MECHANICS

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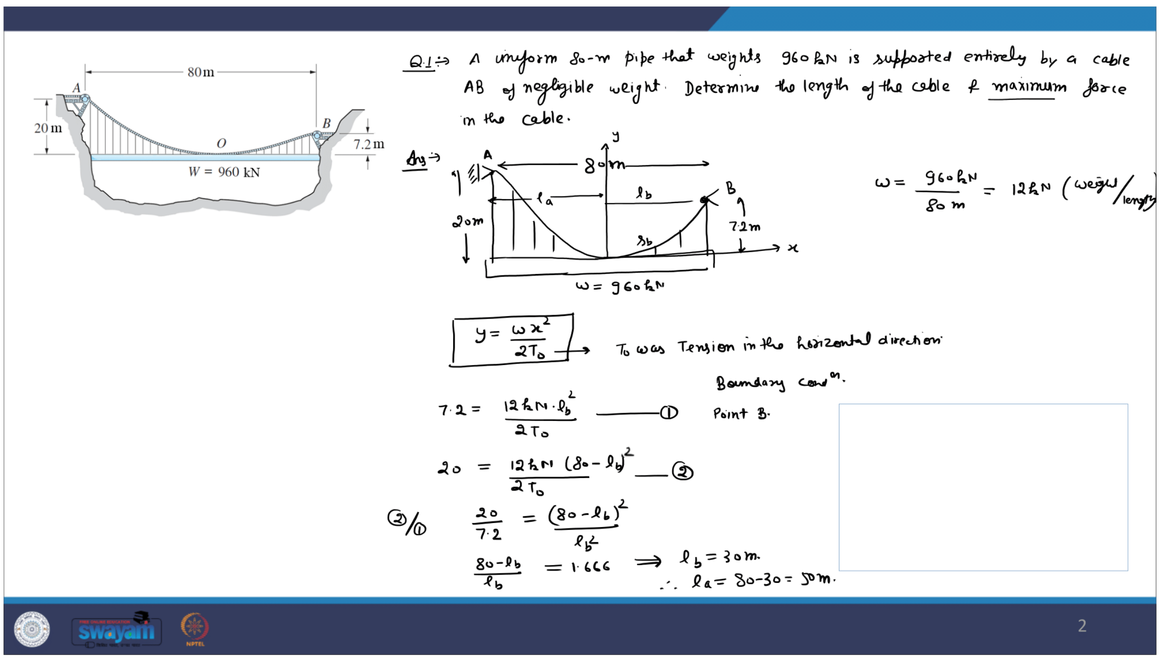
Lecture 13

Examples: Parabolic and Catenary cables

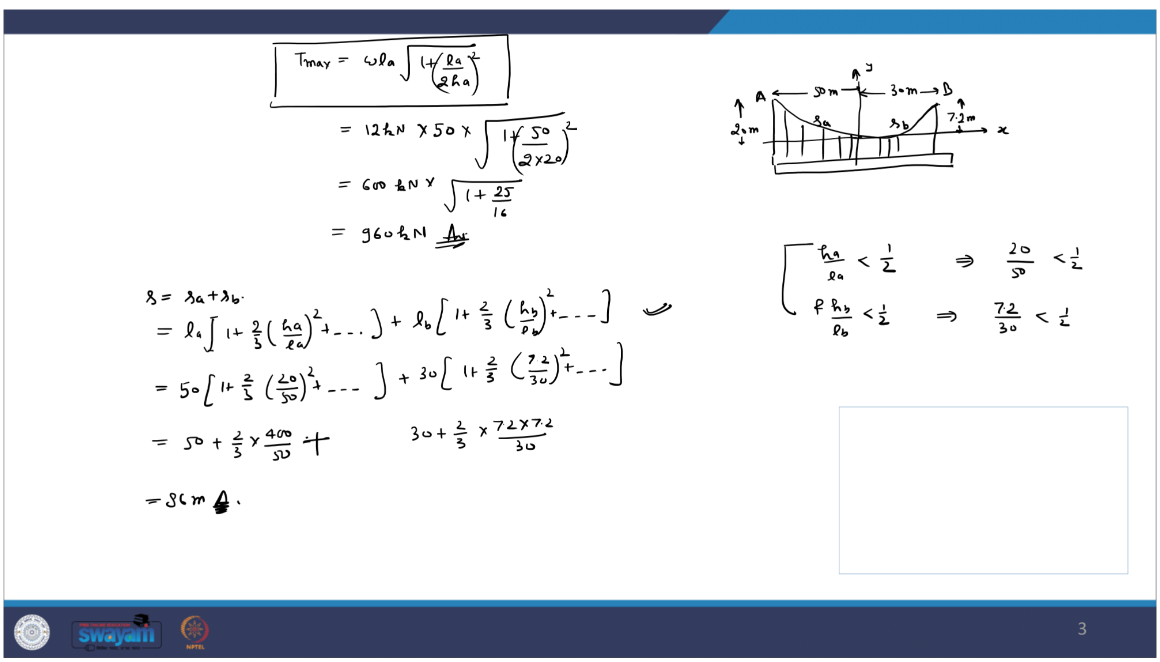
Hello everyone, welcome to the lecture again. In the last couple of lectures, we analyzed cables in two cases. In the first case, the loading was uniform along the horizontal direction and in the second case, the cable was hanging under its own weight. We saw that the equation of the cable in the first case is parabola and in the second case, it was catenary. Now, let us look at various examples.



