MECHANICS

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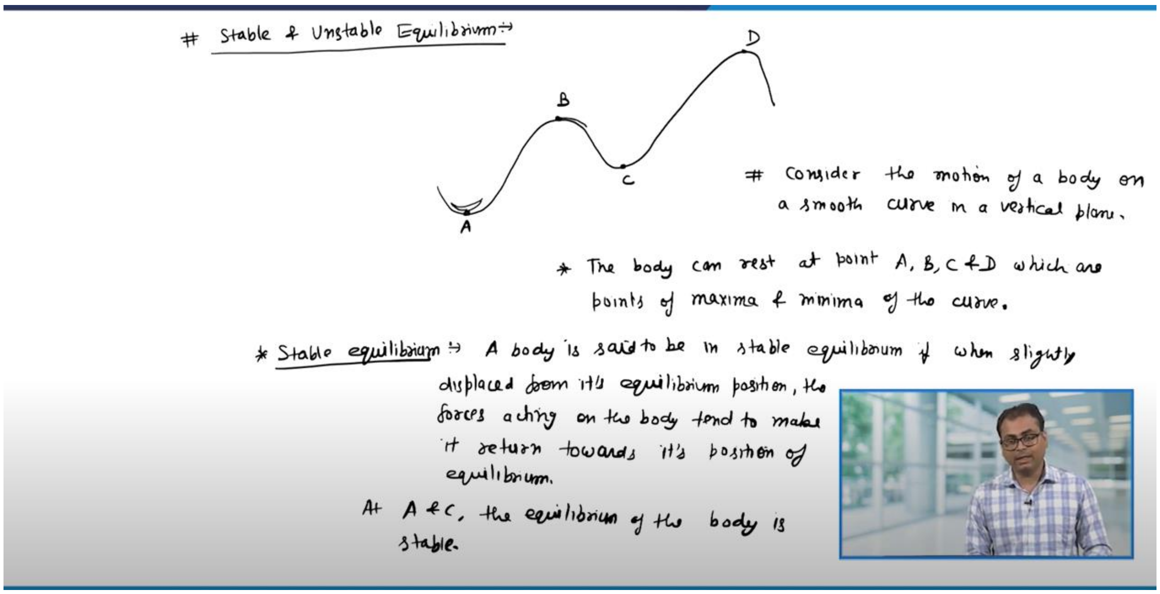
Lecture: 23

Stable and unstable equilibrium

Hello, everyone; welcome to the lecture again.

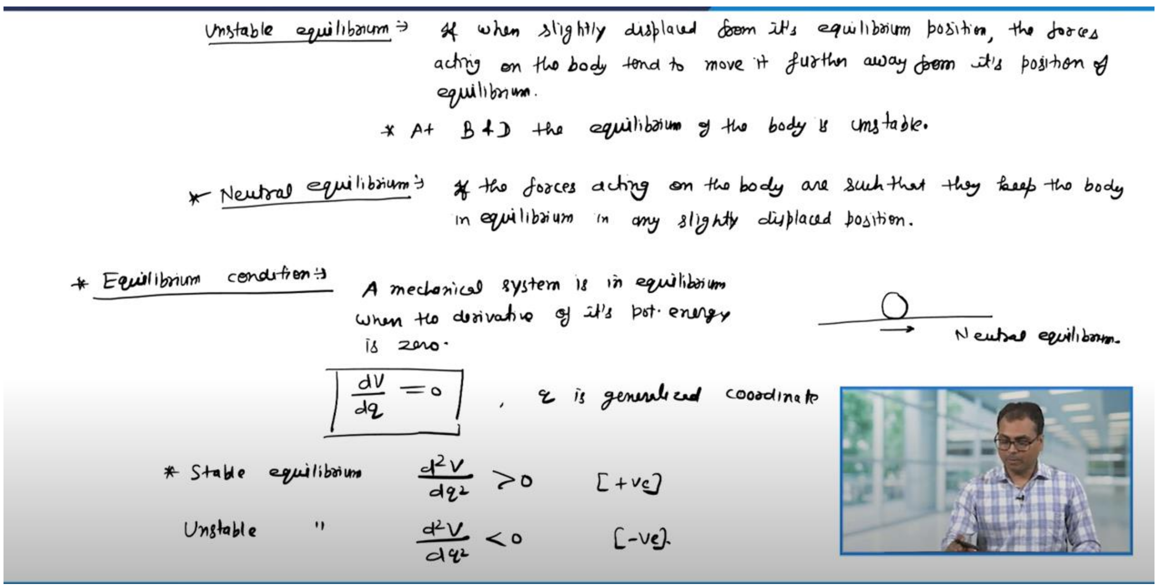


Today, we are going to discuss the stable and unstable equilibrium of rigid bodies. Let us understand it by an example. Let us say that a body is moving on a smooth curve which is in vertical plane.

