MECHANICS

Prof. Anjani Kumar Tiwari

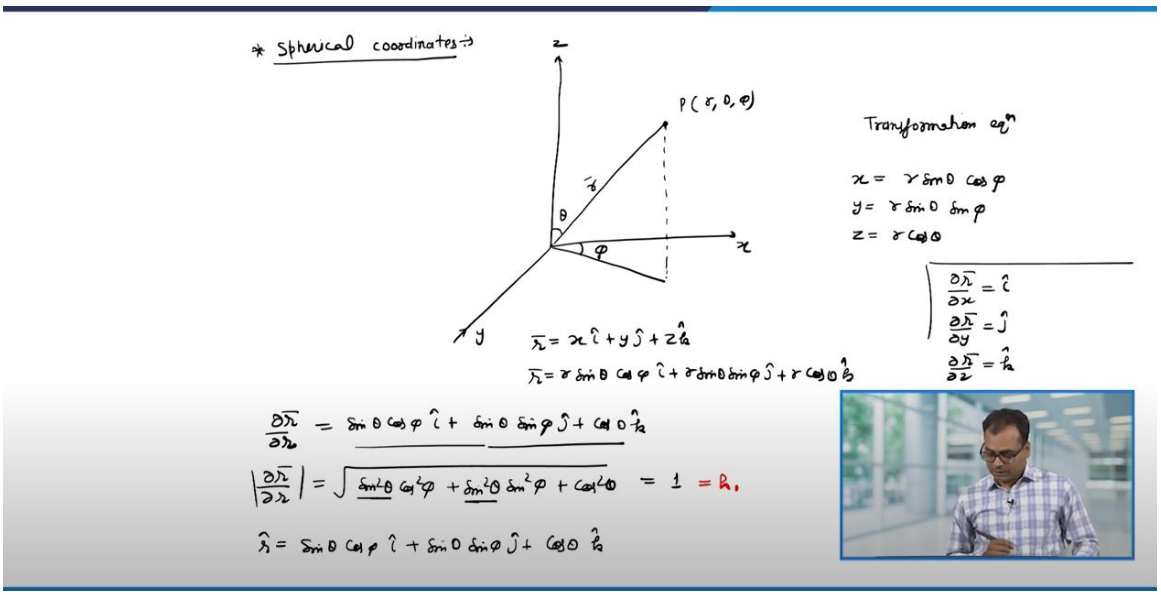
Department of Physics

Indian Institute of Technology, Roorkee

Lecture: 31

Coordinate systems: spherical coordinates

Hello everyone, welcome to the lecture again. In the last class, we discussed the velocity and acceleration of a particle in Cartesian coordinate and planar polar coordinate system.

