

Mechatronics
Prof. Pushparaj Mani Pathak
Department of Mechanical and Industrial Engineering
Indian Institute of Technology, Roorkee

Lecture - 30
Transfer Function and Frequency Response

I welcome you all to this NPTEL online certification course on Mechatronics. Today we are going to talk about the Transfer Function and Frequency Response. In the last lecture, I discussed with you the dynamic response of the system of the first-order system, second-order system, and we have seen the various performance measurement parameters also. Here I am going to discuss with you another way of seeing the dynamic response of the system where we transform our equations from the time domain to the Laplace domain that is s domain, and then we look at the response. We look at the response, of course, in the case of the frequency response. So, I am going to talk about all those things in this lecture. Let us look at the system transfer function. First of all, suppose I have got an amplifier. For that amplifier, I can define a gain as simply output by input. So, what does this means that suppose you have an amplifier with a gain of 5.

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Introduction – System Transfer Function

- For an amplifier we define gain as
- $\text{Gain} = \frac{\text{Output}}{\text{input}}$

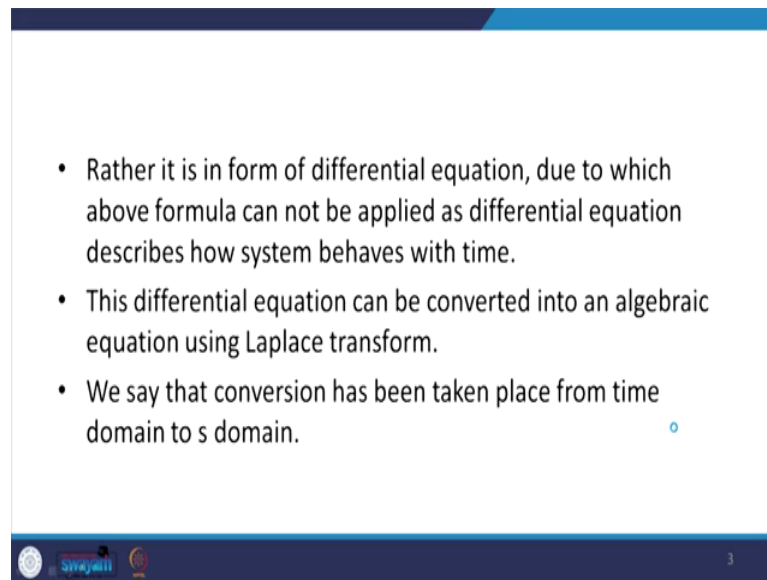
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graph LR; Input["Input  
y(t)"] --> Gain["Gain  
G"]; Gain --> Output["Output  
x(t)"]
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- The physical meaning of this equation is that if the gain of an amplifier is 5, then for an input of 4mV, the output will be 20 mV.
- For many systems the relationship between output and input is not in algebraic form as above.

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Then it means that for a given input of 4 millivolts, the output which you are going to get will be the 20 millivolts. Now for many systems, the relationship between output and input will not be as simple as this one. It will be rather in the algebraic form.

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- Rather it is in form of differential equation, due to which above formula can not be applied as differential equation describes how system behaves with time.
- This differential equation can be converted into an algebraic equation using Laplace transform.
- We say that conversion has been taken place from time domain to s domain.

These algebraic forms or differential equations describe the behavior of the system with time. And if these are differential equations can be converted into the algebraic form using the Laplace transform. And so, as I said, we are moving that way from the time domain to the s domain.

The transfer function is defined as the Laplace transform of the output divided by the Laplace transform of the input. So, this is again a way of defining the transfer function. And a signal in the time domain is depicted by $f(t)$, where a signal in the Laplace domain is dependently depicted by $F(s)$.

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Transfer Function

- The relationship between output and input can be defined using transfer function as
- Transfer function = $\frac{\text{Laplace transform of output}}{\text{Laplace transform of input}}$ ✓
- Signal in time domain indicated by $f(t)$ ✓
- Signal in Laplace domain indicated by $F(s)$ ✓

In the Laplace domain, we write the function always in the capital letter that is the universal convention, whereas, in the time domain, we would write it in terms of the small letter.

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    graph LR
      Input["Input  
Y(s) ✓"] --> TF["Transfer function  
G(s) ✓"]
      TF --> Output["Output  
X(s) ✓"]
  
```

- Let input to a linear system has Laplace transform $Y(s)$ and output of linear system has Laplace transform $X(s)$, then transfer function $G(s)$ of the system is defined as

$$G(s) = \frac{X(s)}{Y(s)} \quad \checkmark$$

- Here all the initial conditions are assumed to be zero, i.e. zero output with zero input.

So, suppose I have got a transfer function $G(s)$ and when this is subjected to input $Y(s)$ and output is $X(s)$. So, this is what it means that,

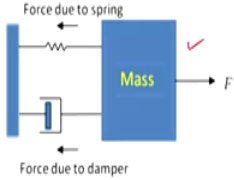
$$G(s) = X(s)/ Y(s)$$

Here we assumed the initial conditions to be zero, that is, zero output with zero input.

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Example

- Consider mass spring damper system shown.
- Dynamics of system is described by
- $m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = F$ ✓
- Taking Laplace transform ✓
- $m s^2 X(s) + c s X(s) + k X(s) = F(s)$ ✓



$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}$$

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Suppose I take the example of a transfer function. So, I can take a second-order system, which is a spring-mass damper system, and for this system we the dynamic equation is given by this one after drawing the free body diagram. And if I take the Laplace to transform for this is what I am going to get, and so, from here, I can find out the value ratio of the output by input. So, output here is X(s), and input is F(s). So, this is the transfer function.

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Laplace Transforms

- It can be used for solving linear differential equation.
- We can convert functions such as sinusoidal and exponential into algebraic functions of a complex variable s.
- Differentiation and integration operations can be replaced by algebraic operations in the complex plane.
- It allows use of graphical technique for predicting the system performance without solving system differential equation.
- When we solve differential equation using this method transients and steady state components of the solution can be obtained simultaneously.

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So, what is Laplace transform? And these Laplace transforms are used to convert the differential equations into an algebraic equation. We can convert functions such as sinusoidal and exponential into the algebraic function of a complex variable s.

And the differentiation and integration operations can be replaced by algebraic operations in the complex plane. It allows the use of graphical techniques for predicting the system performance without solving the differential equation. So, that is one of the very good advantages of the Laplace transform, and when we solve a differential equation using this method, transient and a steady-state component of the solution can be obtained simultaneously, which otherwise has to be done separately, as we have seen in the last lecture.

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The slide is titled "Laplace Transform" and contains the following text:

- Let $f(t)$ be a function of time t such that $f(t)=0$, for $t<0$
- s be a complex variable ✓
- \mathcal{L} be an operational symbol indicating the Laplace transform of function
- Laplace transform of function $f(t)$ is given by
- $\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st} dt$

At the bottom of the slide, there are logos for "Sivajini" and a small number "8".

