MECHANICS

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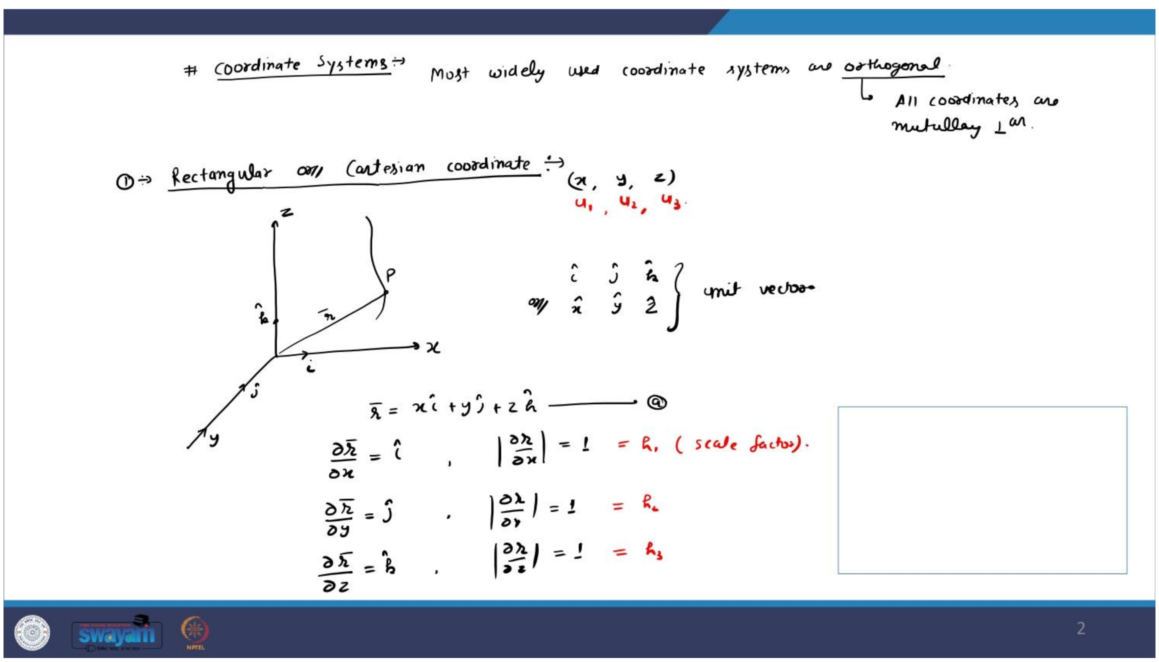
Lecture 30