So, this is the first question statement. A uniform $80\,m$ pipe that weights $960\,kN$ is supported entirely by a cable AB of negligible weight. Determine the length of the cable and maximum force in the cable. So, first, since the cable is supporting a uniform weight along the horizontal direction, therefore, the equation of the cable will be the parabola. So, this comes under the first category. Let me first make the free body diagram of the system. So, we have this cable and this is point A, point B and this is supporting a uniform weight. Let us put the axis at the bottom point of this cable. So, this is the x-axis, the y-axis. This is 80 m. This height is 7.2 m and the weight is 960 kN. This is given 20 m. Let's say this length is l_b and this length is s_b. Now, the weight per unit length which we denote by w will be 960 kN divided by 80 m. So, this will be 12 kN and this is nothing but weight per unit length.

Now, as I said, the equation of the cable will be parabola. So, let us use the equation $y = \frac{wx^2}{2T_0}$ .Now, here T_0 remember this was the tension in the horizontal direction and it was constant. So, T_0 was the tension in the horizontal direction. Now, to determine let's say x, let us use the boundary condition. So, we can see here that for $y = 7.2$ that is for point B. So, we have $7.2 = (12\ kN).\frac{l_b^2}{2T_0}$. So, this is equation number 1. This is for point B. Similarly, for the point A, we have $y = 20 = \frac{(12\ kN)(80-l_b)^2}{2T_0}$.

Now, to determine l_b, let us divide equation number 2 to 1. So, we get $\frac{20}{7.2} = \frac{(80-l_b)^2}{l_b^2}$ and this gives me $\frac{80-l_b}{l_b} = 1.666$ and this will give you $l_b = 30\ m$. Therefore, $l_a = 80 - 30 = 50\ m$. So, therefore, $l_a = 50\ m$.