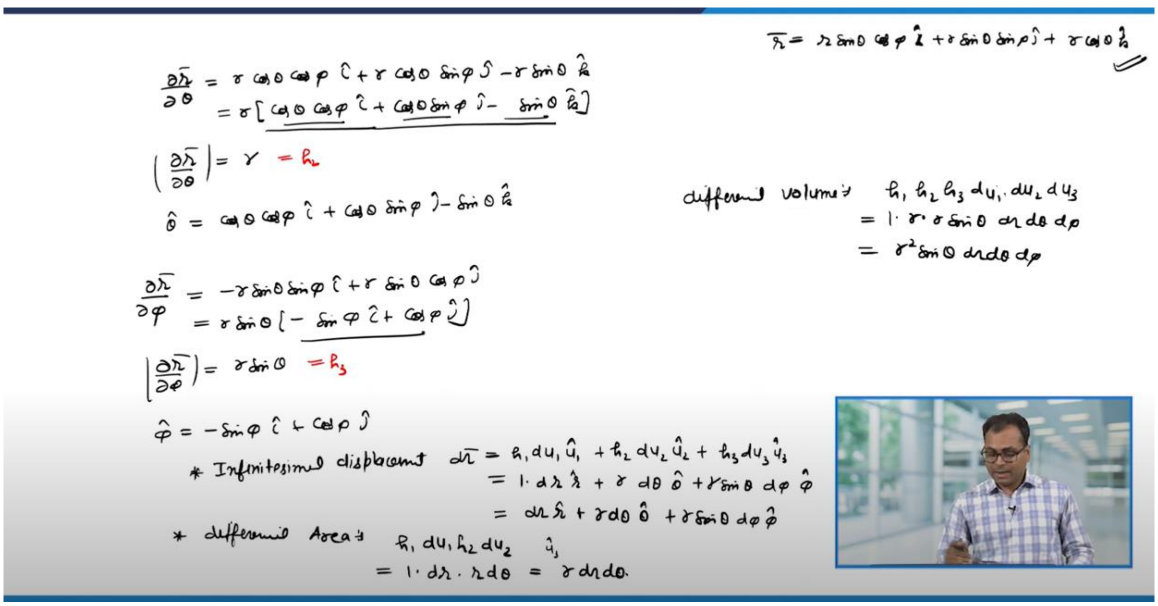


Today, we are going to discuss the spherical coordinates. So, spherical coordinates, let us say we have the x, y and z axis. Let us say there is a point P whose coordinate is let us say r, θ and ϕ, where r is the distance of this point from the origin. θ Is the angle that it makes from the z-axis and ϕ is the angle that its projection on y-x plane makes with the x-axis. So, this angle is ϕ. Now, let us write down the r, θ and ϕ. So, in terms of x, y and z and these are called the transformation equation.

So, this is how we can transform from Cartesian coordinate to a spherical coordinate. So, x is equal to, you can see the figure. So, x is the projection of r on the y-x plane. So, this is $rsin\theta$ and then its projection along the x-axis. So, therefore, it will be $cos\phi$.

Similarly, y will be $rsin\theta sin\phi$ and z will be $rcos\,\theta$. Now, vector r, I can write down as $xi + yj + zk$ and now since i know what is x, y and z, so therefore, r vector can be written as $r = sin\theta cos\phi\hat{\imath} + sin\theta sin\phi\hat{\jmath} + cos\theta\hat{k}$. Now, let us calculate $d\vec{r}/dr$. This should be a unit vector in the r direction because in Cartesian coordinate also, you note that dr/dx was i, dr/dy was j and dr/dz was unit vector along z direction that is $\hat{k}$.

So, $d\vec{r}/dr$ will be $sin\theta cos\phi\hat{\imath} + sin\theta sin\phi\hat{\jmath} + cos\theta\hat{k}$ and $d\vec{r}/dr$, its modulus should be 1. Let us see that. It is $\sqrt{sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}$. Now, from here, you can take $sin^2\theta$ common, then it will be $\cos^2\phi + \sin^2\phi$, that will be 1, and then you will get $sin^2\theta + \cos^2\theta$, that will be 1. Therefore, this is also 1, and therefore, this is our h_1, and $\hat{r}$ is 1. This. It is $sin\theta cos\phi\hat{\imath} + sin\theta sin\phi\hat{\jmath} + cos\theta\hat{k}$.



Now, let us calculate $\delta r/\delta\theta$. That should give you a unit vector in θ direction. So, our vector r is this which is $rsin\theta cos\phi\hat{\imath} + rsin\theta sin\phi\hat{\jmath} + rcos\theta\hat{k}$. So, we are interested in finding out $\delta r/\delta\theta$ that should give you the unit vector in θ direction. So, it is $rcos\theta cos\phi\hat{\imath} + rcos\theta sin\phi\hat{\jmath} - rsin\theta\hat{k}$. I can take r as a common factor. So, it is $cos\theta cos\phi\hat{\imath} + cos\theta sin\phi\hat{\jmath} - sin\theta\hat{k}$ and you can see that $\delta r/\delta\theta$ its modulus comes out to be r.