Coordinate systems: Cartesian and planar polar coordinates

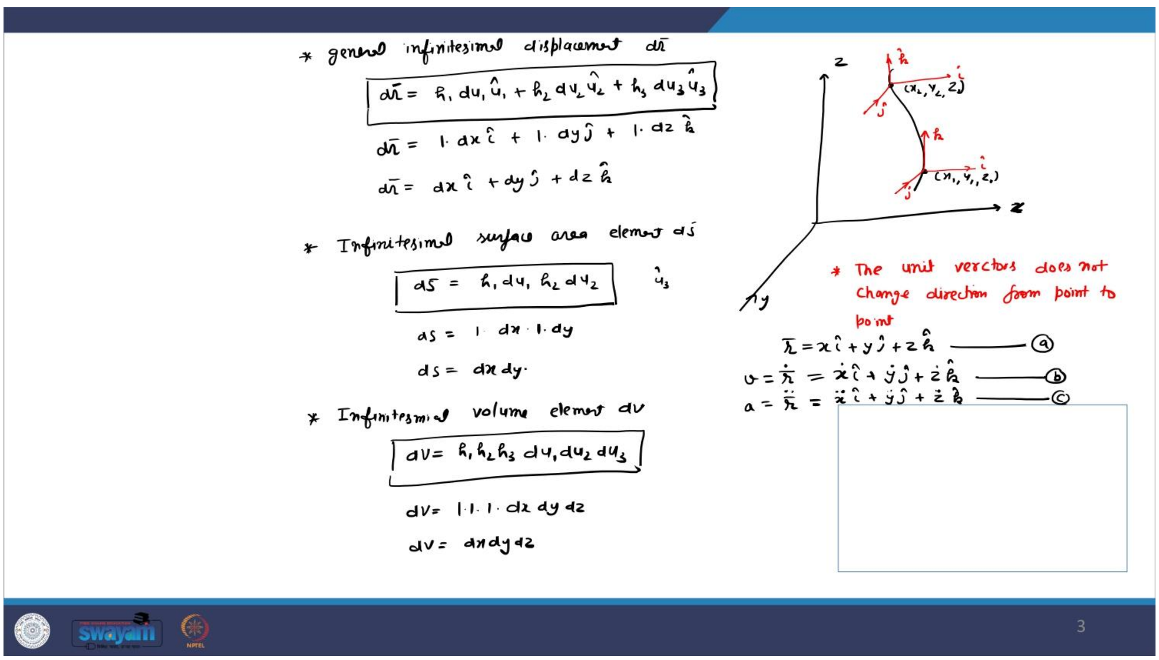
Hello everyone, welcome to the lecture again. Now, we are going to start the second part of this course that is dynamics. In dynamics, depending upon the symmetry of the problem, we have to choose the right coordinate system. Therefore, today we are going to discuss the Cartesian and planar polar coordinates. So, we are going to discuss the coordinate systems and the most widely used coordinate systems are orthogonal that means that the coordinate that are involved in defining the position vector, etc., they are perpendicular to each other. So, that is the meaning of orthogonal. This implies that all coordinates are mutually perpendicular.



Let us first discuss what is Cartesian coordinate or rectangular coordinate. So, as all of us know, in Cartesian coordinate, the axis are x-axis, y-axis and z-axis and the position vector of let's say a particle is moving and let's say the position vector of a particle at some instant is r and let's say this point is P. Now, along the x, y and z direction, you can take the unit vector. So, you can say either $\hat{\imath}$, $\hat{\jmath}$ or $\hat{k}$ or you can take $\hat{x}$, $\hat{y}$ and $\hat{z}$ as the unit vector. So, let's say this is $\hat{\imath}$, this one is $\hat{\jmath}$ and this one is $\hat{k}$. Therefore, when a particle is moving in the Cartesian coordinates, its position vector $\bar{r}$ can be written as

$$\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} -------- (a).$$

Now, in Cartesian coordinate, as you saw that x, y and z are the coordinate axes. Let us call them u_1, u_2 and u_3. Now, when we calculate this quantity $\frac{\delta \bar{r}}{\delta x} = \hat{\imath}$. Similarly, $\frac{\delta \bar{r}}{\delta y} = \hat{\jmath}$ and when you calculate $\frac{\delta \bar{r}}{\delta z} = \hat{k}$. Also note that $\left|\frac{\delta \bar{r}}{\delta x}\right| = 1$ because it is a unit vector. Similarly, $\left|\frac{\delta \bar{r}}{\delta y}\right| = 1$ and $\left|\frac{\delta \bar{r}}{\delta z}\right| = 1$. Let me call them the scale factors along x, y and z direction. So, let me call them h_1, h_2 and h_3 and this h is just a scale factor. Now, in terms of the u_1, u_2, u_3, h_1, h_2 and h_3, we can write down various quantities. For example, what is the general infinitesimal displacement, what is the infinitesimal surface area and what is the infinitesimal volume element, etc.



Now, let me write down the general infinitesimal displacement vector $d\bar{r}$ in terms of h_1, h_2, h_3 and u_1, u_2, u_3. So, the general infinitesimal displacement let us call it

$$d\bar{r} = h_1 du_1 \hat{u}_1 + h_2 du_2 \hat{u}_2 + h_3 du_3 \hat{u}_3.$$

I am not proving this formula, but we will use this formula in various coordinate systems. So, here $h_1 = 1, h_2 = 1, h_3 = 1$ and $u_1 = x, u_2 = y, u_3 = z$.

So, therefore, $d\bar{r} = 1.dx\,\hat{\imath} + 1.dy\,\hat{\jmath} + 1.dz\,\hat{k}$.

