MECHANICS
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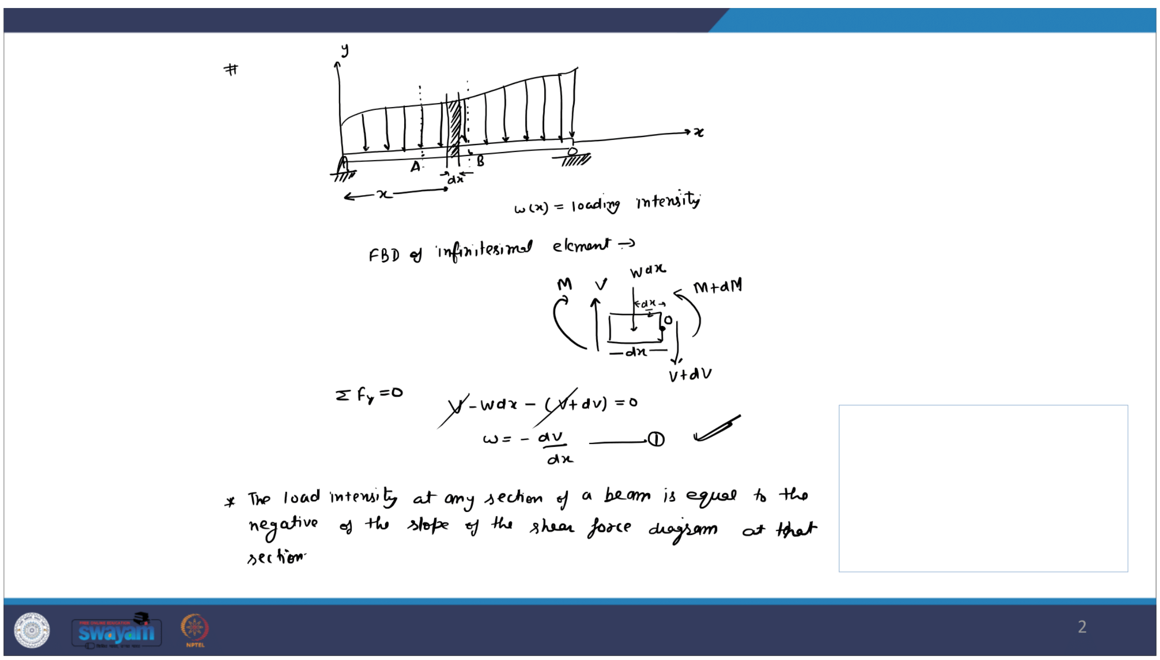
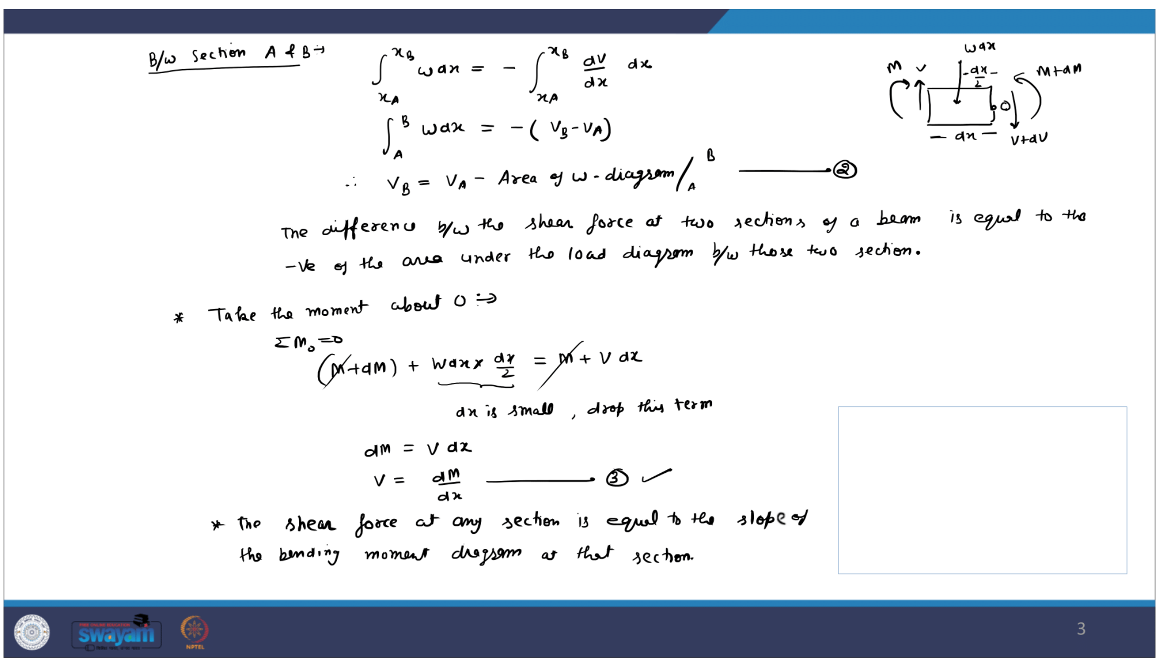
Lecture 19
Shear force and bending moment: distributed load

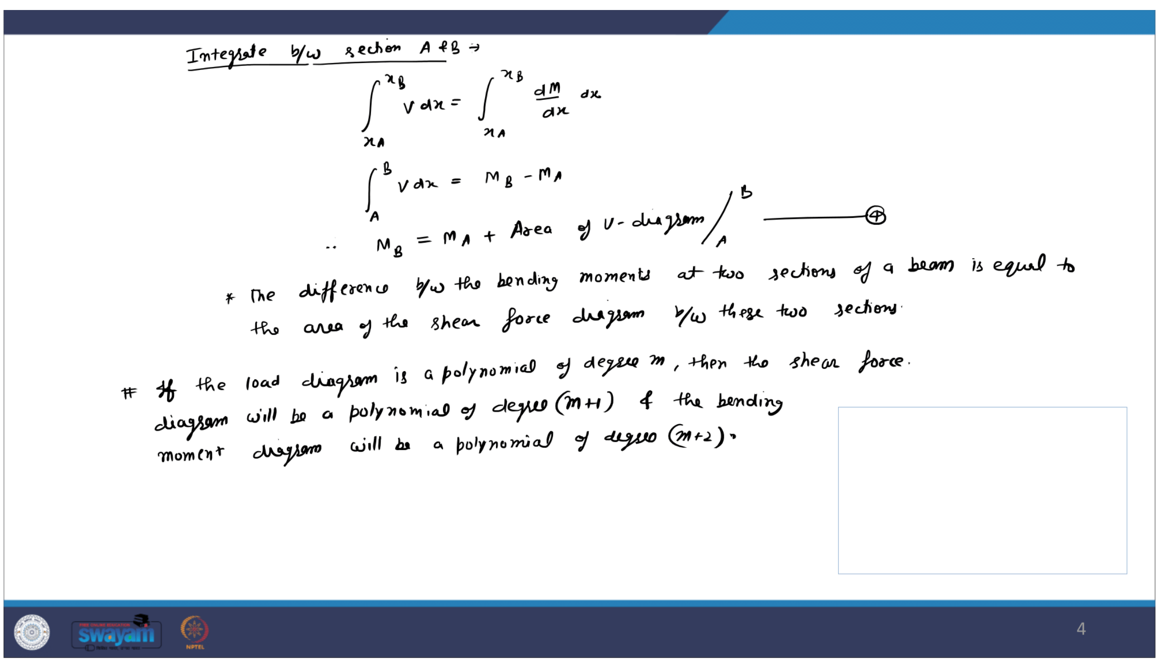
Hello everyone, welcome to the lecture again. In the last lecture, we discussed the shear force and bending moment when the loading was concentrated. We saw that when the beam is subjected to a concentrated load only, in that case, the shear force is constant between the load and the bending moment varies linearly. Today, we are going to discuss the shear force and bending moment in the case of distributed load.