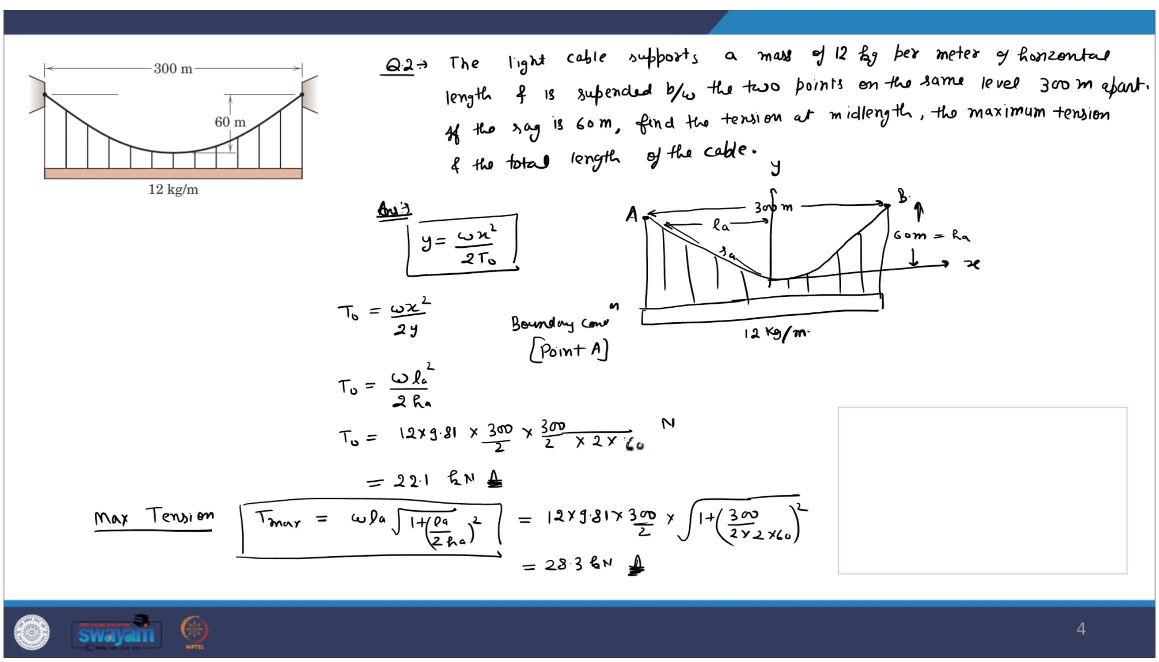


Now, let us find out what will be the maximum tension. So, in the question statement, it has asked the maximum force. So, let us find out the maximum tension in the cable. So, remember $T_{max} = wl_a\sqrt{1 + \left(\frac{l_a}{2h_a}\right)^2}$. Now, here we have used you know $x = l_a$ because that is where the maximum tension will be because that is the highest point in the cable not

l_b. So, let us put the values. So, this is something that we have derived. So, this will be $12\ kN \times 50 \times \sqrt{1+\left(\frac{50}{2\times 20}\right)^2}$. This will be $600\ kN \times \sqrt{1+\frac{25}{16}} = 960\ kN$. Now, let us also find out what is the length of the cable. So, remember we have the following setup. You have this weight, then you have this cable and this is the minimum point, x-axis, y-axis, this is $50\ m$, point A, point B, this is $30\ m$ and let us call this s_a and this equal to s_b. So, this is $7.2\ m$ and this one is $20\ m$. Now, to find out the entire length, we have to sum s_a and s_b , $s = s_a\ + s_b$and we know how to calculate s_a.

So, this will be $s = l_a\left(1+\frac{2}{3}\left(\frac{h_a}{l_a}\right)^2+..\right) + l_b(1+\frac{2}{3}\left(\frac{h_b}{l_b}\right)^2+\cdots)$. Now, remember we can use this formula only if $\frac{h_a}{l_a} < \frac{1}{2}$ and $\frac{h_b}{l_b} < \frac{1}{2}$. Let us see if this is true. So, here our $h_a = 20\ \ m$ and $l_a = 50\ m$ and this is indeed less than half. And in this case, $h_b = 7.2\ m, l_b = 30\ \ m$, which is again less than half.

So, therefore, this formula we can use. So, I have, you know, taken only the second term, but to get the exact value of s, you can take third term and fourth term also. So, this will be $s = 50\left(1+\frac{2}{3}\left(\frac{20}{30}\right)^2+..\right) + 30(1+\frac{2}{3}\left(\frac{7.2}{30}\right)^2)$. So, let us simplify this. This will be $s = 50+\frac{2}{3}\times\frac{400}{50}+..+30+\frac{2}{3}\times\frac{7.2\times 7.2}{30} = 86\ m$. So, this is the length of the cable.



Now, let us look at the second example on the same concept and here the problem statement is following. The light cable supports a mass of 12 kg per meter of horizontal length and is suspended between the two points on the same level $300\ m$ apart. If the sag is $60\ m$, find the tension at mid length, the maximum tension and the total length of the cable.