So, consider the motion of a body on a smooth curve in a vertical plane. Now, if the body is there on this vertical plane, then the body can be in equilibrium at this point, this point, this point or this point. So, let me write it down. The body can rest at point A, B, C and D which are points of maxima and minima of the curve.

Now, let us identify which among these points are the point of stable equilibrium. A body is said to be in stable equilibrium if let us say when slightly displaced from its equilibrium position, the forces acting on the body tend to make it return towards its position of equilibrium. So, this is the definition.

As per the definition, you can see that the stable equilibrium is those point wherein if you displace the body, then the body will have a tendency to return towards its position of equilibrium. Now, from here, it is clear that At A and C, the equilibrium of the body is stable. Because if you move the body away from these points, then it has a tendency to return to the same point. This is not the case for point B, because herein, if you displace the body, then the body is no longer going to come back to the same position. Therefore, point B and D are the point of unstable equilibrium.

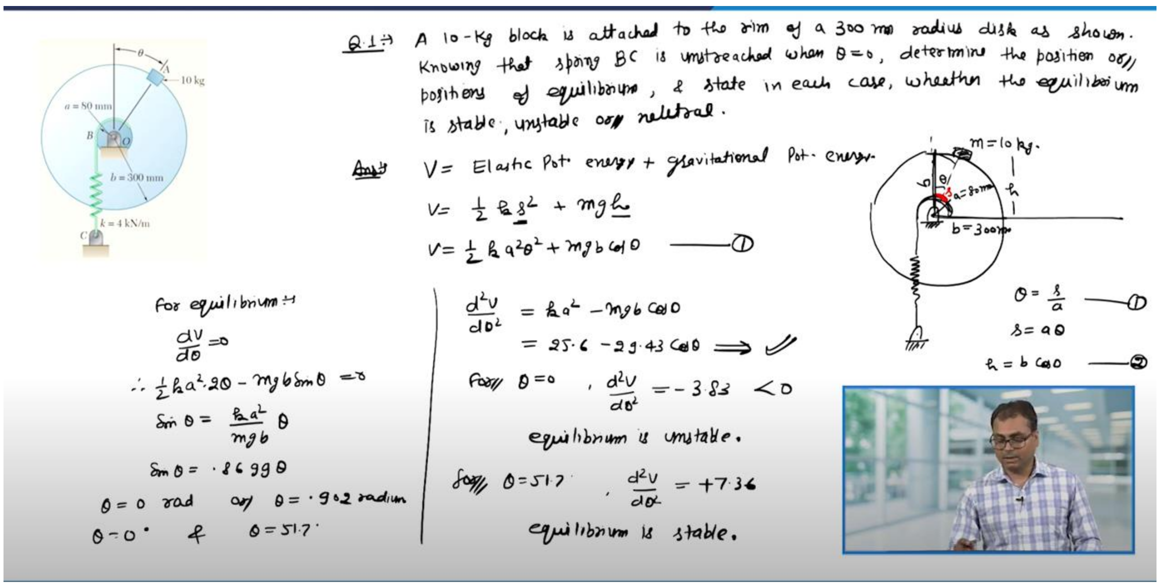


Let me write it down. If when slightly displaced from its equilibrium position, the forces acting on the body tend to move it further away from its position of equilibrium. So, if we refer to the same figure, then point B and D is the point of unstable equilibrium at B and D, the equilibrium of the body is unstable. Now, we can also have a neutral equilibrium. So, the definition is the following. If the forces acting on the body are such that they keep the body in equilibrium in any slightly displaced position.

Let us understand it by an example. Suppose, we have a flat surface and on this surface, you have a body. Now, you displace the body slightly. In that case, the body will remain in the equilibrium. So, it will move, but again, it will be in the equilibrium.

In that case, this is called the neutral equilibrium. Now, what are the conditions for the equilibrium? A mechanical system is in equilibrium when the derivative of its potential energy is zero. Because the potential energy has to be maxima or minima at the equilibrium. Therefore, its first derivative has to be 0.