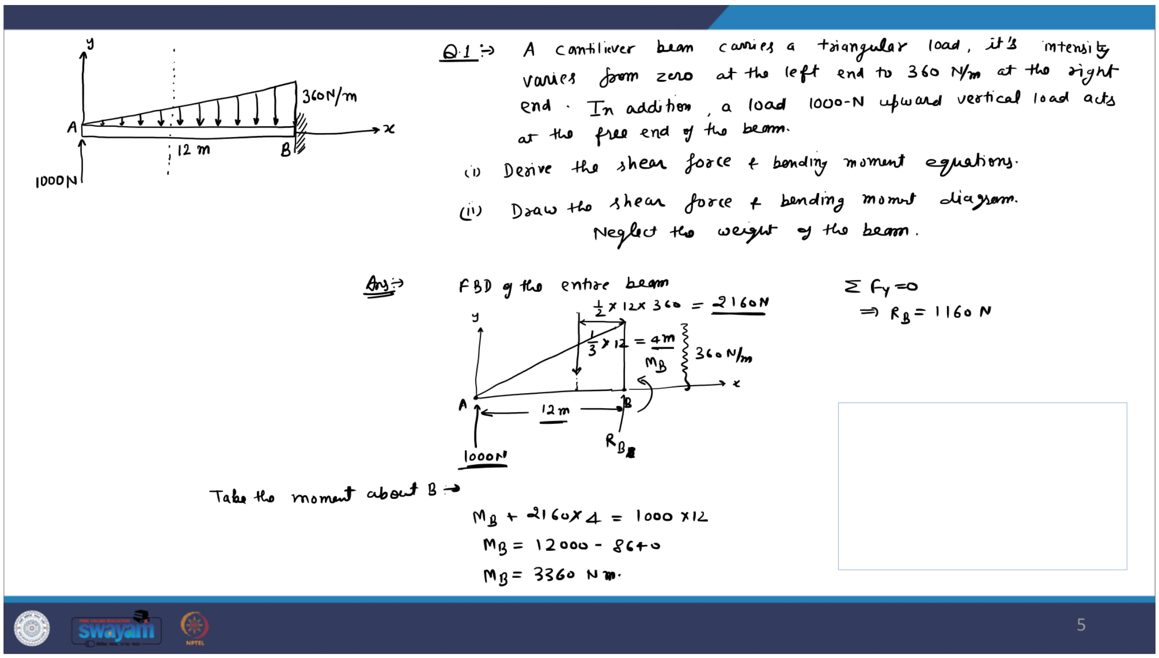
So, the situation is following. Suppose I have a beam and let's say on this end, we have a pin joint and over here we have some roller support. Let's say this is x-axis and this one is y-axis and this beam is under some distributed load. So, let's say the load is like this. Let's say the load per unit length, which is called the loading intensity that is w. So, $w(x)$, which is a continuous function is a loading intensity. Okay. And let's say we want to analyze a section of this beam. Okay. So, let me consider a part of the beam, let's say between A and point B and to analyze that, let me take an infinitesimal part of the beam. Let's say this part is dx and from one end, it is at a distance of x. Now, let us look at the free body diagram of this infinitesimal part dx. So, free body diagram of infinitesimal element dx. So, we have this part. So, this is dx and on the left-hand side, let us follow the positive convention. So, we have the force V and the moment M. And on the right-hand side, we will have, let's say, the force is acting downward. It is $V + dV$ and the bending moment is $M + dM$. Now, this part will also have the load. So, the load will be load intensity w multiplied by the horizontal length which is dx. So, therefore, the load will be wdx and it will act at a distance of $x/2$. So, $dx/2$ because the length is dx. So, therefore, this will be $dx/2$ and also on the other side, it will be $dx/2$. Let us consider that this point is O for the analysis. Now, to find out the shear force V, we can balance the forces in the y direction. So, let us use $\sum F_y = 0$. So, we got $V - wdx - (V + dV) = 0$. What does this mean? This means that, so V can get cancelled, we get $W = -\frac{dV}{dx}$. Let us call this equation number 1. This implies that the load intensity which is w at any section of a beam is equal to the negative of the slope of the shear force because V was the shear force. So, we have dV/dx. So, it is the slope of the shear force diagram at that section, okay. Now, we can find out from here what is V just by integrating it.