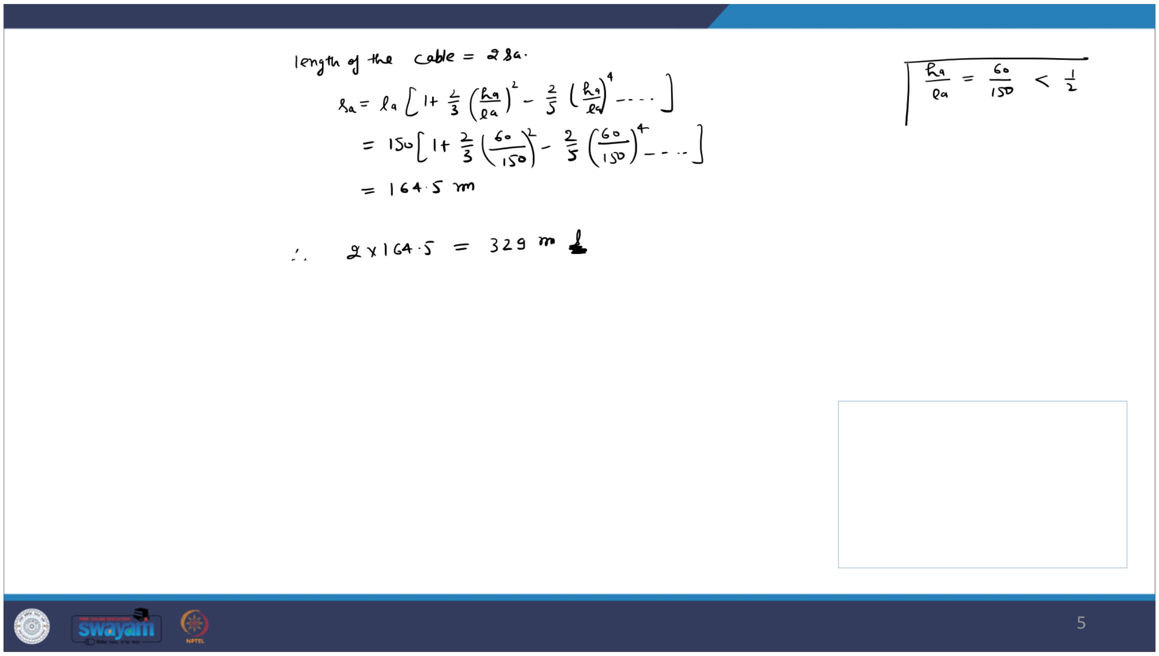
So, again you can see here that the cable is supporting a uniform weight along the horizontal direction. Therefore, the equation of the cable will be the parabola. Now, let me first make the diagram. So, we have this cable and this cable is supporting a uniform weight. This is our x-axis. This one is the y-axis. This is given $300\ m$. Let's say this point is A and this point is B. So, the weight per unit length is $12\ kg/m$. This height is $60\ m$ and let us say this length is l_a and this is of course, h_b equal to h_a. And let us call this equal to s_a and the other one will be s_b. So, let us use the equation of the parabola. This is $y = \frac{wx^2}{2T_0}$. Now, in the question statement, it has asked you to find out the tension at the mid length. Now, the tension at the mid length will be T_0.

So, let us find out what is T_0. So, from here, you can see that $T_0 = \frac{wx^2}{2y}$. Now, to find out T_0, let us use the boundary condition. So, let us use boundary condition and for that I am going to use point A. So, at point A, T_0 will be T_0 because that is just the horizontal component equal to w, $x = l_a$. So, we have $T_0 = \frac{wl_a^2}{2h_a}$ and we know the value of l_a and h_a.