That means dV/dq, where q is a generalized coordinate equal to 0, and V is the potential energy. So, here let me write down that q is generalized coordinate. Okay. Now, recall from the calculus for maximum, its second derivative has to be negative and for the minima, its second derivative has to be positive. Therefore, for stable equilibrium, your d^2V/dq square must be larger than 0. That means it has to be positive, and for unstable equilibrium, your $\frac{d^2V}{dq^2} < 0$. That means it has to be negative.



Now, let us use this concept to examine few problems. So, this is a problem statement of question number 1. A 10 kg block is attached to the rim of a 300 mm radius disc as shown. Knowing that spring BC is unstretched, when θ equal to 0, determine the position or positions of equilibrium and state in each case whether the equilibrium is stable, unstable or neutral, okay.

To analyze the equilibrium, let us look at the system. There is a rim of radius $b = 300mm$ and the mass $m = 10kg$ is held by using a spring. So, there is a spring that prevents the rotation of this disc, and the radius of this inner disc is equal to 80 mm.

Now, it is given that the spring is unstretched when the mass is in the vertical position. So, therefore, when $\theta = 0$, then the spring is in equilibrium. Now, we can find out how much is the stretch in the spring because that will help us to find out the potential energy of the spring. So, let us first see in this position, when this angle is θ, how much the spring has stretched. So θ is given by s/a, wherein let us say s is the stretch in the spring.

So, therefore, s will be $a\theta$. Now, let us also find out how much is the potential energy of this mass m. Let us say in this instant, the height of the mass m from the horizontal is h. So, we can find out how much is h. h Will be $bcos\theta$. θ Because this radius is b. Now, we can write down the total potential energy of the system. So, total potential energy V is the elastic potential energy plus the gravitational potential energy. So, V will be $\frac{ks^2}{2}$.

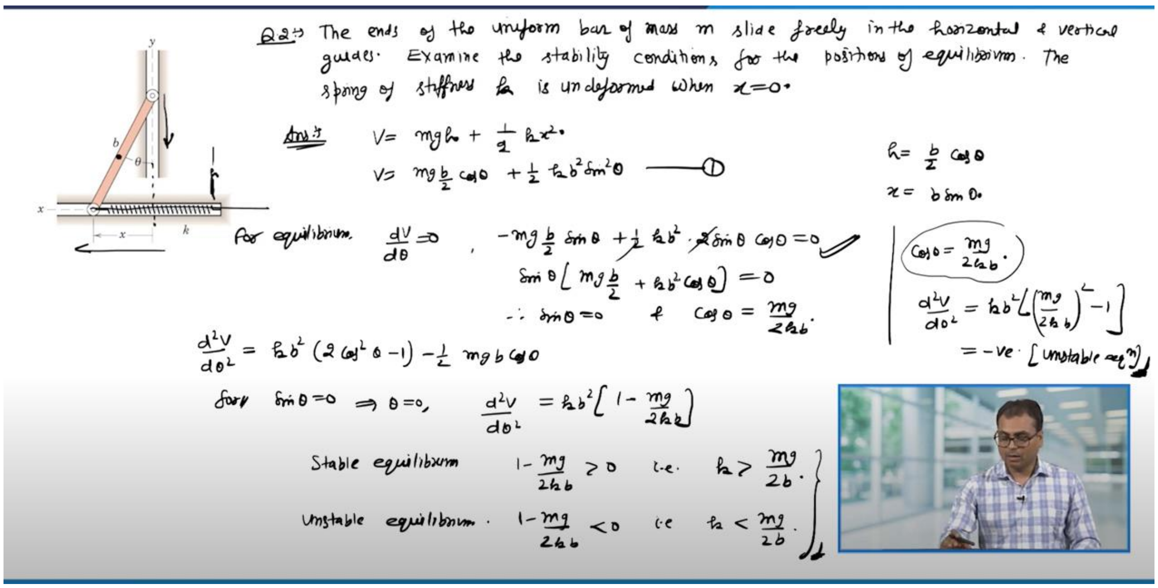
This is the potential energy of the spring plus mgh. And here, note that I have write down the whole thing in terms of θ because θ is our generalized coordinate. So, let us write down this s and h from these equations so that everything will become in the generalized coordinate. So, V will be $\frac{ks}{2}$ is $a^2\theta^2 + mg, h$ is $bcos\theta$. Let us call this equation number 1 and with this equation, now let us study the equilibrium.