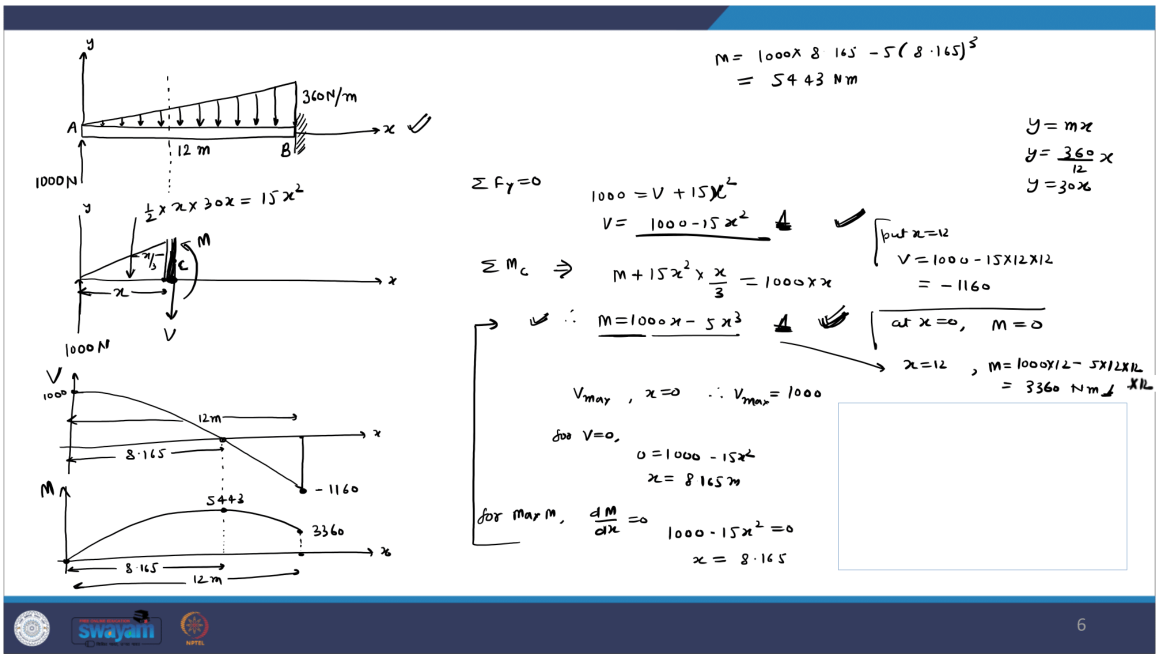
So, let us integrate this between the section A and B, okay. So, let us integrate between the section A and B to find out what is V. So, we have $\int_{x_A}^{x_B} wdx = -\int_{x_A}^{x_B} \frac{dV}{dx} dx$. This gives you $\int_A^B wdx = -(V_B - V_A)$ and from here we can find out that $V_B = V_A -$ Area of the w diagram$|_B^A$. So, we have this load intensity, you find out what is the area of that, that will give you wdx integral. So, this on the left hand side, we have area of w diagram between A and B. Let us call this equation number 2. In text, we can say that the difference between the shear force at two sections of a beam is equal to the negative of the area under the load diagram between those two sections, okay. So, now, so this is about the shear force. Let me again make the diagram of that infinitesimal part. So, we had this as dx, we assume that in this direction V is acting. The bending moment is like that. In this direction, we have $V + dV$ and the moment is $M + dM$. This is wdx acting at a distance of $dx/2$ from the right and we said that let us take this point O for the analysis. So, we got the forces. Now, let us find out the bending moment, okay. And to find out the bending moment, let us take the moment about O. So, again the moment equation has to be balanced. So, let us take $\sum M_O = 0$. So, we have moment about O, $V + dV$ will not contribute. So, we have $(M + dM) + wdx$ multiplied by the perpendicular distance which is $dx/2$ equal to moment $M +$ force V multiplied by the perpendicular distance which is dx. $(M + dM) + wdx \times \frac{dx}{2} = M + V \times dx$. Now, you can see here that M will get cancelled and this term wdx and $dx/2$, it has, you know, dx twice. So, dx is small. Therefore, you know, this term will be even smaller. So, let us drop this term because dx is small. Therefore, we are dropping this. So, we get $dM = Vdx$ or we get $V = dM/dx$. Let us call this equation number 3 and this equation implies that the shear force at any section is equal to the slope of the bending moment diagram at that section, okay, because d M by d x is the slope of the bending moment diagram.