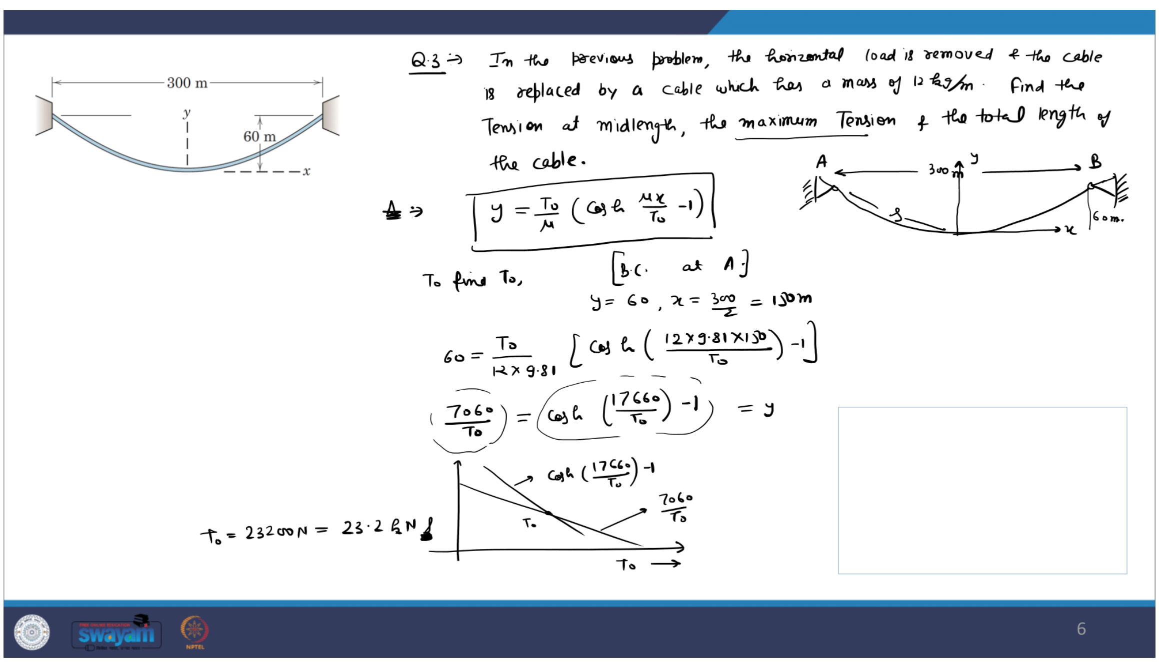
So, let us put it back. So, $T_0 = 12 \times 9.81 \times \frac{300}{2} \times \frac{300}{2\times2\times60}$. So, that much Newton and you can simplify it, it will comes out to be $22.1\ kN$. Now, let us find out what is the maximum tension. So, maximum tension. So, this is T_{max} . So, $T_{max} = wl_a\sqrt{1 + \left(\frac{l_a}{2h_a}\right)^2}$. Let us put the values. So, it will be $T_0 = 12 \times 9.81 \times \frac{300}{2}\sqrt{1 + \left(\frac{300}{2\times2\times60}\right)^2}$ and again you can simplify it, it will comes out to be $28.3\ kN$. So, we have find out the value of T_0 and T_{max}.

Now, let us calculate the length of the cable. So, from here, you can see that the length of the cable will be $2s_a$. So, the length of the cable will be $2s_a$ and let us find out what is s_a. $s_a = l_a(1 + \frac{2}{3}\left(\frac{h_a}{l_a}\right)^2 - \frac{2}{5}\left(\frac{h_a}{l_a}\right)^4 + ..)$ If you want, you can ignore this.

Now, again, let me emphasize to use this formula, I have to make sure that $\frac{h_a}{l_a} < \frac{1}{2}$. So, let us see. My $h_a = 60, l_a = 150$ and this is indeed less than half. So, therefore, I can use this equation.



So, this will be $150(1+\frac{2}{3}\left(\frac{60}{150}\right)^2-\frac{2}{5}\left(\frac{60}{150}\right)^4+\cdots=164.5\ m)$. Therefore, the length will be $2\times 164.5=329\ m$. Now, these two examples, as I said, they were related to a uniform loading along the x direction. Now, let us discuss the case where the cable is hanging under its own weight.



So, this is one such example and let me write down the problem statement. So, this question is related to the previous question. Question number 3. In the previous problem, the horizontal load is removed and the cable is replaced by a cable which has a mass of $12\ kg/m$. So, earlier the cable does not has mass or the mass was negligible. Now, the mass of the cable is $12\ kg/m$. Find the tension at mid length, the maximum tension and the total length of the cable. So, as I said, because this cable is hanging under its own weight, therefore, it will follow the equation of catenary. Let me write down the equation of catenary for the cable that we have derived. So, $y = \frac{T_0}{\mu}\left(cosh\frac{\mu x}{T_0} - 1\right)$. So, to find out T_0, let me use the boundary condition. So, this is to find T_0, we are using the boundary condition and for that, let us say this point is, let me just make the free body diagram once again. So, we have this cable we put the coordinate x is at the bottom point. So, this is x, this is y and from here the cable is hanging.