So, let $f(t)$ be a function of time such that $f(t) = 0$, for $t < 0$ and the s is a complex variable, then and L be the operational symbol indicating the Laplace transform of a function then the Laplace transform of function $f(t)$ is this is how it is given as represented by,

$$L[f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st} dt$$

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- The process of finding the function $f(t)$ from the Laplace transform $F(s)$ is called inverse Laplace transformation.
- Inverse Laplace transform is represented as \mathcal{L}^{-1}
- Inverse Laplace transform can be found from $F(s)$ by inversion integral as
- $\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)e^{st} ds$, for $t > 0$
- Here c is the abscissa of convergence, is a real constant and is chosen larger than the real parts of all singular points of $F(s)$

The process of finding the function $f(t)$ from the Laplace transform $F(s)$ is called the inverse Laplace transform. That is, if you want to come back from the Laplace domain to the time domain, what we have to do is that we have to perform the inverse Laplace transformation, and this inverse Laplace transformation is represented by L^{-1} And inverse Laplace can be found by inversion integral like this. So, this way, we can do the inverse.

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Basic Laplace Transforms for Common Input

- Unit impulse at time $t = 0$ has a transform of 1
- A unit step signal defined by

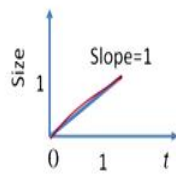
$$1(t) = 0, \text{ for } t < 0$$

$$= 1, \text{ for } t > 0$$
- $\mathcal{L}[1(t)] = \int_0^{\infty} e^{-st} dt = \frac{1}{s}$

So, the basic Laplace transform for the common input, which we have seen in my previous lecture. Such as if we have a uniform impulse at $t = 0$, it has a Laplace transform equal to 1, and if it is a unit step signal defined by less than t for a $t < 0$, it is 0 and time > 0 , it is 1 this way then the Laplace transform for that is $1/s$.

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- Unit ramp function
 $f(t) = 0, \text{ for } t < 0,$
 $= t, \text{ for } t \geq 0$
 $\mathcal{L}[t] = \int_0^{\infty} t e^{-st} dt = \frac{1}{s^2}$ ✓
- Unit amplitude sine wave signal ✓
 $f(t) = 0, \text{ for } t < 0,$
 $= \sin \omega t, \text{ for } t \geq 0$
 $\mathcal{L}[\sin \omega t] = F(s) = \int_0^{\infty} \sin \omega t e^{-st} dt = \frac{\omega}{s^2 + \omega^2}$ ✓



And for unit ramp one, the Laplace transform is $\frac{1}{s^2}$. And for unit amplitude sine wave signal, Laplace transform is

$$L[\sin \omega t] = \frac{\omega}{(s^2 + \omega^2)}$$

where ω is the frequency of the sinusoidal signal.

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- Unit amplitude cosine wave signal
 $f(t) = 0, \text{ for } t < 0,$
 $= \cos \omega t, \text{ for } t \geq 0$
 $\mathcal{L}[\cos \omega t] = F(s) = \int_0^{\infty} \cos \omega t e^{-st} dt = \frac{s}{s^2 + \omega^2}$

Similarly, the Laplace transform for,

$$L[\cos \omega t] = \frac{s}{(s^2 + \omega^2)}$$

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Basic rules in working with Laplace transform

- $\mathcal{L}[Af(t)] = AF(s)$
- $\mathcal{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$
- Laplace transform of 1st derivative of a function is
- $\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$, however with transfer function all initial values are taken to be zero.
- $\mathcal{L}\left[\frac{d^2f(t)}{dt^2}\right] = s^2F(s) - sf(0) - \frac{df(0)}{dt}$
- Laplace transform of an integral of a function is
- $\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s)$

So, there are certain basic rules in working with Laplace transform that are given here.

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Inverse Laplace Transform

F(s)	f(t)
$\frac{1}{s+a}$	e^{-at}
$\frac{a}{s(s+a)}$	$1 - e^{-at}$
$\frac{b-a}{(s+a)(s+b)}$	$e^{-at} - e^{-bt}$
$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$
$\frac{a}{s^2(s+a)}$	$t - \frac{(1-e^{-at})}{a}$


- When algebraic manipulation has been done in s domain then the results can be brought back to time domain using inverse Laplace transform.
- Some important inversions are

And if we look at the inverse of the Laplace transform. So, if these are the Laplace transform, here is the inverse of that.

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First Order System

- $a_1 \frac{dx}{dt} + a_0 x = b_0 y$
- LT of this with all initial conditions zero is
- $a_1 sX(s) + a_0 X(s) = b_0 Y(s)$
- $G(s) = \frac{X(s)}{Y(s)} = \frac{b_0}{a_1 s + a_0}$
- $G(s) = \frac{X(s)}{Y(s)} = \frac{b_0/a_0}{(a_1/a_0)s + 1} = \frac{G}{\tau s + 1}$
- Where G is gain of the system when there is steady state condition
- (a_1/a_0) is time constant of the system.



So, now, let us look at the first-order system. So, this is the first-order system, as we have seen in the previous lecture. So, Laplace transforms for this with initial condition zero. I can simply write like a_1 . This one I can write as $sX(s)$. This is $a_0 X(s)$ is equal to $b_0 Y(s)$. So, I can write the relationship between output,

$$\frac{X(s)}{Y(s)} = \frac{b_0}{a_1 s + a_0}$$

$$\frac{X(s)}{Y(s)} = \frac{G}{\tau s + 1}$$

where G is the gain of the system when there is a steady-state condition, and $\frac{a_1}{a_0}$ is the time constant of the system.

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- If first order system is subjected to unit step input then
- $G(s) = \frac{X(s)}{Y(s)} = \frac{G}{\tau s + 1}$
- $X(s) = G(s)Y(s)$
- $X(s) = \frac{G}{\tau s + 1} \cdot 1$
- $X(s) = \frac{G \left(\frac{1}{\tau}\right)}{s \left(s + \frac{1}{\tau}\right)}$
- $X(s) = \frac{G \left(\frac{1}{\tau}\right)}{s \left(s + \frac{1}{\tau}\right)}$
- Taking inverse Laplace transform
- $x = G \left(1 - e^{-t/\tau}\right)$

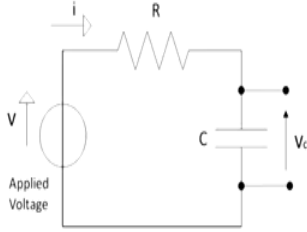
Now, if the first-order system is subjected to a unit step input, then we can take this $Y(s) = 1/s$. So, I can take write this $X(s) = G(s)Y(s)$. So, this is $1/s$ I am doing it I can write it in this form, and if I take the inverse Laplace of this, then this is what we get. So, you see that by taking the inverse Laplace transform, we are coming back to the time domain.