Now, to find out what is the value of M, let us again integrate this between A and B. So, let us integrate between section A and B. So, $\int_{x_A}^{x_B} Vdx = -\int_{x_A}^{x_B} \frac{dM}{dx} dx$. So, this gives you $\int_A^B Vdx = M_B - M_A$ or we can write down $M_B = M_A +$ Area of V-diagram$|_B^A$. Let us call this equation number 4 and we can again write it down in the text. This implies that the difference between the bending moment at two sections of a beam is equal to the area of the shear force diagram between these two sections, okay. Now, from all this analysis, we can also see that in general, if the load diagram is a polynomial of degree, let's say, m, then the shear force diagram will be a polynomial of degree $m + 1$ and the bending moment diagram will be a polynomial of degree $m + 2$ because we have this equation So, V will be wdx and if w is a polynomial of, let's say, m degree in that case V because that is the integral of that it will be of the order of $m + 1$ and similarly for the bending moment diagram because that is the integral of V.



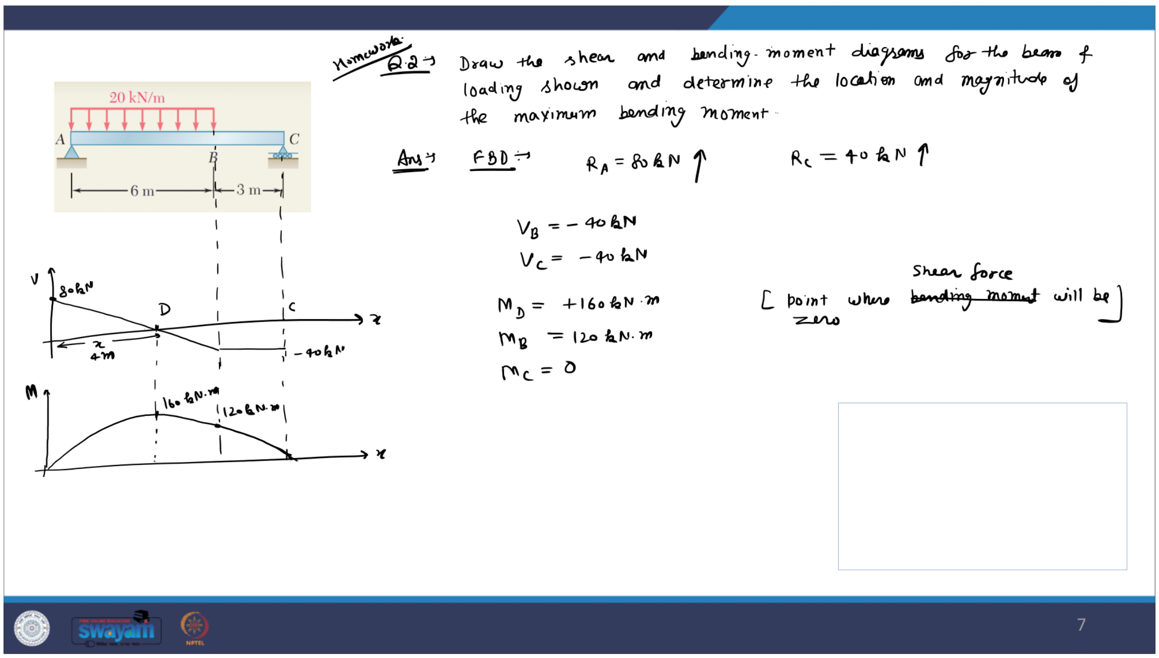
Now, with this basic concept let us now look at you know couple of examples. So, the first problem statement is following. A cantilever beam carries a triangular load and its intensity varies from 0 at the left end to 360 N/m at the right end. In addition, a load of 1000 N upward act at the free end of the beam and in the question, it has asked you to derive the shear force and bending moment equations and it has also asked you to draw the shear force and bending moment diagram. It is also given that you can neglect the weight of the beam. So, first of all, let us see here the load is a linear function of x. So, therefore, it is obvious that the shear force will be one order higher. So, that will be, it will depend upon the x^2. Similarly, the bending moment will depend upon the x^3. So, let us see that. Before we analyze this, first let us consider the free body diagram of the entire beam to find out the reaction forces. So, let us look at the free body diagram. of the entire beam. So, we have point A, point B, this is our x-axis, this one is y-axis and the loading is like this. Here, we have an external force of 1000 N. Let us put reaction force B_y over here and the bending moment M_B. Now, we can find out the area under the curve. So, that will give us the total loading. So, that will be. So, first of all, it is going to act. So, this is 12 m from the right hand side, it is going to act at a distance of $1/3$ of the base. So, this is $\frac{1}{3} \times 12 = 4\,m$. So, this distance is going to be $4\,m$. Now, how much is the loading intensity? So, load will be $\frac{1}{2} \times 12$ into this is given. So, this is $360\ N/m$. So, therefore, $\frac{1}{2} \times 12 \times 360 = 2160\ N$. So, that much force is going to act downward. So, this completes the free body diagram of the entire beam. Now, let us use the force balance equation first to find out what is the reaction force at point B. So, let me call it R_B. So, we have $\sum F_y = 0$. This gives you $R_B = 1160\ N$ because we have 1000 force acting upward and we have $2160\ N$ force acting downward. So, therefore, $R_B = 2160 - 1000 = 1160\ N$. Now, to find out the moment, this M_B let us take the moment about B. So, in that case, the R_B is not going to do anything because it is passing through point B. So, we have M_B plus we have 2160 multiplied by the perpendicular distance. We have just calculated it is $4\,m$ equal to 1000 into 12. $M_B + 2160 \times 4 = 1000 \times 12$ So, this gives you $M_B = 12000 - 8640 = 3360\ Nm$. So, these are all the forces that are acting.