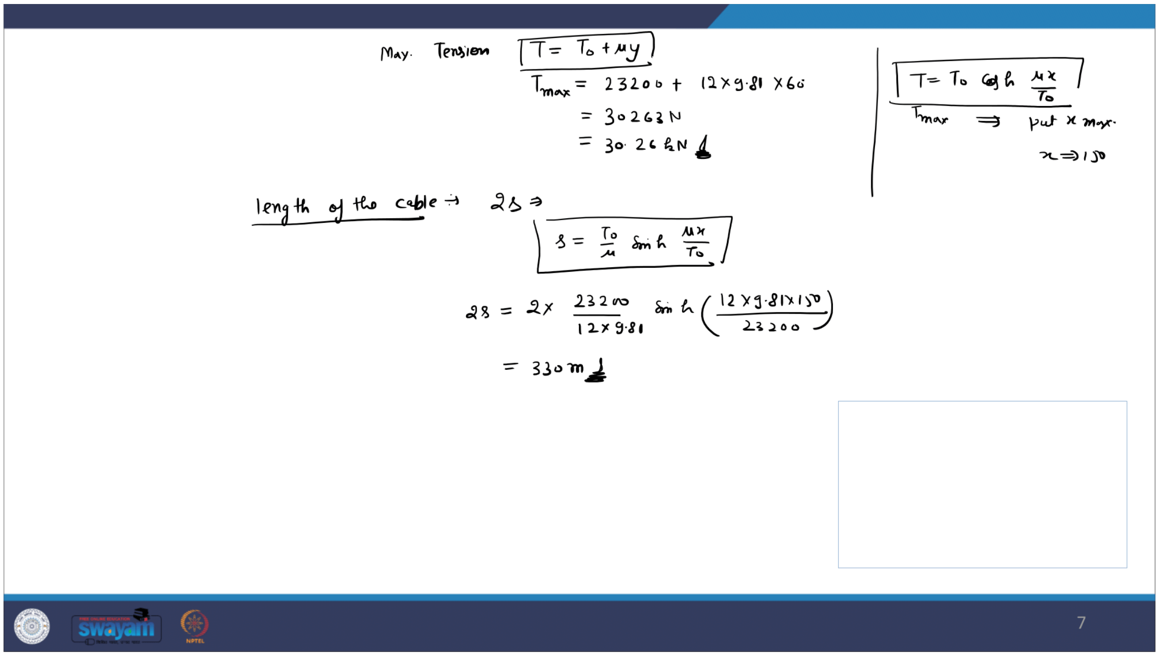
Let me call this point A, this point B and this is given that it is $60\ m$, this is $300\ m$, So, let us use the boundary condition at let us say A point. So, at A, you can see that $y = 60$ and $x = \frac{300}{2} = 150\ m$. Now, let us put it back into the above equation. So, we have $60 = \frac{T_0}{12\times 9.81}(\cosh\frac{12\times 9.81\times 150}{T_0} - 1)$. Let us simplify it. So, this will be $\frac{7060}{T_0} = \cosh\left(\frac{17660}{T_0}\right) - 1$. Now, this equation is not easy to solve.

So, therefore, what we can do is we can solve it graphically and for that, let us plot both the side and then the intersection, you know, of both the plots will give you the solution for T_0. So, let us assume that this is y. So, what we are going to do is we will plot this, we will plot that and then their intersection will give you T_0. So, this is how this equation can be solved. So, on this side, you have T_0 and you can plot $\frac{7060}{T_0}$.

So, let us say this is one plot and for the second one, this is the plot. So, this you can do in the computer and the intersection as I said, this will give you the value of T_0. So, let us say this is $\cosh\left(\frac{17760}{T_0}\right) - 1$ and this is $\frac{7060}{T_0}$, okay. And if you do that, you find out the intersection. So, the intersection gives you $T_0 = 23200\ N$ which is nothing but $23.2\ kN$. So, this is T_0. Now, let us find out the maximum tension in the cable. So, again we have derived the formula for the maximum tension.