Therefore, $d\bar{r} = 1.dx\,\hat{\imath} + 1.dy\,\hat{\jmath} + 1.dz\,\hat{k}$ and this is something that we already know.

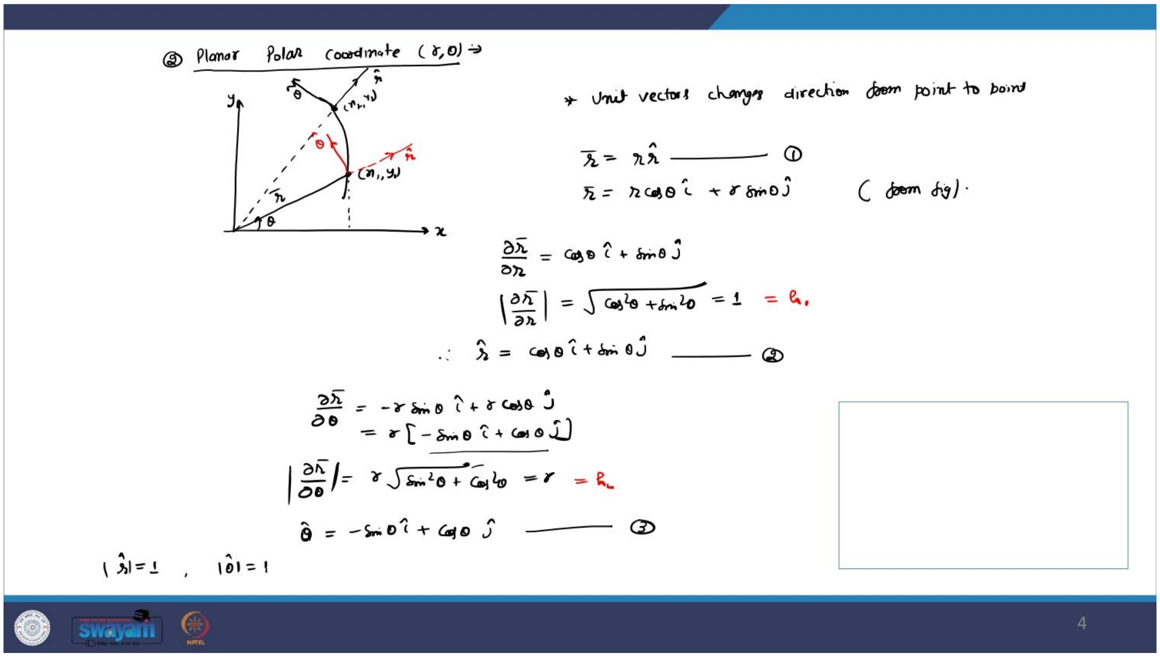
Now, let me write down the infinitesimal surface area in terms of h_1 , h_2, h_3. Let's say it is ds. So,

$ds = h_1 du_1 h_2 du_2\,\hat{u}_3$ and its direction will be perpendicular to u_1 and u_2. So, it will be in $\hat{u}_3$ direction. Let us see ds will be $h_1 = 1, h_2 = 1$ and $du_1 = dx, du_2 = dy$ and its direction will be perpendicular to xy because we are calculating the area in xy. Therefore, it will be in $\hat{z}$ direction. I am not writing down the direction here $ds = dxdy$ and again this is something that we already know.

Now, let me write down infinitesimal volume element dV in terms of h_1, h_2, h_3. So,

$dV = h_1 h_2 h_3 du_1 du_2 du_3$. So, in Cartesian coordinate, $dV = dxdydz$. And we already know again that the volume element in Cartesian coordinate is $dxdydz$. Now, let us look at very important point. In Cartesian coordinate, let's say the particle is moving. And at some instant, it is at this point, let's say it is some x_1, y_1, z_1. And after some time, it is at other point, let's say x_2, y_2, z_2. Then look at the unit vectors. So, here the unit vectors are $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ and when it goes to different point, then again the unit vectors are $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$. This tells us that the unit vectors does not change direction from point to point. So, when the particle moves from one point to other points, the unit vector, they do not change direction in Cartesian coordinate. So, let us look at equation number (a) again, which is position vector $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ in Cartesian coordinate. Therefore, the velocity or $\dot{r} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{z}$. Note that because $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$, they do not change. Therefore, they are constant and we do not need to take their differentiation. Now, let me write down the acceleration $a = \ddot{r} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}$. Now, let me emphasize that the expression of velocity and acceleration is quite simple in Cartesian coordinate because the unit vectors do not change when the particle or object move from one point to other.

