Mechatronics
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Lecture - 30
Transfer Function and Frequency Response

I welcome you all to this NPTEL online certification course on Mechatronics. Today we are going to talk about the Transfer Function and Frequency Response. In the last lecture, I discussed with you the dynamic response of the system of the first-order system, second-order system, and we have seen the various performance measurement parameters also. Here I am going to discuss with you another way of seeing the dynamic response of the system where we transform our equations from the time domain to the Laplace domain that is *s* domain, and then we look at the response. We look at the response, of course, in the case of the frequency response. So, I am going to talk about all those things in this lecture. Let us look at the system transfer function. First of all, suppose I have got an amplifier. For that amplifier, I can define a gain as simply output by input. So, what does this means that suppose you have an amplifier with a gain of 5.

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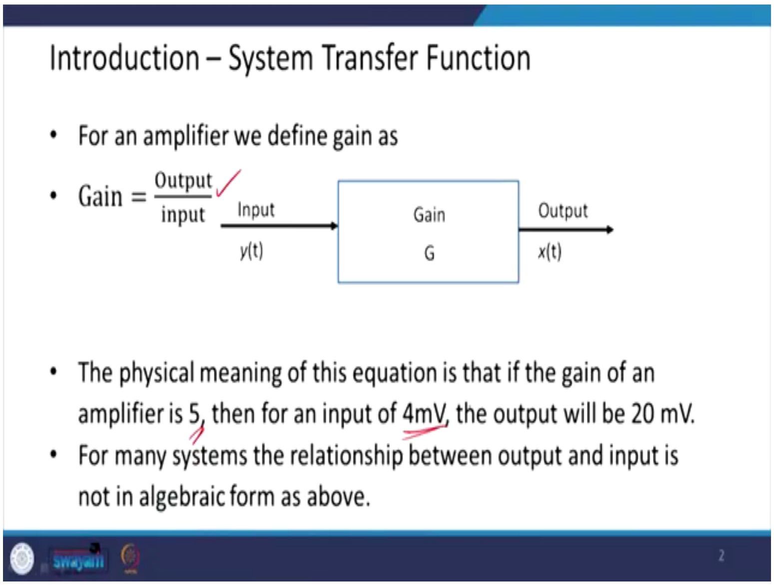

Then it means that for a given input of 4 millivolts, the output which you are going to get will be the 20 millivolts. Now for many systems, the relationship between output and input will not be as simple as this one. It will be rather in the algebraic form.

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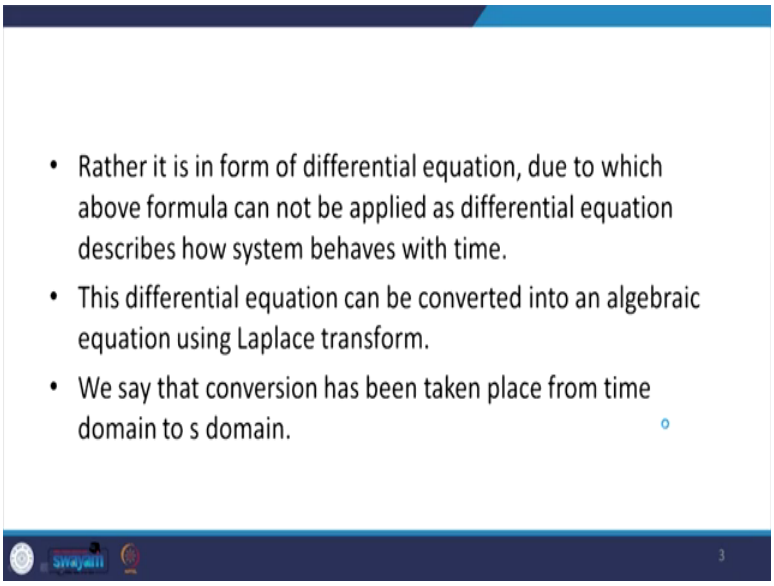

These algebraic forms or differential equations describe the behavior of the system with time. And if these are differential equations can be converted into the algebraic form using the Laplace transform. And so, as I said, we are moving that way from the time domain to the s domain.