Now, to find out the shear force and bending moment, let us draw a part of this beam. Now, note that because the loading intensity is continuous, the beam does not need to be divided into many segments, okay. Only one expression of V and M will apply to the entire beam because it is a continuous loading and, you know, single loading. So, let us look at the free body diagram of the left hand part of this beam. So, let me just make it once again. So, this was the entire beam. Let me make it here. So, we have our x-axis and this is the part that we are interested in. So, therefore, it is like this. Let's say this is you know, at a distance x. The n part is at a distance x from the left-hand side. This is our y-axis and here I have external loading of $1000\ N$. And here, let us say that V is acting like this and M is going to act like that. Now, from a distance of, you know, $x/3$, from the right side, the entire weight of the section is going to act and it will be $\frac{1}{2} \times x$ into this height. Now, this height, I can find out using the equation $y = mx$ where m is the slope and this m, I can find out from the entire beam. So, $y = \frac{360}{12}x$. So, therefore, $y = 30x$. So, therefore, this is $\frac{1}{2} \times x \times 30x = 15x^2$. Now, let us balance the forces to find out what is V. So, let us use $\sum F_y = 0$. So, this gives you $1000 = V + 15x^2$ and this gives you V=$1000 - 15x^2$. So, you can see that V varies as a function of x^2 because the loading is a linear function of x. Now, to find out the shear or the bending moment. So, it will be summation M. Let us take this point as C. So, let us take $\sum M_C = 0$. So, it will be M plus $15x^2$ into the perpendicular distance, which is $x/3$ equal to 1000 into perpendicular distances x. $M + 15x^2 \times \frac{x}{3} = 1000 \times x$ Note that V is not going to, you know, take part in this because we are taking the moment about C and V is passing through C. So, therefore, $M = 1000x - 5x^3$. Now, let us plot this V and M as a function of x. So, let's say this is x-axis, this is V. So, you can see here that the plot of $1000 - 15x^2$ because of $-15x^2$ it is going to decrease. So, therefore, it will go like that and we can find out what is the maximum values and what is the minimum values. For the maximum value of V, let us put $x = 0$. So, for V_{max}, let us put $x = 0$. This will give you $V_{max} = 1000$. So, therefore, this is 1000. Now, for $V = 0$, let us find out what is the value of x. So, again let us use this equation. Put $V = 0$. So, $0 = 1000 - 15x^2$. This will give you $x = 8.165\ m$. Therefore, this distance is 8.165. So, at this moment, the shear force will be 0. Now, on the other hand, so, let's say over here, let us find out what is its value. So, for that, we know that here $x = 12\ m$. So, therefore, again in this equation, let us put $x = 12$. So, we will get $V = 1000 - 15 \times 12 \times 12$ and this will give you -1160. So, therefore, this value will be minus 1160. So, this completes the shear force diagram. Now, let us look at the bending moment diagram. So, this is our x-axis and because of x minus, you know, $ax - bx^3$. So, the plot will be like that. So, you can plot it and you can see it will look like this. Now, let us find out the, you know, various, you know, points on this plot. So, first of all, we can see that from this equation at $x = 0$, my $M = 0$. So, therefore, this point is 0, this is M. Now, let us see what is the maximum value of M. For that, we have to take this equation and differentiate this with respect to x, put that equal to 0, that will give you the maximum value of, you know, the point of M and then you can find out what is the maximum value of M. So, let us do that.

So, we have this equation, let us for maximum value of M, let's say $\frac{dM}{dx} = 0$. So, that will be $1000 - 15x^2 = 0$. From here, you can find out what is the value of x. So, $x = \sqrt{\frac{1000}{15}} = 8.165\ m$.

So, therefore, this is also 8.165 m and at that we have the maximum value of M. To find out the maximum value of M, let us put this value of x in this equation. So, let us do that. So, we will have $M = 1000 \times 8.165 - 5 \times (8.165)^3$ and this will give you 5443 Nm. So, therefore, this will be 5443 Nm. Now, I also know that at this point $x = 12$. So, again in this equation, I can put $x = 12$. So, let us put the value of x equal to 12 in this equation and this will give you $M = 1000 \times 12 - 5 \times 12 \times 12 \times 12 = 3360\ Nm$. So, therefore, this value will be 3360 Nm. So, this completes the question and we have find out what is the, you know, bending moment and shear forces.



Now, let me give you a homework problem and the problem statement is following. So, this is a homework problem. Draw the shear and bending moment diagrams for the beam and loading shown and determine the location and magnitude of the maximum bending moment. So, just to give you the hint, you can consider the free body diagram of the entire beam and this will give you the $R_A = 80\ kN$ that will act upward. Similarly, the $R_C = 40\ kN$ and that will also act upward. You can find out the shear force at point B. So, shear force at point B, $V_B = -40\ kN$. The shear force at C , $V_C = -40\ kN$. You can find out the bending moment. So, the bending moment at point D,$M_D = 160\ kN.m$. Now, D is the point where in the bending moment will be 0. Okay. So, this is the point where bending moment will be 0. You can find out the bending moment at point B, $M_B = 120\ kN.m$ and the bending moment at point C, $M_C = 0$. The diagram will look like this. So, let me plot

V. So, point D is the point where the shear force will be 0. So, let me just plot this. So, this is the point. Let us call it D. Here the value will be $80\ kN$ and then it will look like this, this will be $-40\ kN$ and this is the distance let's say $x = 4\ m$ and the shear force diagram will be like that. So, it will attain a maxima at point D and then it will be like this. So, from here to here, it will be a straight line. So, here it will be $120\ kN.m$ and here it will attain the maxima $160\ kN.m$. So, this question you can try by yourself. So, with this, let me stop here. Thank you.