Let us now look at planar polar coordinates. Let's say we have x-axis, y-axis and a particle is moving. Let's say it is at some point x_1, y_1 and later it is at some point x_2, y_2. In polar coordinate, the position of the particle is denoted by r and θ. So, let's say the position vector is r and it is making an angle θ from the x-axis. Therefore, the unit vectors along the r and θ will be $\hat{r}$ and perpendicular to it will be $\hat{\theta}$. Now, when the particle moves from let's say x_1, y_1 to some other point x_2, y_2, then you can see that the direction of $\hat{r}$ and the direction of $\hat{\theta}$ has changed. So, let me write it the unit vectors changes direction from point to point for planar coordinate system.



Here, planar means the body is moving in a plane. Now, let me write down $\bar{r}$.

So, $\bar{r} = r\hat{r} --------(1)$.

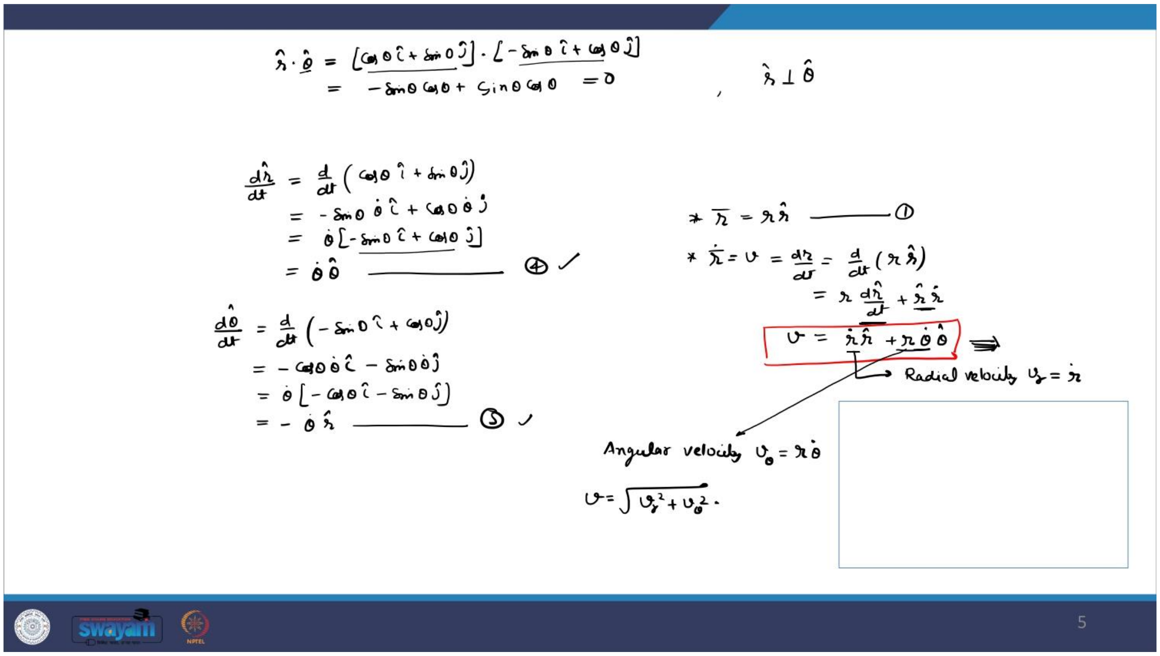
Now, from the figure, you can write down $\bar{r}$ in terms of magnitude r and θ.

So, $\bar{r} = rcos\theta\hat{\imath} + rsin\theta\,\hat{\jmath}$. This is from the figure.