The transfer function is defined as the Laplace transform of the output divided by the Laplace transform of the input. So, this is again a way of defining the transfer function. And a signal in the time domain is depicted by f (t), where a signal in the Laplace domain is dependently depicted by F(s).

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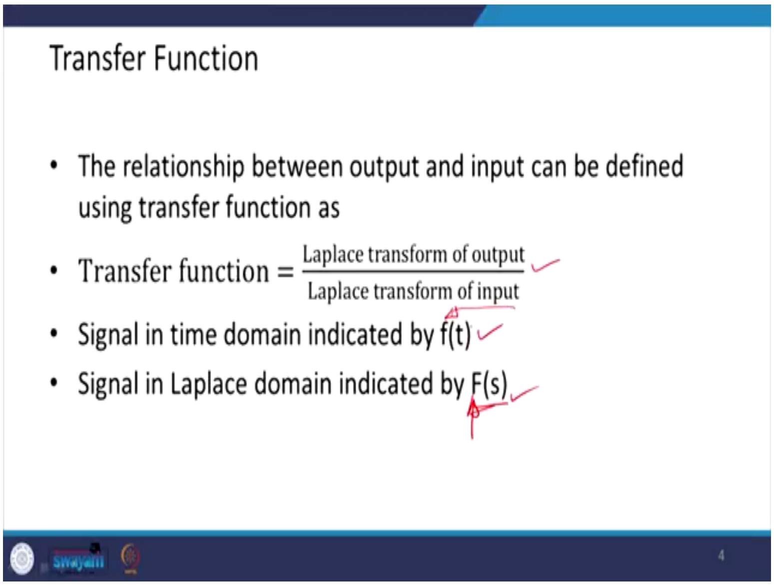

In the Laplace domain, we write the function always in the capital letter that is the universal convention, whereas, in the time domain, we would write it in terms of the small letter.

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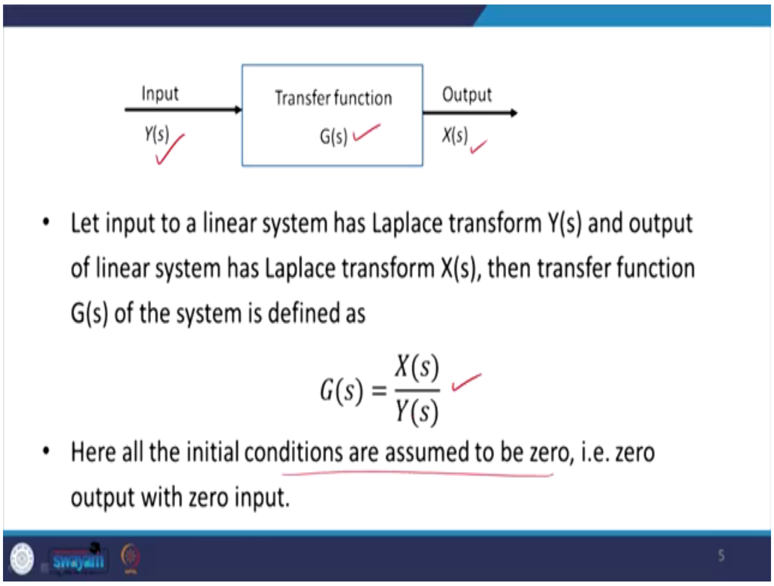

So, suppose I have got a transfer function G(s) and when this is subjected to input Y(s) and output is X(s). So, this is what it means that,

$$G(s) = X(s)/ Y(s)$$

Here we assumed the initial conditions to be zero, that is, zero output with zero input.