So, here is what we have done we have this expression in which the first-order system dynamic equation is in the form of a first-order differential equation. So, with the help of Laplace transform, we converted it into the algebraic expression, and then after algebraic conversion into algebraic expression, we went back to the time domain by using the inverse Laplace transform.

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Example: Resistor-capacitor system

- $V = CR \frac{dV_c}{dt} + V_c$ ✓
- $V(s) = CRsV_c(s) + V_c(s)$
- $G(s) = \frac{V_c(s)}{V(s)} = \frac{1}{CRs+1}$ ✓



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So, again the resistor-capacitor system is an example of the first ordered system. So, we can write the expression for this one, and this could be simplified in this way.

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TF of 2nd order system

- For a 2nd order system, the relationship between input y and output x is given by differential equation of the form
- $a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0x = b_0y,$ ✓
- Here a_2, a_1, a_0 and b_0 are constants.
- Taking the Laplace transform of equation with all initial conditions as zero, gives
- $a_2s^2X(s) + a_1sX(s) + a_0X(s) = b_0Y(s),$

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Now, let us look at the second-order system. So, in the case of a second-order system, the relationship between the input and output is given by this one, as we have seen in the previous lecture. Where these a_2, a_1, a_0 and b_0 are constant. Now, if I take the Laplace of this is a second derivative. So, this is $s^2X(s)$ is first derivative, so $sX(s)$, and here it will be only $X(s)$ and $Y(s)$.

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- $a_2s^2X(s) + a_1sX(s) + a_0X(s) = b_0Y(s)$,
- $G(s) = \frac{X(s)}{Y(s)} = \frac{b_0}{a_2s^2 + a_1s + a_0}$ ✓
- Alternative way of writing the differential equation for a 2nd order system is
- $a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0x = b_0y$ ✓
- For this system ω_n and ζ can be defined as
- $\omega_n^2 = \frac{a_0}{a_2}$ and $\zeta^2 = \frac{a_1^2}{4a_2a_0}$

So, now I can find out the relationship between X(s) and Y(s), and it is this one. So, here you can see that we have a quadratic term in s over here. There is an alternate way of writing this differential equation of the second-order system, and that we can do by defining these terms ω_n^2 and ζ^2 like this.

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- Simplifying yields
- $\frac{d^2x}{dt^2} + 2\zeta\omega_n \frac{dx}{dt} + \omega_n^2x = (b_0/a_0)\omega_n^2y$,
- Where ζ is the damping ratio and ω_n is the natural angular frequency with which system oscillates.
- Laplace transform is of form
- $G(s) = \frac{X(s)}{Y(s)} = \frac{(b_0/a_0)\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ ✓
- When this system is subjected to unit step input i.e.,
- $Y(s) = \left(\frac{1}{s}\right)$ ✓

And if I simplify write this expression in terms of ζ and ω_n I can write it like this, and if I take the Laplace, then I can write X(s)/ Y(s) is equal to this one. So, this way, I can write, and if I subject it to a unit step input, then the transfer function for input is 1/ s.

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- $X(s) = \frac{(b_0/a_0)\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$
- $X(s) = \frac{(b_0/a_0)\omega_n^2}{s(s+p_1)(s+p_2)}$
- Where p_1 and p_2 are the roots of the equation
- $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$
- $p = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2}$

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So, I can substitute in this one, and I can see the response for that. So, I have 1/s term over here. So, this is my output in the s domain, and this is my output in the s domain. I can write this always because it is a quadratic equation. So, they are going to be the two poles. So, I can write it in the form of $s + p_1$ and $s + p_2$ and where these p_1 and p_2 are the poles are the roots of the equation. So, I can find out these p_1 and p_2 values by equating this is equal to 0 and then writing the solution of this second-order equation yes.

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- $p = -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2}$
- $p = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$
- $p_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$
- $p_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$
- **Overdamped case ($\zeta > 1$)**

$$X(s) = \frac{b_0\omega_n^2}{s(s+p_1)(s+p_2)}$$

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So, this is what is that one, and I can simplify this one this is there, and then I can take write the expression response for $\zeta > 1$ in terms of the p_1 and p_2 .


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By applying inverse Laplace transform to the partial fractions, we get

$$x(t) = \frac{b_0 \omega_n^2}{a_0 p_1 p_2} \left[1 - \frac{p_2}{p_2 - p_1} e^{-p_2 t} + \frac{p_1}{p_2 - p_1} e^{-p_1 t} \right]$$

If, $\zeta = 1$,


- $p = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$
- $p_1 = p_2 = -\omega_n$

$$X(s) = \frac{\frac{b_0 \omega_n^2}{a_0}}{s(s + \omega_n)(s + \omega_n)} = \frac{\frac{b_0 \omega_n^2}{a_0}}{s(s + \omega_n)^2}$$


And then, by applying the inverse Laplace transform to the partial fraction, we can get back to the time domain. So, this is my expression in the time domain similarly. If I have $\zeta = 1$, then this is what my expression is going to be. So, these are my roots. Both are going to be equal. So, this is my response in the s domain.

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- Expanding by partial fraction
- $X(s) = \frac{b_0 \omega_n^2}{a_0} \left[\frac{1}{s} - \frac{1}{s + \omega_n} - \frac{\omega_n}{(s + \omega_n)^2} \right]$ ✓
- $x(t) = \frac{b_0 \omega_n^2}{a_0} \left[1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t} \right]$ ✓
- **Underdamped case ($\zeta < 1$)**
- $x = \frac{b_0}{a_0} \left[1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1 - \zeta^2}} \sin(\omega_n \sqrt{1 - \zeta^2} t + \phi) \right]$



And again, by partial fraction, I can convert it into the time domain. So, this is how we get the response of the system in the time domain using the Laplace transform. Similarly, if it's an underdamped case, then this is my response which I am going to get.

I can explain one example indicating the state of damping of a system having the transfer function this one and subjected to the unit step input function. So, the unit step input function means your $Y(s) = 1/s$, and $G(s) = X(s)/Y(s)$. So, your $X(s)$ is $1/s$, and you can see that this is $s+4$, and $s+4$.

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Example

- Indicate the state of damping of a system having the transfer function $G(s) = \frac{1}{(s^2+8s+16)}$ and subjected to a unit step function.
- For unit step input $Y(s) = \frac{1}{s}$, $G(s) = \frac{X(s)}{Y(s)}$
- $X(s) = \frac{1}{s(s+4)(s+4)}$
- Roots of $s^2+8s+16$ are $p_1=-4$ and $p_2=-4$,
- The roots are real and equal, hence system is critically damped.

So, in this case, your roots are going to be equal to this one. So, the roots are real and equal, and hence the system is critically damped. So, this way, we can tell the behavior of the system.

