it, then the kinetic energy becomes $T_2$ and the relation is $T_1 + U_{12} = T_2$.




Now, based on these concepts, let us look at the some of the problems and the first problem statement is following. A satellite is launched in a direction parallel to the surface of the earth with a velocity of $30000 \frac{km}{hr}$ from an altitude of $400\ km$. Determine the velocity of the satellite as it reaches its maximum of $4000\ km$ and it is given that the earth radius is $6370\ km$. Now, in this question, we have been asked to find out the velocity of the satellite. Note that the satellite is performing a central motion. Therefore, the angular momentum will be conserved. So, let me state that the satellite is acted on by a central force. Therefore, the angular momentum is conserved. So, angular momentum as we have discussed it is the moment of momentum. Let us apply that at point $A$ and $B$ to find out the velocity at $B$. So, we have $L$ which is moment of momentum $r_A \times mv_A = r_B \times mv_B$. Now the mass does not change, therefore, that will get cancelled and we have $r_A$. So, $r_A$ you have to measure from the centre of the earth. So, again the situation is following. You have the earth centre here. This point is $A$ and that point is $B$. So, this is your $r_A$, this is your $r_B$ and $r_A$ and $r_B$ are from the centre. Therefore, $r_A = 6370 + 400$ . And this is equal to $6770\ km$ and $r_B = 6370 + 4000$. And that comes out to be $10370\ km$. Now, let us put it above. So, we have $6770 \times 30000 = 10370 \times v_B . v_B = 19590\ km/hr$.

Now, let us look at another problem statement. An automobile of mass $1000\ kg$ is driven




down a 5° inclined at a speed of $72\ km/hr$ when the brakes are applied causing a constant total braking force of $5000\ N$ and we have been asked to determine the distance travelled by the automobile as it comes to a stop. So, we have the following situation. We have inclination which is 5° and a car is moving on it. The weight of the car is going to act downwards. So, $W = 1000\ kg = 9810\ N$. So, we multiply it by $g$. And the braking force of $5000\ N$ is applied. So, since the car is moving in that direction, therefore the braking force will act in this direction and its value is $5000\ N$. Now, the normal force $N$ will be perpendicular to the 5° inclination. So, this is the direction of $N$ and now we can use the work and energy. Let's say this car stops after traveling a distance $d$ and we have to find out what is this $d$. So, let us use the principle of work energy which is $T_1 + U_{12} = T_2$. Let me call this equation number 1. The $T_1$, which is the initial kinetic energy, it is $\frac{1}{2} m v_1^2$, which is $\frac{1}{2} \times 1000 \times \left(72 \times \frac{1000}{3600}\right)^2$. So, this gives you $200000\ J$. Now, $T_2$ is the final kinetic energy. And since the car comes into the rest, therefore, $T_2 = 0$. Let us look at $U_{12}$.

So, $U_{12} = F.d$ and the breaking force of $5000$ is acting over a distance of $d$. So, therefore, $d$ and it will be minus because the force and the displacement, they are in opposite direction plus we have the component of $W$ along the inclination. So, it will be $Wsin5°$ multiplied by the distance $d$. So, this comes out to be $-5000 \times d + Wsin5° \times d$ and we know the value of $sin5°$. We put it here, then you get $-4145d$. Let us put everything in equation number 1. So, we have $200000 - 4145\ d = 0$ and this gives you $d = 48.3\ m$.

So, with this let me stop here. See you in the next class. Thank you.