So, we know that for equilibrium, my $\frac{dV}{d\theta}$ has to be 0. Therefore $\frac{ka^2}{2}2\theta$. So, I am differentiating equation number 1 $-mgbsin\theta$ has to be 0, and this gives us $sin\theta = \frac{ka^2}{mgb}\theta$. Note that the values of a, and b and m, k, etc. are given in the question.

Therefore, from here you can find out that $sin\theta$ is 0.8699θ and this gives you $\theta = 0\, radian$ or $\theta = 0.902\, radian$ which will give you $\theta = 0^0$ and $\theta = 51.7^0$. Now, let us look at the second derivative. So, let us find out what is $d^2V/d\theta^2$. This will be $ka^2 - mgbcos\theta$.

Again, the values of a, b, m, etc. are given. So, therefore, this will be $25.6 - 29.43cos\theta$. And for $\theta = 0$, you can put over here, you will see that $d^2V/d\theta^2$ becomes-3.83, which is, of course, negative.

Therefore, for $\theta = 0$, the equilibrium is unstable. Now, let us look at the case. For $\theta = 51.7^0$. In this case, $d^2V/d\theta^2$, again you can put it here, $\theta = 51.7^0$ and you will see that $d^2V/d\theta^2$ comes out to be plus 7.36. That means, in this case, the equilibrium is stable.



Now, let us look at another question. So, this is question number two and the problem statement is following. The ends of the uniform bar of mass m slide freely in the horizontal and vertical guides examine the stability condition for the positions of equilibrium and it is given that the spring of stiffness k is un-deformed when $x = 0$.

So, you can imagine this assembly. When this bar is going to move down, in that case, this spring is going to stretch. So, therefore, this spring is preventing the bar to come down. Therefore, it can be in equilibrium and we are going to examine what are the conditions on the equilibrium.

So, first of all, we have to write down what is the potential energy of the system. So, the potential energy V will be let us say I fix my axis over here. So, this is the level and from here I am going to calculate the potential energy. Let us say this height is h. So, this is where the center of mass of this road is.

So, the potential energy V will be mgh and the spring is stretched in this case by amount x. Therefore, the energy associated with the spring will be $kx^2/2$. Now, again the first thing is to write down these equations in generalized coordinate. Here the generalized coordinate is θ because with θ I can identify what is the state of the system is or where all the parts of the systems are. So, you can see here that if this is h, then h will be$\frac{b}{2}cos\theta$. And $x = bsin\theta$.

So, let us put that in this equation, we will get$V = \frac{mgb}{2}cos\theta + \frac{kb^2}{2}\sin^2\theta$. This is our equation number 1, wherein the potential energy is in the generalized coordinate. Now, for the equilibrium equation, we have$\frac{dV}{d\theta} = 0$. So, let us differentiate equation number 1.

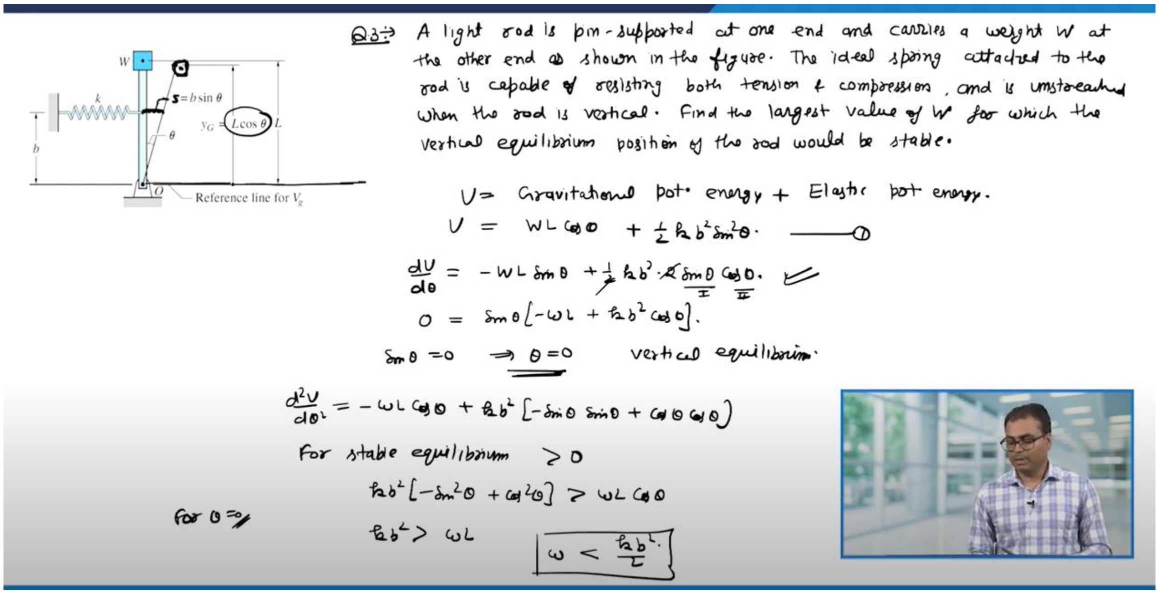
We get $-\frac{mgb}{2}sin\theta + \frac{kb^2}{2}2sin\theta cos\theta = 0$. Now, let us take $sin\theta$ outside. So, we have $mgb/2$. This 2 will get cancelled with that 2 plus $kb^2cos\theta$ equal to 0.