So, again you can do a square root this square plus that square plus that square and then you have r^2 that comes out to be r. Let us call this as h_2. Scale factor along θ direction. Therefore, our unit vector $\hat{\theta}$ should be this quantity which is $cos\theta cos\phi\hat{\imath} + cos\theta sin\phi\hat{\jmath} -$

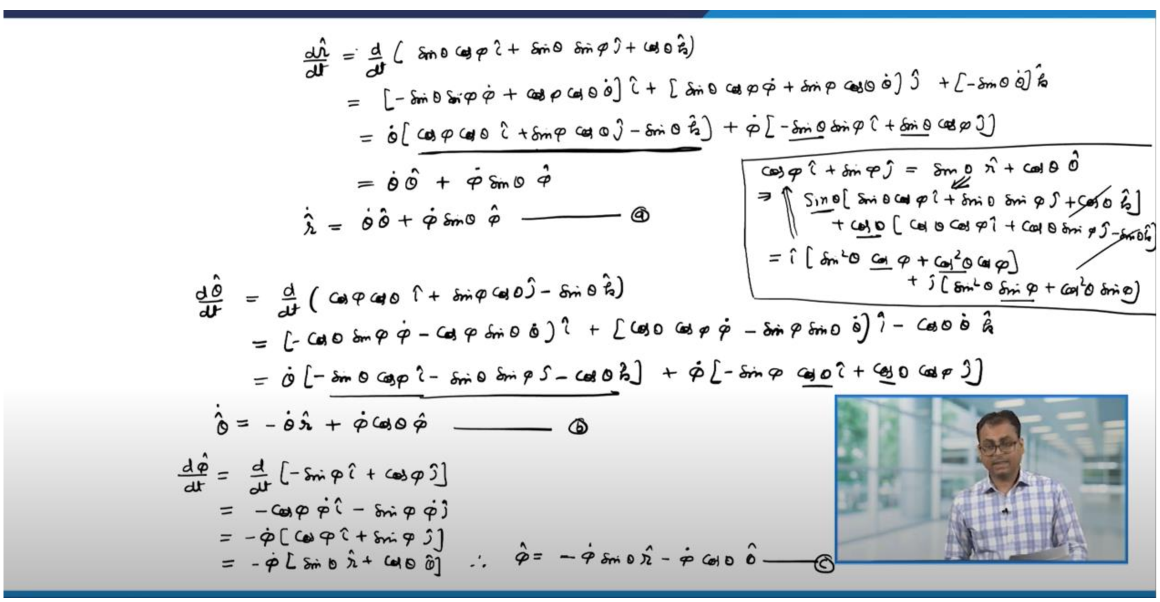
$sin\theta\hat{k}$. Now, let us find out $\delta\vec{r}/\delta\phi$. This would give you a unit vector along the ϕ direction. So, again let us use this and differentiate it with respect to ϕ.

We have $-rsin\theta sin\phi\hat{\imath} + rsin\theta cos\phi\hat{\jmath}$. I can take $rsin\theta$ as a common vector, then we get $-sin\phi\hat{\imath} + cos\phi\hat{\jmath}$ and you can see that $\delta r/\delta\phi$ mode is $rsin\theta$. So, therefore, this is our h_3 and whatever is left is the unit vector along the ϕ direction. So, $\hat{\phi}$ is $-sin\phi\hat{\imath} + cos\phi\hat{\jmath}$. Now, we can write down the infinitesimal displacement vector dr in terms of $h_1h_2h_3$ and $u_1u_2u_3$ because in spherical coordinate system, these are our u_1u_2 and u_3.

Therefore, the infinitesimal displacement dr is will be $h_1du_1\widehat{u_1} + h_2du_2\widehat{u_2} + h_3du_3\widehat{u_3}$ and h_1 is 1 du_1 is dr and $\hat{r}$ plus h_2 is ru_2 is θ so this is $d\theta\hat{\theta}$ plus h_3 is $rsin\theta$ and u_3 is ϕ, so $d\phi\hat{\phi}$. Therefore, the infinitesimal displacement vector in spherical coordinate becomes $dr\hat{r} + rd\theta\hat{\theta} + rsin\theta d\phi\hat{\phi}$. Now, let us look at the differential area element.

This is h_1du_1 and h_2du_2 and its direction is in the u_3 direction. So, it becomes $1dr$ and $rd\theta$. So, this becomes $rdrd\theta$. Now, let us look at differential volume.