Now, let me again calculate what is $\frac{\delta\bar{r}}{\delta r}$. Earlier, note that we calculated $\frac{\delta\bar{r}}{\delta x}$. where x was the coordinate axis. So, here the coordinates are r and θ. So, let us calculate $\frac{\delta\bar{r}}{\delta r} = cos\theta\hat{\imath} + sin\theta\hat{\jmath}$. And it should be a unit vector. Therefore, $\left|\frac{\delta\bar{r}}{\delta r}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1 = h_1$. Therefore, we have $\hat{r} = cos\theta\hat{\imath} + sin\theta\hat{\jmath} -----(2)$.

Now, let us calculate $\frac{\delta\bar{r}}{\delta\theta}$. So, $\frac{\delta\bar{r}}{\delta\theta}$ should be a unit vector along θ. So, let us see that $\frac{\delta\bar{r}}{\delta\theta} = -rsin\theta\hat{\imath} + rcos\theta\hat{\jmath}$ and we can take r as a common factor. So, it is $r(-sin\theta\hat{\imath} + cos\theta\hat{\jmath})$. Now, $\left|\frac{\delta\bar{r}}{\delta\theta}\right| = r\sqrt{\sin^2\theta + \cos^2\theta} = r = h_2$ and then the unit vector becomes $\hat{\theta} = -sin\theta\,\hat{\imath} + cos\theta\hat{\jmath} -----(3)$.

So, now we got the unit vector in terms of $\hat{\imath}$ and $\hat{\jmath}$ which are the unit vector in the Cartesian coordinate. You can check that $|\hat{r}| = 1$ and the $|\hat{\theta}| = 1$, okay.



Now, let us see whether r and θ are orthogonal or not. So, for that, we have to take the dot product of $\hat{r}$ and $\hat{\theta}$. So, let us take

$\hat{r}.\hat{\theta} = (cos\theta\hat{\imath} + sin\theta\hat{\jmath}).(-sin\theta\,\hat{\imath} + cos\theta\hat{\jmath}) = -sin\theta cos\theta + sin7theta\ cos]theta = 0$. So, that means that $\hat{r} \perp \hat{\theta}$.

Now, we are going to find out the expression for the velocity and acceleration in planar polar coordinate system. But for that, we have to first find out how $\hat{r}$ and $\hat{\theta}$ changes with time. So, therefore, let us find out what is $\frac{d\hat{r}}{dt}$.

$$\frac{d\hat{r}}{dt} = \frac{d}{dt}(cos\theta\hat{\imath}+sin\theta\hat{\jmath})$$
$$= -sin\theta\dot{\theta}\hat{\imath} + cos\theta\dot{\theta}\hat{\jmath}$$
$$= \dot{\theta}(-sin\theta\hat{\imath} + cos\theta\hat{\jmath})$$
$$= \dot{\theta}\hat{\theta} - - - - - (4)$$