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Effect of pole location on transient response

Consider a first order system with

$$TF = \frac{1}{(s+1)}$$

Let the input be impulse

Then $\frac{1}{(s+1)} = \frac{X(s)}{Y(s)} \implies X(s) = \frac{1}{(s+1)}$

Hence $x = e^{-t}$

and as $t \rightarrow \infty, x \rightarrow 0$

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Then let us look at the effect of pole location on the transient response. So, if we consider a first-order system is something like this and if the input is impulse. Then I can write this as $1/(s+1)$ because $Y(s)$ is going to be one only. So, if I take the inverse Laplace of this, then I will get the time responses to this one, and what does this mean that? This means that as $t \rightarrow \infty, x \rightarrow 0$.

So you see that this is $s = -1$. So, your root is equal to root is a in the left half of the s plane or the pole is in the left half of the s plane.

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2. Let $TF = \frac{1}{(s-1)}$ and subjected to unit impulse

Then $\frac{1}{(s-1)} = \frac{X(s)}{Y(s)} \implies X(s) = \frac{1}{(s-1)}$

Hence $x = e^t$

and as $t \rightarrow \infty, x \rightarrow \infty$

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And if your transfer function is $1/(s-1)$ and subjected to unit impulse, then again this one, and if I write the take the inverse Laplace of this and go in the time domain, this is $x = e^t$. And this means that as $t \rightarrow \infty$, your response will be going $x \rightarrow \infty$. So, this is an example of an unstable system. So we can conclude that if your root is here, in this case, you see where is the pole s is equal to 1. It means that if your pole is on the right half of the s plane, your system is going to be unstable, and if your pole is on the left half of the s plane, your system is going to be stable. So, this is what we can conclude from that.

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• For 2nd order system with

$$\text{TF } G(s) = \frac{\frac{b_0 \omega_n^2}{a_n}}{(s^2 + 2\zeta \omega_n s + \omega_n^2)} \text{ when subjected to a unit impulse input i.e., } Y(s)=1$$

$$X(s) = \frac{\frac{b_0 \omega_n^2}{a_n}}{(s^2 + 2\zeta \omega_n s + \omega_n^2)} = \frac{\frac{b_0 \omega_n^2}{a_n}}{(s+p_1)(s+p_2)}, \text{ where } p_1$$

and p_2 are roots of characteristic equation

For a second-order system. Similarly, we know the response is this one. When subjected to a unit impulse, I can see that $Y(s)=1$. I can write $X(s)$ as this one, or I can write it in terms of the two roots, where p_1 and p_2 are the roots of this characteristic equation.

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- $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0.$

$$p = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$p = -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2}$$

Depending on the value of the ζ , p can be real or imaginary and imaginary term involves oscillations.

So, again here I can write these roots like this one as we have seen, and depending on the value of ζ , p can be real or imaginary terms that involves oscillation. So, that way, we can predict the behavior.

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- Example: Consider a 2nd order system with TF

$$G(s) = \frac{1}{(s - (-2+j))(s - (-2-j))}$$

Here, $p = -2 \pm j.$

- When the system subjected to unit impulse

$$G(s) = \frac{X(s)}{Y(s)} \implies X(s) = G(s)$$

- $X(s) = \frac{1}{(s - (-2+j))(s - (-2-j))}$

$x(t) = e^{-2t} \sin t$

So, if I consider a second-order system something like this, so, its roots are you can see that $-2+j$ and $-2-j$ over here. And when this is subjected to unit impulse, your $X(s) = G(s)$. And here, if I take the inverse Laplace of this, this is what I am going to get.

And so, in this case, we can see that there is an exponentially decaying envelope on this one on the *sin* function over here. And this way, oscillations are going to gradually decrease, and your system is going to be stable.

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$$G(s) = \frac{1}{(s - (2 + j))(s - (2 - j))}$$
 Here, $p = 2 \pm j$.
 When the system subjected to unit impulse

$$G(s) = \frac{X(s)}{Y(s)} \Rightarrow X(s) = G(s)$$

 • $X(s) = \frac{1}{(s - (2 + j))(s - (2 - j))}$
 • $x(t) = e^{-2t} \sin t$

And if you have a root like this. So, this one, then your poles are $2 + j$. So, when this system is subjected to unit impulse again, you are going to get $X(s) = G(s)$ because unit impulse $Y(s) = 1$ and if I take the inverse Laplace of this is I am going to get the response. So, here you can see that we have an exponentially increasing term over here, and this is enveloping the sinusoidal functions. So, that makes the system unstable.

(Refer Slide Time: 20:18)

Compensation

- The output from a system might be unstable
- Or response may be slow
- Or there is too much overshoot
- System response to an input can be altered by including compensators.
- A compensator is a block which is incorporated in a system so that it alters the overall TF of the system to get the desired characteristic

Now we have seen the response of the system for input. Now, as I said, the output from a system might be unstable, the response may be very slow, or there may be too much of an overshoot. So, such system behavior can be altered by including the compensator. So, a compensator is a block that is incorporated into a system so that it alters the overall transfer function of the system to get the desired characteristic.

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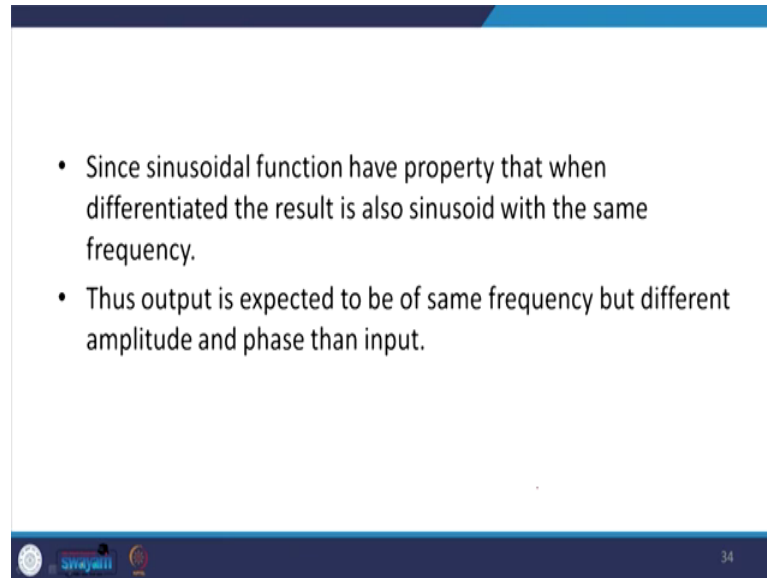
Frequency Response

- Sinusoidal input is an important input in design and analysis of system
- Consider a 1st order system described by
- $a_1 \frac{dx}{dt} + a_0 x = b_0 y$
- Here y is input and x is output
- Let unit amplitude sinusoidal input be given by $y = \sin \omega t$
- So $a_1 \frac{dx}{dt} + a_0 x = b_0 \sin \omega t$

So, after seeing that let us try to learn the frequency response. So, till now, I had talked about the step input to a system, but our input to the system could be sinusoidal as well. And the sinusoidal inputs are an important input in the design and analysis of systems. So,

if the first order system which we have considered over here. If I have a sinusoidal input $likey = \sin\omega t$, this is how my system expression is going to be.