Maximum tension you can use $T = T_0 + \mu y$ And for the maximum tension, you have to maximize y. Let us see what is the maximum value of y. The maximum value of $y = 60$. So, therefore, $T_{max} = T_0$ and T_0 we have just find out. This is $23200 + \mu y$, μ is the weight per unit length. So, this will be $23200 + 12 \times 9.81 \times 60$. So, this comes out to be $30263\ N$ or this is $30.26\ kN$. Note that you can also find out the tension by using $T =$

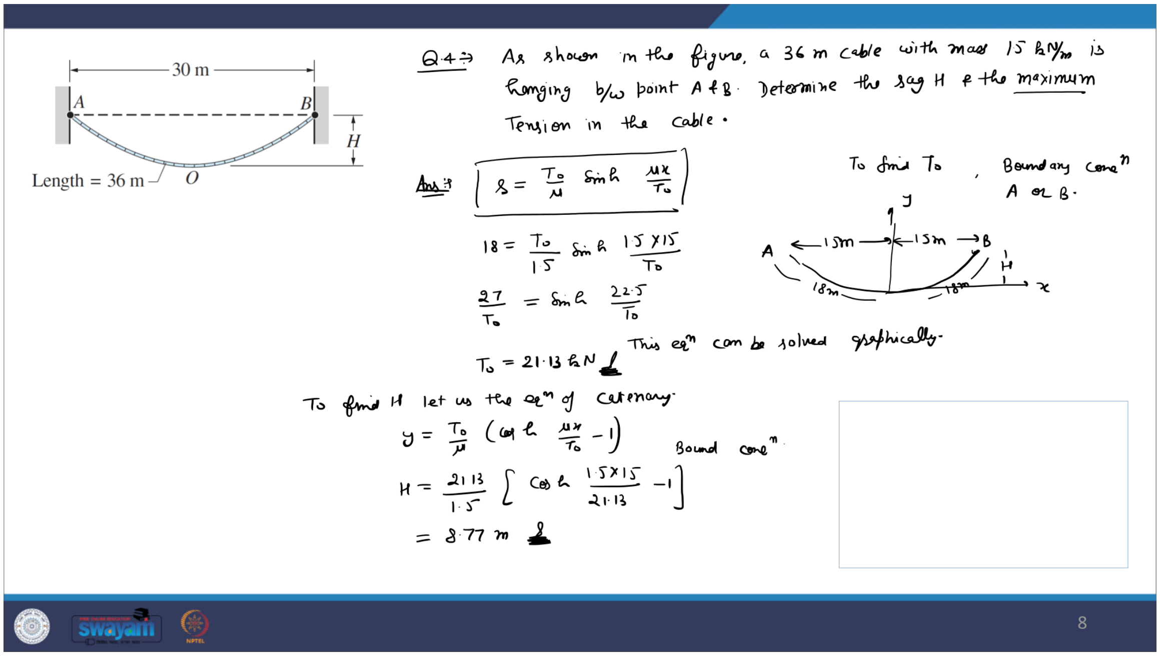
$T_0 \cosh\left(\frac{\mu x}{T_0}\right)$ and here for T_{max}, you have to put x maximum and the maximum value of x=150. So, you can also find out T_{max} using this formula.



Now, let us find out the length of the cable. So, the length of the cable, you can see it will be $2s$. So, let us say this is s or s_a, then it will be $2s_a$. So, the length will be $2s$ and we know the formula for s because $s = \frac{T_0}{\mu}\sinh\left(\frac{\mu x}{T_0}\right)$. Again, we have derived this formula.