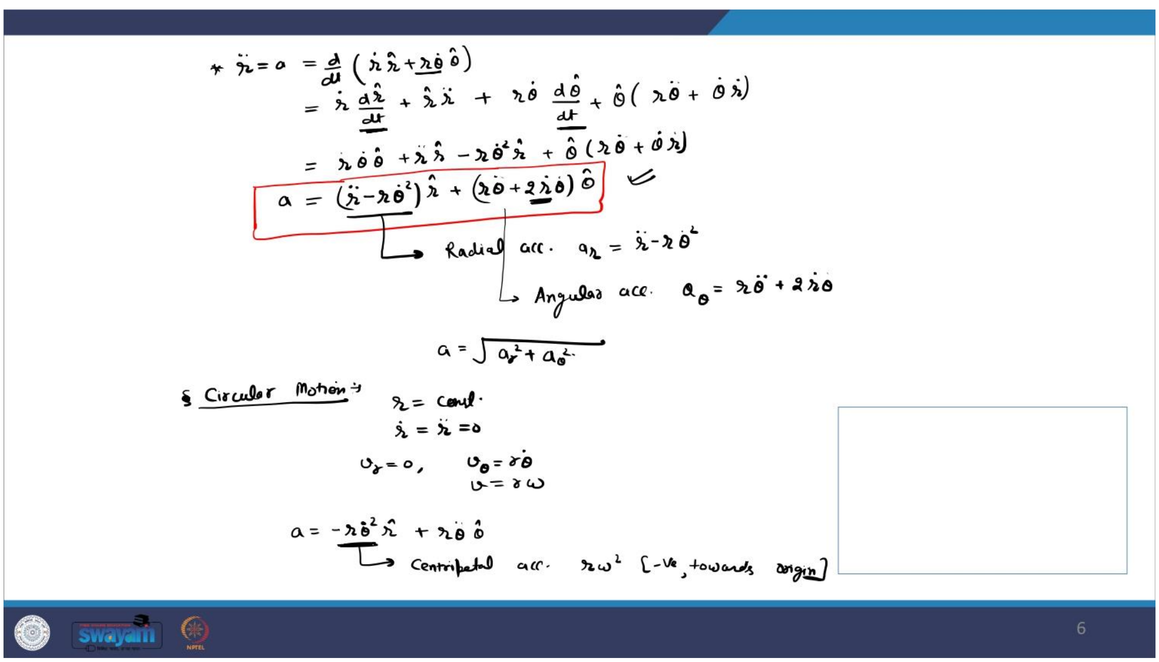
Now, let us also see what is $\frac{d\hat{\theta}}{dt}$.

So, $\frac{d\hat{\theta}}{dt} = \frac{d}{dt}(-sin\theta\hat{\imath} + cos\theta\hat{\jmath})$
$$= -cos\theta\dot{\theta}\hat{\imath} - sin\theta\dot{\theta}\hat{\jmath}$$
$$= \dot{\theta}(-cos\theta\hat{\imath} - sin\theta\hat{\jmath})$$
$$= -\dot{\theta}\hat{\theta} - - - - - (5)$$

Now, let us find out the expression for velocity and acceleration in polar coordinate system. So, we have equation number (1), wherein we wrote $\bar{r} = r\hat{r}$. This was our equation number (1), remember. This was our starting point. Therefore,

$$\dot{\bar{r}} = v = \frac{dr}{dt} = \frac{d}{dt}(r\hat{r})$$
$$v = r\frac{d\hat{r}}{dt} + \hat{r}\dot{r}$$
$$v = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

This is the expression for the velocity in this coordinate. Here, the first term is along $\hat{r}$. So, therefore, the radial velocity $v_r = \dot{r}$ and the velocity along $\hat{\theta}$ is called the angular velocity $v_\theta = r\dot{\theta}$. And if you want to find out what is the total velocity, then total velocity will be $v = \sqrt{v_r^2 + v_\theta^2}$.