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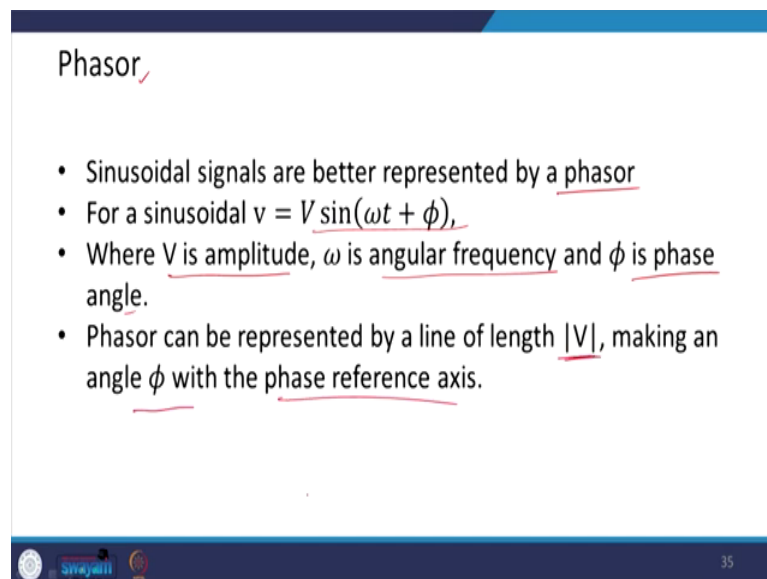
Slide 34 contains two bullet points:

- Since sinusoidal function have property that when differentiated the result is also sinusoid with the same frequency.
- Thus output is expected to be of same frequency but different amplitude and phase than input.

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Now, since the sinusoidal function have the property that when differentiated, the result is also sinusoidal with the same frequency. Thus the output is expected to be of the same frequency but may be of different amplitude and phase than that of the input. So, there, the frequency response is generalized with the help of a phasor, and the sinusoidal signals are better represented by a phasor.

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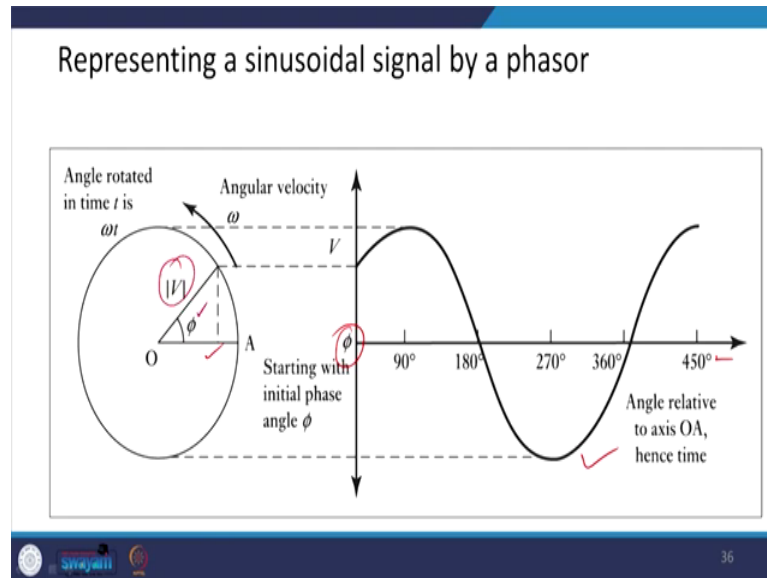
Slide 35 is titled "Phasor" and contains four bullet points:

- Sinusoidal signals are better represented by a phasor
- For a sinusoidal $v = V \sin(\omega t + \phi)$,
- Where V is amplitude, ω is angular frequency and ϕ is phase angle.
- Phasor can be represented by a line of length |V|, making an angle ϕ with the phase reference axis.

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So, for a sinusoidal $v = V \sin(\omega t + \phi)$ V is the amplitude and ω is the angular frequency, and ϕ is the phase angle. So, the phasor can be represented by a line of length magnitude of V , making an angle ϕ with the phase reference axis.

(Refer Slide Time: 22:46)



So, here you can see that you have a sinusoidal signal over here. And starting with initial phase angle ϕ and you have the angle related to x is OA hence time this one. So, this is how it can be represented by a phasor of magnitude V and the angle ϕ .

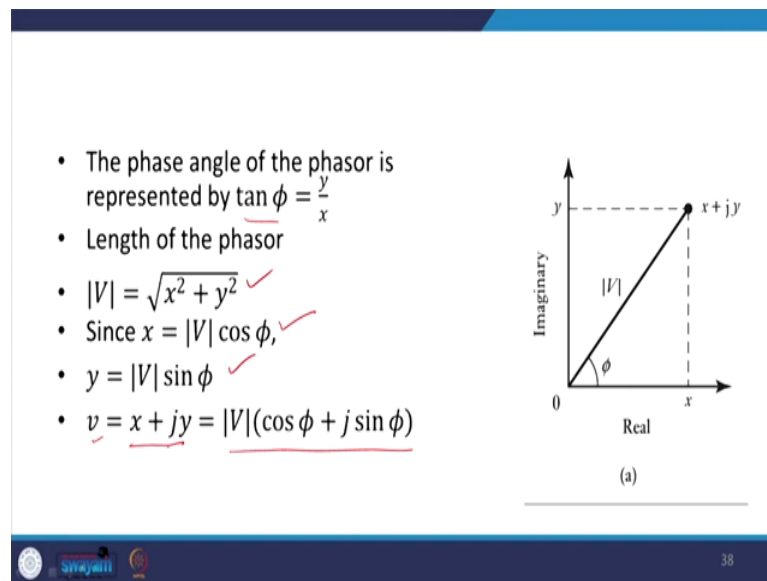
(Refer Slide Time: 23:10) ϕ

- Phasor can be described by complex number.
- Complex no is represented by $x+jy$
- On a graph with imaginary component as the y axis and real part as the x axis, x and y are the Cartesian coordinates of the point which represent the complex number.
- We join this point to the origin to represent the phasor.

(a)

So, as I was saying, so, phasor can be described by a complex number, and the complex number is represented by $x+jy$. So, here you have a phasor of magnitude V and at an angle ϕ . This I can describe by complex number $x+jy$. So, here on the graph with imaginary component as the y axis and the real part at the x -axis x and y are the Cartesian coordinate of the point which represents the complex number. We joined this point to the origin to represent the phasor.