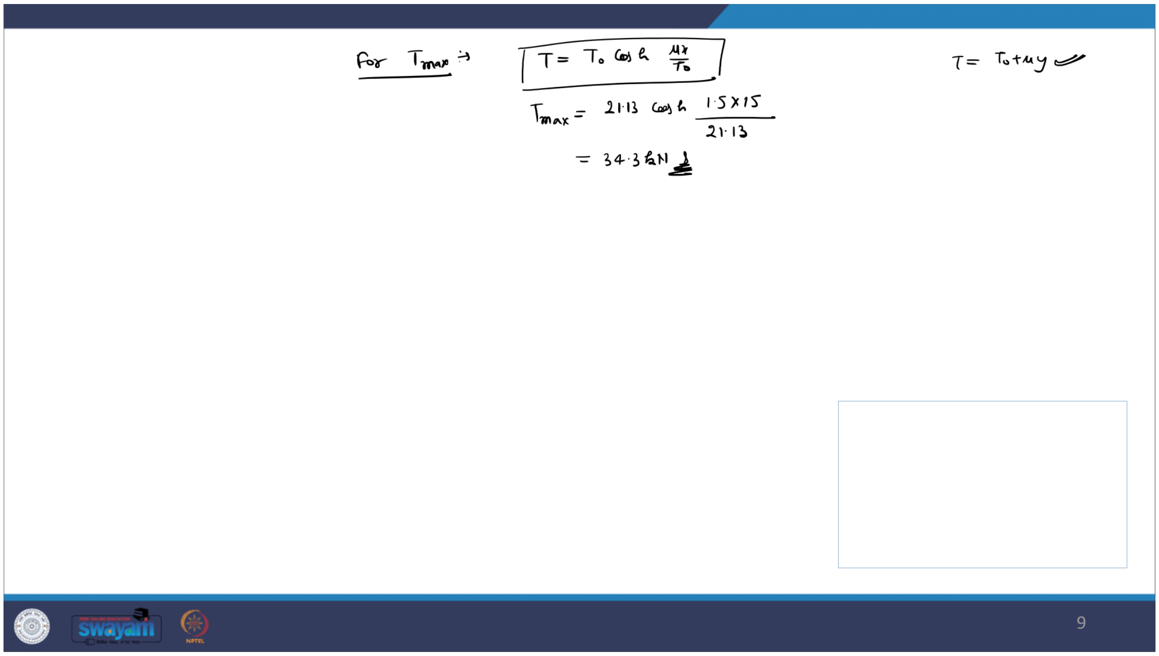
So, therefore, $2s = 2 \times \frac{23200}{12\times 9.81} \times \sinh\left(12 \times 9.81 \times \frac{150}{23200}\right)$. And you can solve it using calculator. This will be $330\ m$.

Now, let us look one more example on the cable, which is suspended under its own weight. So, here the problem statement is as shown in the figure, a 36 meter cable with mass 1.5 kilo Newton per meter is hanging between point A and B. determine the sag H and the maximum tension in the cable. So, let us find out the sag.



Now, to find out the sag, let us first find out what is T0. And to find out T0, because here s is given, the length of the cable is given. So, let us use the formula for the length of the cable. So, we have also derived this s equal to T0 divided by mu sin hyperbolic mu x over T0. And as I said, to find T0 because that is constant. This is tension along the horizontal direction. So, let us use the boundary condition. You can either use A or B because that is symmetric. So, at A or B, s is 18. So, we put our, you know, you have this cable, this is point A, this is point B. Remember, you have to put the origin at the minima. So, therefore, this will be 18 m, this will be 18 m and this is 15 m and this is also 15 m. So, $s = 18 = \frac{T_0}{15}\sinh\left(\frac{15\times15}{T_0}\right)$. Now, let us simplify this. So, this will be $\frac{27}{T_0} = \sinh\left(\frac{22.5}{T_0}\right)$ and again this equation can be solved graphically. So, either you can use computer or you can plot both the side in the graph and you can find out the intersection. The intersection will give you the value of T_0 and $T_0 = 21.13\ kN$. Now, let us find out H. So, to find out H, let us use the equation of catenary. So, the equation of the catenary was $y = \frac{T_0}{\mu}(\cosh\left(\frac{\mu x}{T_0}\right) - 1)$ and to find out y, we have to use the boundary condition. So, let us use the boundary condition either at point A or point B. So, at point A or B, $y = H = \frac{21.13}{1.5}\left(\cosh\left(\frac{1.5\times1.5}{21.13}\right) - 1\right) = 8.77\ m$. So, this can be solved in calculator and it will be 8.77 .

Now, let us find out the maximum tension in the cable. So, maximum tension in the cable for T_{max}, $T = T_0 \cosh\left(\frac{\mu x}{T_0}\right)$. You can also use, you know, $T = T_0 + \mu y$



but this is the one that we used in the, you know, previous question. So, this time, let us use this formula. So, $T_{max} = 21.13 cosh\left(\frac{1.5 \times 1.5}{21.13}\right) = 34.3\ kN$.

So, with this, let me stop here. Thank you.