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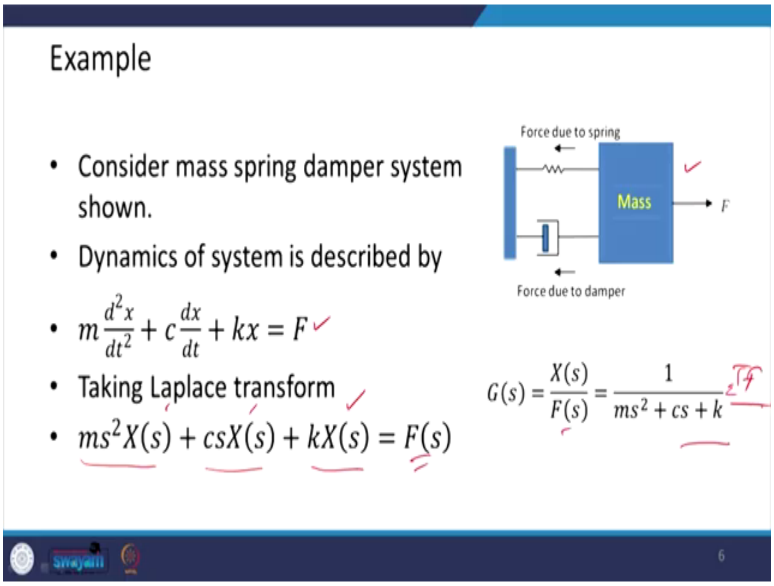

Suppose I take the example of a transfer function. So, I can take a second-order system, which is a spring-mass damper system, and for this system we the dynamic equation is given by this one after drawing the free body diagram. And if I take the Laplace to transform for this is what I am going to get, and so, from here, I can find out the value ratio of the output by input. So, output here is X(s), and input is F(s). So, this is the transfer function.

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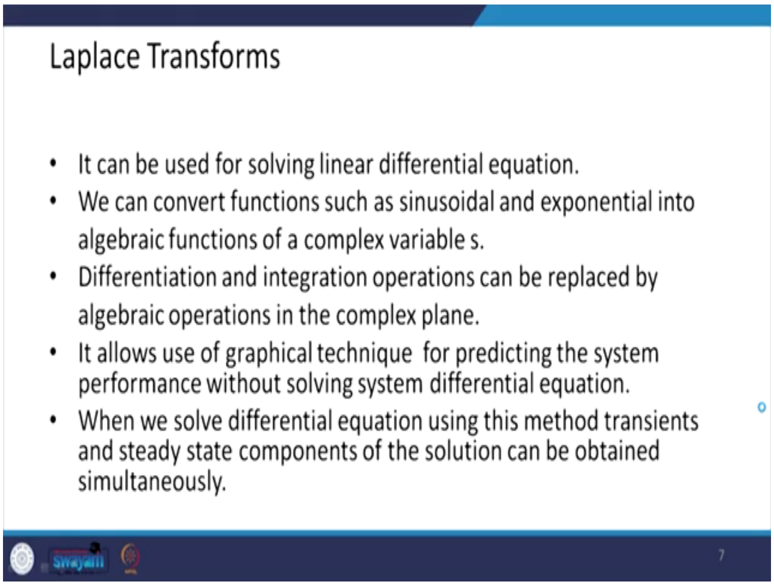

So, what is Laplace transform? And these Laplace transforms are used to convert the differential equations into an algebraic equation. We can convert functions such as sinusoidal and exponential into the algebraic function of a complex variable *s*.

And the differentiation and integration operations can be replaced by algebraic operations in the complex plane. It allows the use of graphical techniques for predicting the system performance without solving the differential equation. So, that is one of the very good advantages of the Laplace transform, and when we solve a differential equation using this method, transient and a steady-state component of the solution can be obtained simultaneously, which otherwise has to be done separately, as we have seen in the last lecture.

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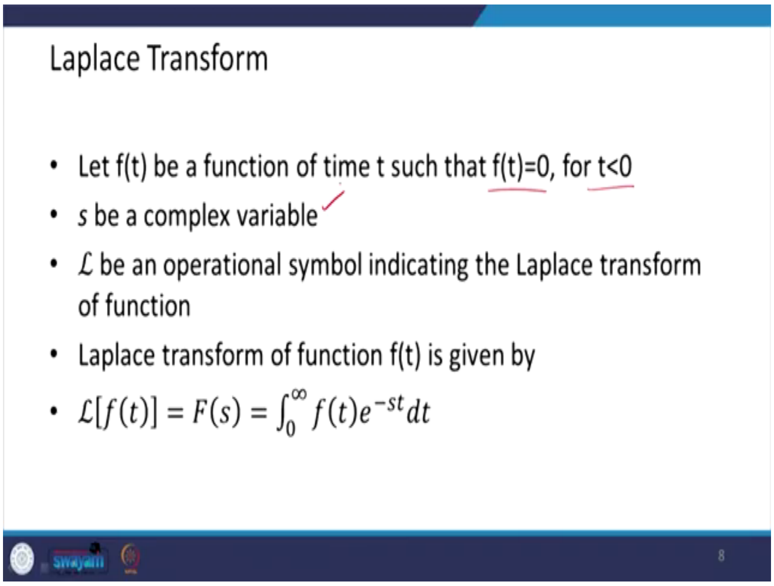