This gives us $sin\theta = 0$ and $cos\theta = mg/kb$. Now, to find out whether the equilibrium is stable or unstable, let us calculate the second derivative. So, $d^2V/d\theta^2$ will be $kb^2(2\cos^2\theta - 1) - \frac{1}{2}mgbcos\theta$. This I calculated from here and for $sin\theta = 0$ that means $\theta = 0$.

We have $\frac{d^2V}{d\theta^2} = kb^2(1 - \frac{mg}{2kb})$ and for stable equilibrium, that is $d^2V/d\theta^2$ has to be positive. For that, your $1 - \frac{mg}{2kb}$ has to be positive. So, this has to be larger than 0. That is, my k has to be larger than $mg/2b$.

Now, for unstable equilibrium, we have $1 - mg/2kb$; it should be negative; that is, my k has to be less than $mg/2b$. Now, let us look at the second case wherein $cos\theta = mg/2kb$. We have$\frac{d^2V}{d\theta^2} = kb^2$. Now, let us put $cos\theta = mg/2kb$.

So, we will have $\left(\frac{mg}{2kb}\right)^2 - 1$. Now, this quantity will always be negative. Because $mg/2kb$ is nothing but $cos\theta$ and the value of $cos\theta$ cannot be larger than 1. So, therefore, this will always be negative. That means, in this case, the equilibrium will be unstable.



Now, let us look at one more problem and the problem statement is following. A light rod is pin-supported at one end and carries a weight W at the other end, as shown in the figure. The ideal spring attached to the rod is capable of resisting both tension and compression and is unstretched when the rod is vertical.

In the question statement, it has asked you to find out the largest value of W for which the vertical equilibrium position of the rod would be stable. So, the first thing is to write down the total potential energy of the system and for that we have to find out the reference. In the figure itself, the reference line is given. So, therefore, let us write down what is the total energy of the system.

So, U which is the total potential energy, it will be the gravitational potential energy plus the elastic potential energy. Now, from the geometry, you can see that the gravitational potential energy is $WLcos\theta$ because in this situation, the height of the weight W is $Lcos\theta$ plus the elastic potential energy. For that, I have to find out how much the spring has stretched. So, you can see that in this situation, the spring has stretched by this amount and from the geometry, I can find out this distance $s = bsin\theta$.

So, therefore, the potential energy corresponding to the spring will be $\frac{ks^2}{2}$ and s is nothing but $bsin\theta$. So, therefore, $b^2 \sin^2 \theta$. Also note that here this equation is in the generalized coordinate θ. So, now let us call it equation number 1 and find out $dU/d\theta$.

So, this will be $-wlsin\theta + \frac{kb^2}{2} 2sin\theta cos\theta$. Now, for the equilibrium, let us put $\frac{dU}{d\theta} = 0$. So, we have 0 equal to $sin\theta(-WL + kb^2cos\theta)$, and this gives you that $sin\theta = 0$ or $\theta = 0$, and this is the vertical equilibrium and we were asked to analyze this case.

Let us first calculate $d^2U/d\theta^2$. So, that will be from here, it will be $-wLcos\theta + kb^2(-sin\theta sin\theta + cos\theta cos\theta)$. So, let me take first function, second function to differentiate it. And we know that for a stable equilibrium, this dU, $d^2U/d\theta^2$ has to be larger than 0.

So, therefore, we get $kb^2 - \sin^2\theta + \cos^2\theta$ should be larger than $WLcos\theta$ and let us put $\theta = 0$ for vertical equilibrium. So, for $\theta = 0$, we get Kb^2 should be larger than WL or W has to be smaller than kb^2/L. This is the condition for stable equilibrium in the vertical position. With this let me stop here see you in the next class. Thank you.