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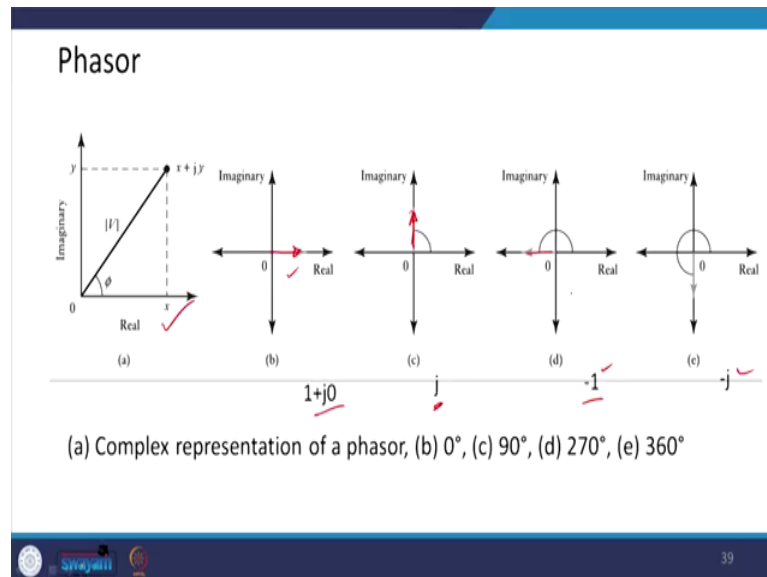


Now, the phase angle of the phasor is represented by the $\tan \phi = y/x$, and the length of the phasor is,

$$|V| = \sqrt{x^2 + y^2}$$

And since $x = |V| \cos \phi$ and $y = |V| \sin \phi$, I can write this $v = x+jy$, and this is how it can be written. So, this is how I can represent a phasor as a complex number. So, here you can see the phasor. So, if a complex representation of phasor at here.

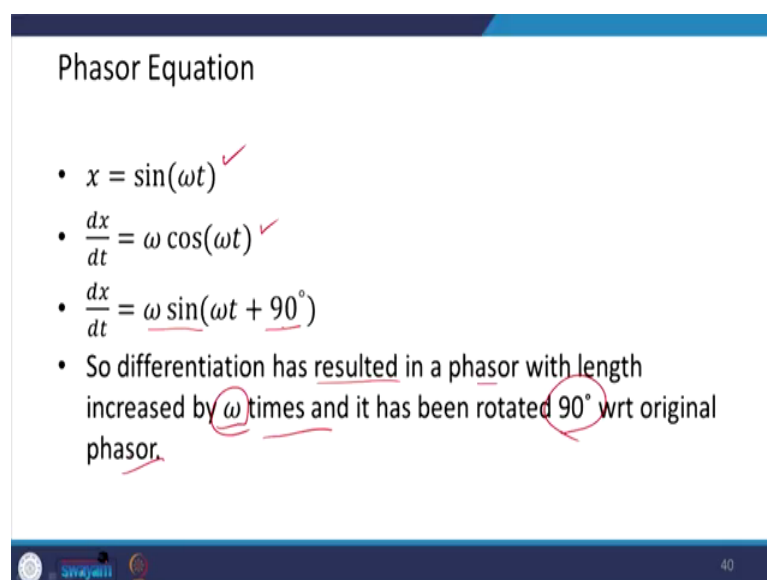
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So, if it is 0 degrees, then this is $1 + j0$ in the complex one if it is over here. So, this is only j real component is not there. And if it is over here, then we have a real component. An imaginary is 0 . So, this is -1 , and here it has got only imaginary, so $-j$. So, here you can see that if the angle is 0 or so, this is this one.

So, when we are turning it by 90° , we are getting this one. So, we are moving from the real to the imaginary if we turn by it 90° or imaginary to real so that way. So, the phasor equation, as we have seen $x = \sin \omega t$. So, the first derivative is $\omega \cos \omega t$, and this I can write as $\omega \sin (\omega t + 90^\circ)$

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So, what has happened is that the differentiation has resulted in a phasor with length increased by ω times, and it has been rotated 90° with respect to the original phasor.

(Refer Slide Time: 26:04)

- So in complex notation differentiation means multiplication of original phasor by $j\omega$ as multiplication by ω will multiply the magnitude by ω and multiplication by j will rotate phasor by 90° wrt previous phasor.
- Thus differential equation
- $a_1 \frac{dx}{dt} + a_0 x = b_0 y$
- Can be written as phase equation as
- $j\omega a_1 X + a_0 X = b_0 Y$
- $\frac{X}{Y} = \frac{b_0}{j\omega a_1 + a_0}$

So, in complex notation, differentiation means the multiplication of the original phasor by ω as multiplication by ω will multiply the magnitude by ω . And multiplication by j will produce the rotate phasor by 90° with respect to the previous phasor. Thus this differential equation I can write as a phasor equation. So, this differentiation dx/dt I am replacing by $j\omega$. So,

$$j\omega a_1 X + a_0 X = b_0 Y$$

(Refer Slide Time: 26:54)

- $\frac{X}{Y} = \frac{b_0}{j\omega a_1 + a_0}$
- $G(s) = \frac{X(s)}{Y(s)} = \frac{b_0}{sa_1 + a_0}$
- Frequency response function or frequency transfer function for steady state condition can be defined as
- $G(j\omega) = \frac{\text{Output phasor}}{\text{Input phasor}}$

So, the transfer function was,

$$\frac{X}{Y} = \frac{b_0}{j\omega a_1 + a_0}$$

So, what has happened is the frequency response function or frequency transfer function for a steady-state condition can be defined as,

$$G(j\omega) = \frac{\text{Output phasor}}{\text{Input phasor}}$$

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Frequency response for a first order system

- A first order system has TF
- $G(s) = \frac{1}{1+\tau s}$ (τ is time constant)
- Frequency response function
- $G(j\omega) = \frac{1}{1+j\tau\omega}$
- $|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2\tau^2}}$
- $\tan \phi = \frac{y}{x} = -\tau\omega$

So, the frequency response of the first order system if I look at. So, this,

$$G(s) = \frac{1}{1 + \tau s}$$

So, in the frequency response here, you can see that what is done is this s is being replaced by this $j\omega$. So, here what we do is that this is in the Laplace domain. If I want to go into the frequency response, I just replace this $s/j\omega$ over here, and then I find out the magnitude I find out the phase of it.