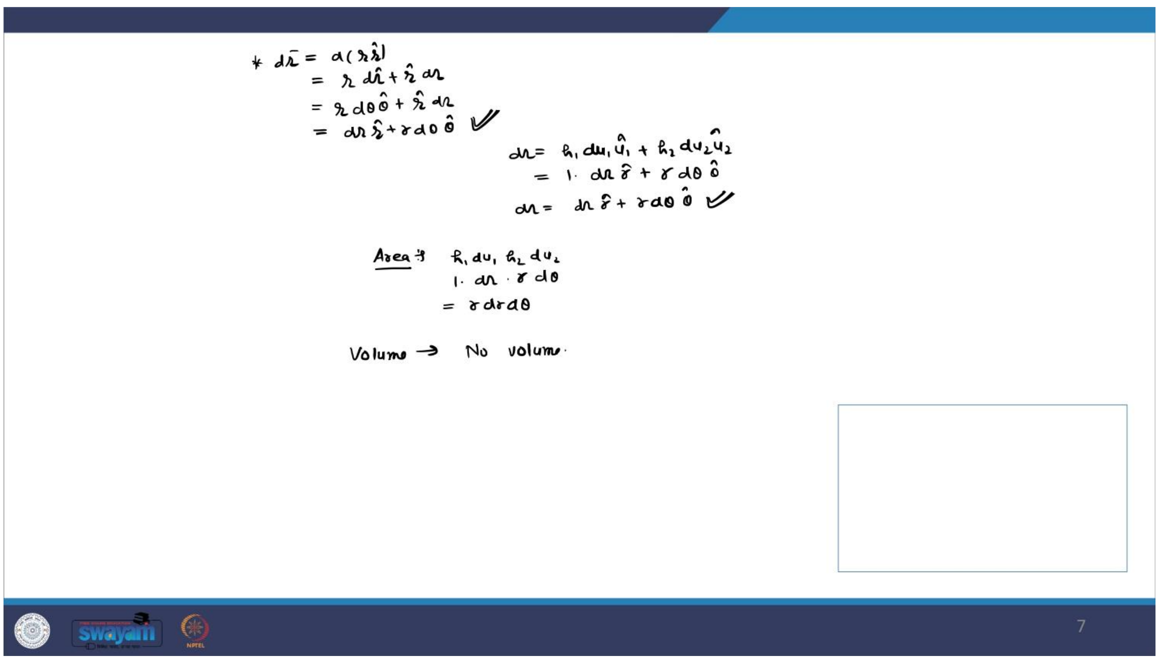


Now, let us find out the expression for the acceleration $\ddot{r}$ which is a which is $\frac{d\dot{r}}{dt}$ and $\dot{r}$ we already have. This is your $\dot{r}$. So,

$$\ddot{r} = a = \frac{d}{dt}\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right)$$
$$= \dot{r}\frac{d\hat{r}}{dt} + \hat{r}\ddot{r} + r\dot{\theta}\frac{d\hat{\theta}}{dt} + \hat{\theta}(r\ddot{\theta} + \dot{\theta}\dot{r})$$
$$= \dot{r}\dot{\theta}\hat{\theta} + \hat{r}\ddot{r} - r\dot{\theta}^2\hat{r} + \hat{\theta}(r\ddot{\theta} + \dot{\theta}\dot{r})$$
$$= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{\theta}\dot{r})\hat{\theta}$$

This is the expression for the acceleration. This term is along r. Therefore, it is called the radial acceleration. And let us denote it by a_r. So, $a_r = (\ddot{r} - r\dot{\theta}^2)$. And this term is angular acceleration. Let us denote it by a_θ. So, $a_\theta = (r\ddot{\theta} + 2\dot{\theta}\dot{r})$. And if you want to calculate the total acceleration, its magnitude, so it will be $a = \sqrt{a_r^2 + a_\theta^2}$. Now, let us analyze the expression of velocity and acceleration for circular motion. So, for circular motion $r = constant$. Therefore, $\dot{r} = \ddot{r} = 0$.

Now, let us look at the velocity.
$v_r = 0, v_\theta = r\dot{\theta}$
So, we know that $v = r\omega$ for circular motion.
Now, let us look at acceleration a. Acceleration a is, let us look at this expression. Here, $\dot{r} = 0$. So, $a = -r\dot{\theta}^2\hat{r} + r\ddot{\theta}\hat{\theta}$. Here, this term is called the centripetal acceleration and it is $-r\omega^2$. So, the minus sign that you see it is because this acceleration is towards the origin.



Now, let us come back to the planar polar coordinate system and derive the expression of infinitesimal displacement $d\bar{r}$ surface area ds.
So,

$$d\bar{r} = d(r\hat{r})$$
$$= rd\hat{r} + \hat{r}dr$$
$$= rd\theta\hat{\theta} + \hat{r}dr$$
$$= dr\hat{r} + rd\theta\hat{\theta}$$

Also, you can find out this using the expression of h_1 and u_1. Remember,

$dr = h_1 du_1 \hat{u}_1 + h_2 du_2 \hat{u}_2$

$dr = 1. dr\, \hat{r} + rd\theta\hat{\theta}$

$dr = dr\hat{r} + rd\theta\hat{\theta}$

Now, let me write down the expression for the area. Remember, it was

$ds = h_1 du_1 h_2 u_2$

$ds = 1. dr. r. d\theta$

$ds = rdrd\theta$

This is something that you already know.

Now, for the volume, since it is planar motion, therefore, there is no volume. With this, let me stop here. See you in the next class. Thank you.