So, we know that it is $h_1, h_2, h_3, du_1, du_2, du_3$ and this becomes $1rrsin\theta drd\theta d\phi$. So, this becomes $r^2sin\theta drd\theta d\phi$. Now, we are going to find out the expression for the velocity and acceleration in a spherical coordinate system. But as you know that as the particle moves, then the unit vector along the $r\theta$ and ϕ direction are going to change.



Therefore, we have to find out how $\hat{r}\hat{\theta}$ and $\hat{\phi}$ changes with time. So, let us calculate what is $d\hat{r}/dt$. This will be d/dt and $\hat{r}$. The expression for $\hat{r}$, we already know. It is this.

So, it is $sin\theta cos\phi\hat{\imath} + sin\theta sin\phi\hat{\jmath} + cos\theta\hat{k}$. Now, i, j and $\hat{k}$, they do not change with time. Therefore, we can take them outside. So, it will be $\left(-sin\theta sin\phi\dot{\phi} + cos\phi cos\theta\dot{\theta}\right)\hat{\imath} + \left(sin\theta cos\phi\dot{\phi} + sin\phi cos\theta\dot{\theta}\right)\hat{\jmath} + \left(-sin\theta\dot{\theta}\right)\hat{k}$ and let us collect the term of $\dot{\theta}$.

So, it will be $\dot{\theta}\left(cos\phi cos\theta\hat{\imath} + sin\phi cos\theta\hat{\jmath} - sin\theta\hat{k}\right) + \dot{\phi}(-\,sin\theta sin\phi\hat{\imath} + sin\theta cos\phi\hat{\jmath})$. Now, this term in the bracket is nothing but $\hat{\theta}$ you can see here and we have $\dot{\phi}$. I can take $sin\theta$ as a common factor and then whatever is left is $\hat{\phi}$. So, we have $\hat{\phi}$.

Therefore, $\dot{\hat{r}}$ becomes $\dot{\theta}\hat{\theta} + \dot{\phi}sin\theta\hat{\phi}$. Let us call it equation number A and let us now calculate what is $\delta\hat{\theta}/\delta t$. So, this will be $\frac{d}{dt}\hat{\theta}$ and this is the $\hat{\theta}$. So, it becomes $cos\phi cos\theta i + sin\phi cos\theta j - sin\theta k$ and it will be $\left(-cos\theta sin\phi\dot{\phi} - cos\phi sin\theta\dot{\theta}\right)\hat{\imath} + (cos\theta cos\phi\dot{\phi} - sin\phi sin\theta\dot{\theta})\hat{\jmath} - cos\theta\dot{\theta}\hat{k}$.

And now, I can collect the coefficient of $\dot{\theta}$ and $\dot{\phi}$. So, the coefficient of $\dot{\theta}$ will be$(-sin\theta cos\phi\hat{\imath} - sin\theta sin\phi\hat{\jmath} - cos\theta\hat{k}) + \dot{\phi}(-sin\phi cos\theta\hat{\imath} + cos\theta\hat{\jmath}cos\phi\hat{\jmath})$. Therefore, $\dot{\hat{\theta}}$ becomes, so this quantity over here is nothing but$-\hat{r}$. So, you can see this. So, therefore, I can write it down as minus $\dot{\theta}\hat{r}$ and here I can take $cos\theta$ as a common factor and then whatever is left is $\hat{\phi}$.