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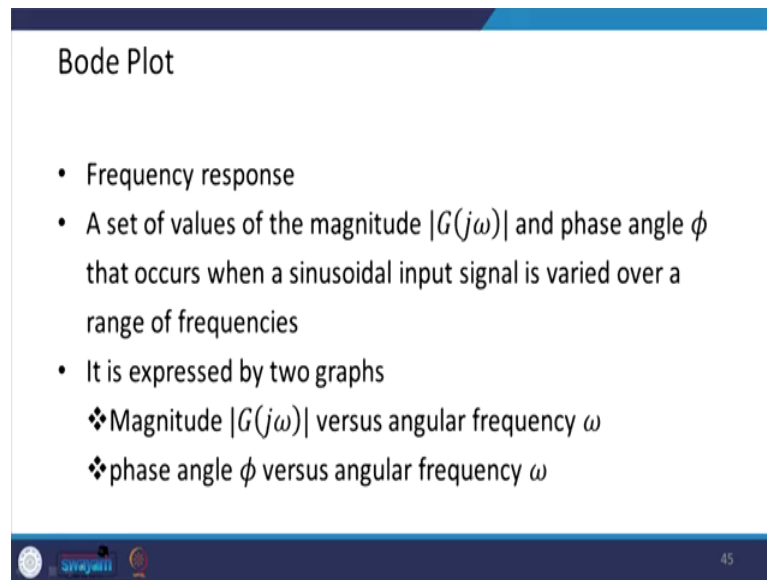
Frequency response for a second order system

- 2nd order system with TF
- $G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ ✓
- Frequency domain response function ✓
- $G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$
- $G(j\omega) = \frac{1 - (\frac{\omega}{\omega_n})^2 - j2\zeta(\frac{\omega}{\omega_n})}{[1 - (\frac{\omega}{\omega_n})^2]^2 + 2\zeta(\frac{\omega}{\omega_n})}$ ✓

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And for the second-order function system, this is the transfer function. So, again frequency domain response function I can write here by replacing s with j omega, I can write it, and then I can write it off this form in the complex form.

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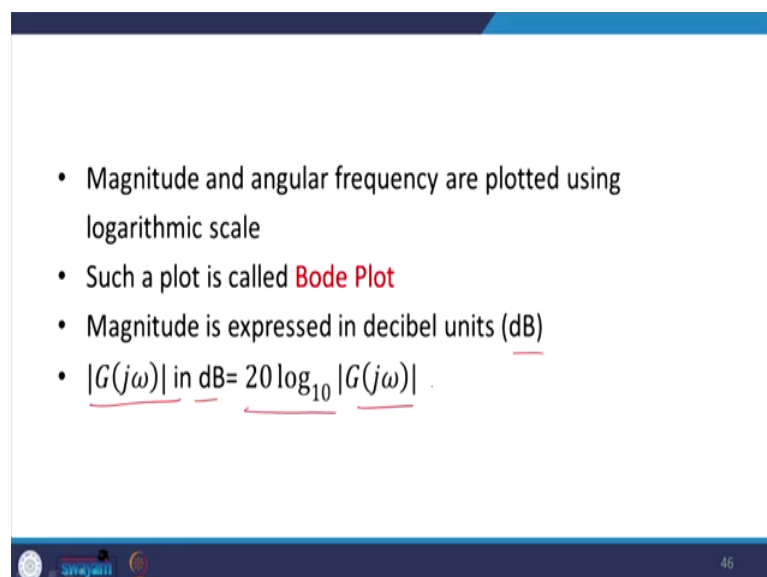
Bode Plot

- Frequency response
- A set of values of the magnitude $|G(j\omega)|$ and phase angle ϕ that occurs when a sinusoidal input signal is varied over a range of frequencies
- It is expressed by two graphs
 - ❖ Magnitude $|G(j\omega)|$ versus angular frequency ω
 - ❖ phase angle ϕ versus angular frequency ω

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Bode plot is used for the frequency response. So, a set of values of the magnitude $|G(j\omega)|$ phase angle ϕ occur when a sinusoidal input signal is varied over a range of frequencies. And it is expressed by two graphs that is one is the magnitude versus angular frequency, and another is the phase angle versus the angular frequency. So, magnitude and angular frequency are plotted using the logarithmic scale, and such a plot is called the bode plot.

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- Magnitude and angular frequency are plotted using logarithmic scale
- Such a plot is called **Bode Plot**
- Magnitude is expressed in decibel units (dB)
- $|G(j\omega)|$ in dB = $20 \log_{10} |G(j\omega)|$

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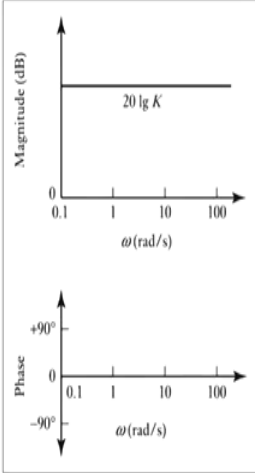
And the magnitude is expressed in the decibel unit. So, this,

$$|G(j\omega)| \text{ in dB} = 20 \log_{10}|G(j\omega)|$$

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Examples of Bode Plots

- Bode plot for system having TF $G(s) = K$ ($K = \text{constant}$)
- Frequency response function $G(j\omega) = K$
- Magnitude $|G(j\omega)| = K$
- $|G(j\omega)| \text{ in dB} = 20 \log_{10} K$
- Phase angle $\phi = 0$

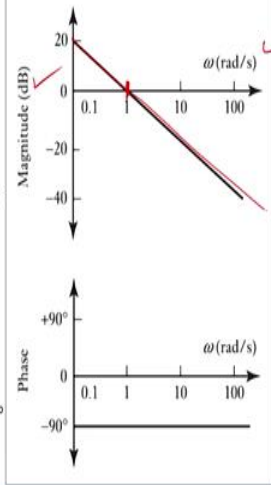


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So, let us take an example of the bode plot. Suppose I have got the bode plot of a system having transfer function $K = \text{constant}$. So, the frequency response $G(j\omega) = K$ only, and the magnitude of that will be K only. So, the magnitude in terms of decibel will be $20 \log_{10} K$ And the phase angle is going to be 0 . So, this is the bode plot for this one $G(s) = K$.

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- Bode plot for system having TF $G(s) = \frac{1}{s}$
- Frequency response function $G(j\omega) = \frac{1}{j\omega}$ so $G(j\omega) = -j/\omega$
- Magnitude $|G(j\omega)| = 1/\omega$
- $|G(j\omega)| \text{ in dB} = 20 \log_{10} 1/\omega$
- $|G(j\omega)| \text{ in dB} = -20 \log_{10} \omega$
- Phase angle $\tan \phi = \frac{-1/\omega}{0} \rightarrow \phi = -90^\circ$



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Now, if my $G(s) = 1/s$, then again, I can find out $G(j\omega) = \frac{1}{j\omega}$, and I can write it in the complex form like this. And its magnitude will be $1/\omega$, or I can write it like this.

So, if I plot it with ω , the magnitude this is what I am going to get, and you that at omega is equal to 1, this is going to be equal to 0. So, this plot is intersecting over here. Similarly, I can find out phase as,

$$\tan \phi = -\frac{1}{\omega} \rightarrow -90^\circ$$

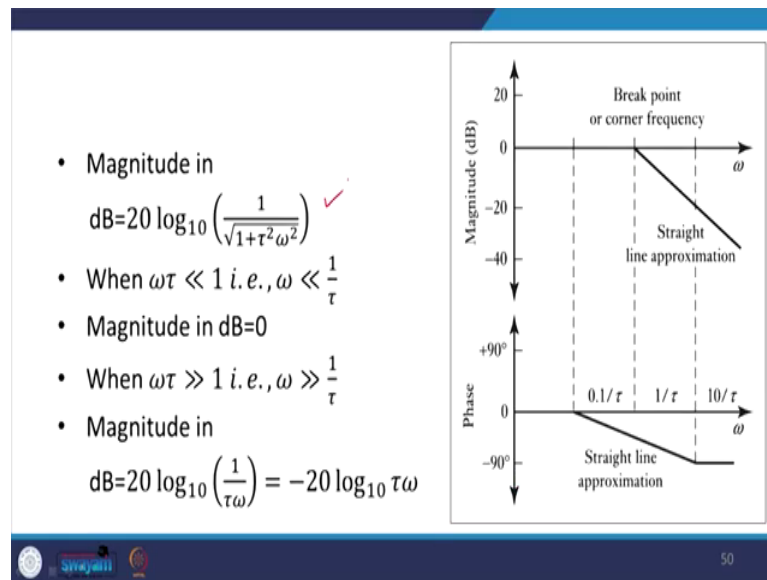
so I can plot it like this.

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- **Bode plot for first-order system**
- System having TF $G(s) = \frac{1}{\tau s + 1}$ ✓
- Frequency response function $G(j\omega) = \frac{1}{j\omega\tau + 1} = \frac{1}{1 + j\omega\tau}$ ✓
- Magnitude $|G(j\omega)| = \frac{1}{\sqrt{1 + \tau^2\omega^2}}$ ✓
- Magnitude in dB = $20 \log_{10} \left(\frac{1}{\sqrt{1 + \tau^2\omega^2}} \right)$ ✓

And the bode plot for the first order system if I want $G(s) = \frac{1}{\tau s + 1}$. So, I replace this s by j ω over here, and I write it in the form of the complex notation that is $x + j y$. So, I write it in this form over here, and then I can find out the magnitude as, $\sqrt{x^2 + y^2}$, and this can be written in the decimal form like this.

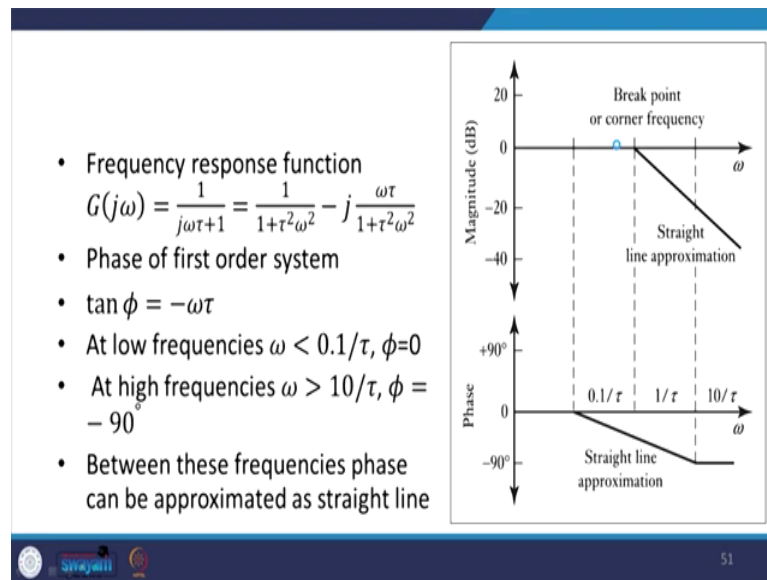
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And so the magnitude is this one. Now here you see if this $\omega\tau \ll 1$ or what we can see that $\omega \ll \frac{1}{\tau}$. Then what happens? I can neglect this term. So, this becomes $\log_{10} 1$. So, this magnitude is going to be 0. So, this is what I am going to get for less than 1 value this magnitude is going to be equal to 0.

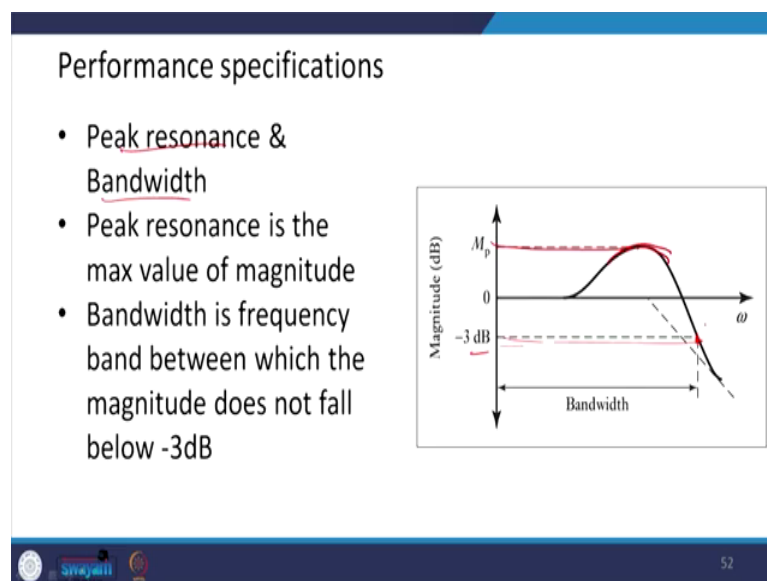
If $\omega\tau \gg 1$, $\omega \gg \frac{1}{\tau}$. So, in this case, the magnitude will be turning out to be like this, which is a straight line approximately. So, this is my plot this frequency breakpoint or the corner frequency it is called.

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And similarly, I can have this one the phase also. So, this is the frequency response function. So, for the phase for this is a $\tan^{-1} \frac{y}{x}$ if I do that. So, it will be $-\omega\tau$, and we can see that as the $\omega < 0.1/\tau$. This value is going to be 0, and at a higher frequency, this is going to be -90° . So this is what we are going to get an in-between this is for high value, this is for low value, and in between, we can approximate with a straight line.

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So, the performance specification for the system can be given by the two parameters, is the peak resonance and the bandwidth. The peak resonance is the maximum value of the

magnitude. So, the maximum value of the magnitude over here is the peak resonance, and bandwidth is a frequency band between which the magnitude does not fall below -3 dB.

So if this is the -3 dB, so, this is going to be the bandwidth. And this concept is used in the case of the census the bandwidth of the census. So, these are the references for your further reading.

Thank you.