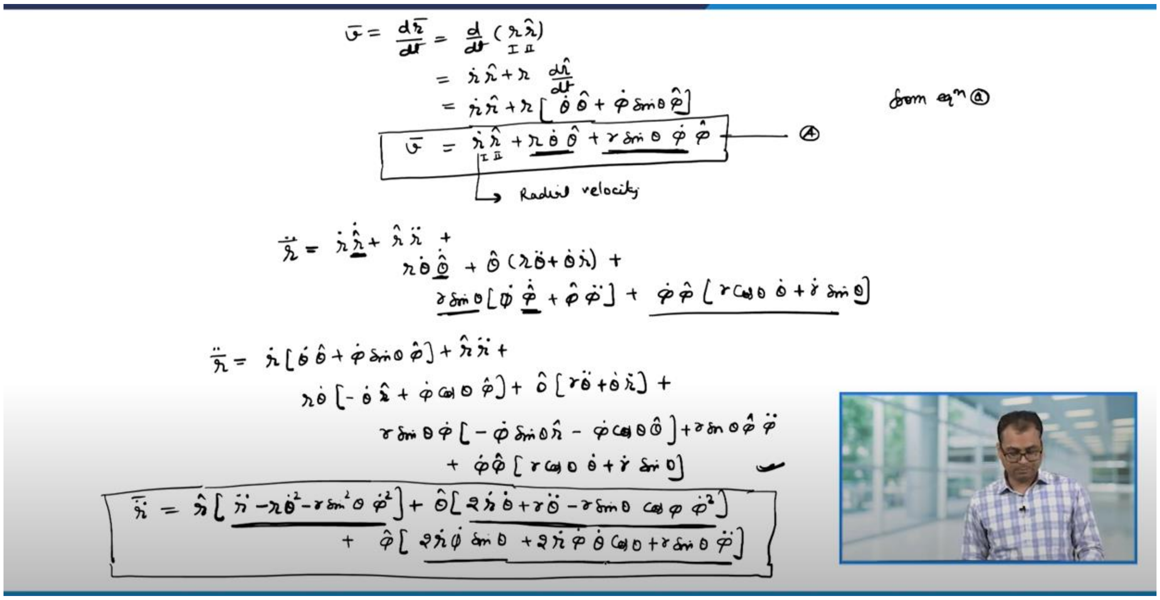
So, it becomes $\dot{\phi}cos\theta\hat{\phi}$. Let us call it equation number P. Now, let us calculate $d\hat{\phi}/dt$ and it is d/dt. The value of $\hat{\phi}$ was$-sin\phi\hat{\imath} + cos\phi\hat{\jmath}$. Therefore, this becomes$-cos\phi\dot{\phi}\hat{\imath} - sin\phi\dot{\phi}\hat{\jmath}$. And I can take $\dot{\phi}$ or $-\dot{\phi}$ as a common factor.

So, this becomes $cos\phi\hat{\imath} + sin\phi\hat{\jmath}$. And I will show you in a second that$cos\phi\hat{\imath} + sin\phi\hat{\jmath} = sin\theta\hat{r} + cos\theta\hat{\theta}$. Therefore, this becomes $-\dot{\phi}sin\theta\hat{r} + cos\theta\hat{\theta}$. Therefore, $\dot{\phi}$ becomes$-\dot{\phi}sin\theta r\hat{r} - \dot{\phi}cos\theta\hat{\theta}$.

Let us call it equation number C. Now, let us come back to this identity. For that, let us look at the right hand side. The right hand side, I can write down as $sin\theta$ and the value of $\hat{r}$ I know $\hat{r}$ was $sin\theta cos\phi\hat{\imath} + sin\theta sin\phi\hat{\jmath} + cos\theta\hat{k}$.

So, we are solving this side plus $cos\theta$ and $\hat{\theta}$ was $cos\theta cos\phi\hat{\imath} + cos\theta sin\phi\hat{\jmath} - sin\theta\hat{k}$. Now, this $sin\theta$ get cancel with that $cos\theta$ because there is this $sin\theta$ and $cos\theta$ multiplication. Therefore, we get$\hat{\imath}(\sin^2\theta cos\phi + \cos^2\theta cos\phi) + \hat{\jmath}(\sin^2\theta sin\phi + \cos^2\theta sin\phi)$. And this becomes this because you can take $cos\phi$ as a common factor. So, it is $\sin^2\theta + \cos^2\theta$, which is 1.

So, it becomes i $cos\phi$ plus here also you can take $sin\phi$ as a common factor. So, it becomes $\sin^2\theta + \cos^2\theta$, which is again 1. So, we get $\hat{\jmath}sin\phi$.



So, now we have find out the time derivative of the unit vectors along r direction, θ direction and ϕ direction which we are going to use to find out the velocity and acceleration in spherical polar coordinates. So, let us first look at the velocity.

V is $d\vec{r}/dt$. This is d/dt, $\vec{r}$ is r, $\hat{r}$ and this I can differentiate. So, this becomes $\dot{r}\hat{r} + rd\hat{r}/dt$. So, this I can write down as $\dot{r}\hat{r} + r$. Now, $d\hat{r}/dt$ i have already calculated from equation number a. So, let us use that.

So, we have $\dot{\theta}\hat{\theta} + \dot{\phi}sin\theta\hat{\phi}$. This is from equation number (a), okay? So, this becomes $\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + rsin\theta\dot{\phi}\hat{\phi}$. This is the velocity in spherical polar coordinates. So, as you can see, this is the velocity along $\hat{r}$.

So, this is radial velocity. This is the velocity along $\hat{\theta}$ direction and this is the velocity along the $\hat{\phi}$ direction. Now, let us calculate the acceleration. So, the acceleration is $\ddot{r}$. $\ddot{r}$ is equal to, so let me take this equation. Let us call it equation number (A) and find out its derivative. So, this is first function. This is second function. So, its derivative is first function as it is derivative of the second one and then second function as it is the derivative of the first one and similarly for the second and third term.

So, we have plus $r\dot{\theta}\dot{\hat{\theta}} + \hat{\theta}r\ddot{\theta} + \dot{\theta}\dot{r}$ plus we have this term now. So, it is $rsin\theta\dot{\phi}\dot{\hat{\phi}} + \hat{\phi}\ddot{\phi} + \dot{\phi}\hat{\phi}rcos\theta\dot{\theta} + \dot{r}sin\theta$. Now, let us put the value of $\dot{\hat{r}}\dot{\hat{\theta}}$ and $\dot{\hat{\phi}}$ from equation number a, b and c. So, we have $\ddot{\vec{r}} = \dot{r}$.

The value of $\hat{r}$ is $\dot{\theta}\hat{\theta} + \dot{\phi}sin\theta\hat{\phi} + \hat{r}\ddot{r}$ plus, now let us put the value of $\dot{\hat{\theta}}$, so we have $r\dot{\theta}\dot{\hat{\theta}}$ is$-\dot{\theta}\hat{r} + \dot{\phi}cos\theta\hat{\phi}$ plus we have $\hat{\theta}r\ddot{\theta} + \dot{\theta}\dot{r}$, plus we have $rsin\theta$, so $rsin\theta$, and we have $\dot{\phi}$, so let us put that $\dot{\phi}$ also outside, $\dot{\phi}$ and the value of $\dot{\hat{\phi}}$ is minus $\dot{\phi}sin\theta\hat{r} - \dot{\phi}cos\theta\hat{\theta}$ plus we have this term. So, it is $rsin\theta\hat{\phi}\ddot{\phi}$ plus the last term $\dot{\phi}\hat{\phi}rcos\theta\dot{\theta} + \dot{r}sin\theta$.

Now, we can collect the coefficient of $\hat{r}\hat{\theta}$ and $\hat{\phi}$. So, we have $\ddot{r} = \hat{r}$. Its coefficient from this equation is $\ddot{r} - r\dot{\theta}^2 - rsin^2\theta\dot{\phi}^2$. Plus the coefficient of $\dot{\hat{\theta}}$ is $2r_0\dot{\theta} + r\ddot{\theta} - rsin\theta cos\phi\dot{\phi}^2$. Plus the coefficient of $\hat{\phi}$ is $2\dot{r}\dot{\phi}sin\theta + 2\dot{r}\dot{\phi}\dot{\theta}cos\theta + rsin\theta\ddot{\phi}$ and this is our master equation to calculate the acceleration. So, here the first terms gives you the acceleration along the $\hat{r}$ direction. This gives you the acceleration along the $\hat{\theta}$ direction and this term will give you the acceleration along the $\hat{\phi}$ direction. With this, let me stop here.

See you in the next class. Thank you.