So, let f(t) be a function of time such that f(t) = 0, for t < 0 and the s is a complex variable, then and L be the operational symbol indicating the Laplace transform of a function then the Laplace transform of function f(t) is this is how it is given as represented by,

$$L[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt$$

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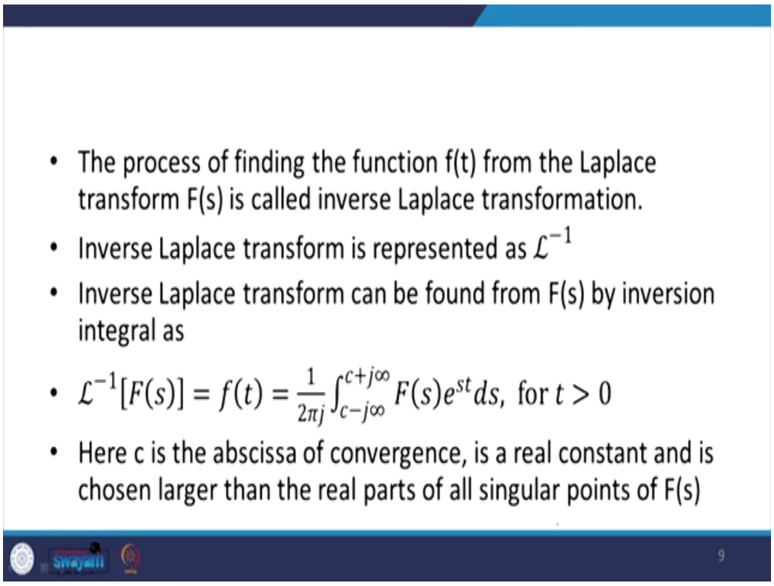

The process of finding the function f(t) from the Laplace transform F(s) is called the inverse Laplace transform. That is, if you want to come back from the Laplace domain to the time domain, what we have to do is that we have to perform the inverse Laplace transformation, and this inverse Laplace transformation is represented by L^{-1}And inverse Laplace can be found by inversion integral like this. So, this way, we can do the inverse.

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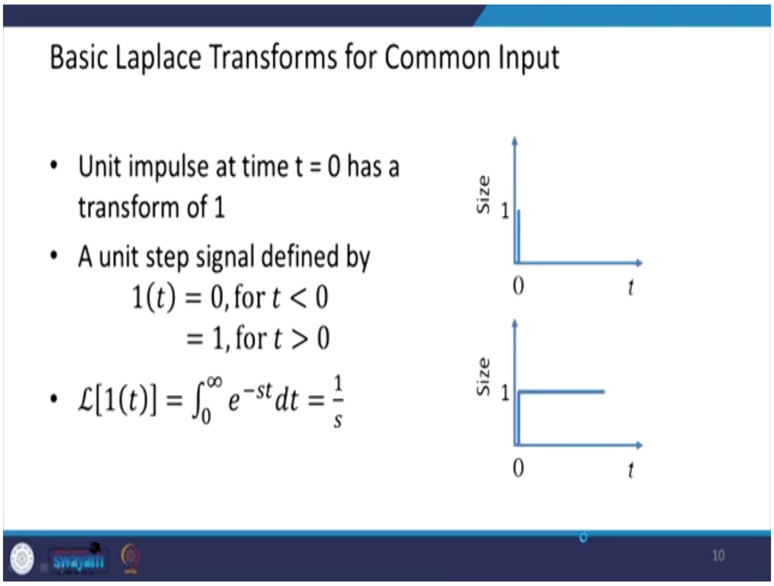

So, the basic Laplace transform for the common input, which we have seen in my previous lecture. Such as if we have a uniform impulse at t = 0, it has a Laplace transform equal to 1, and if it is a unit step signal defined by less than t for a t< 0, it is 0 and time > 0, it is 1 this way then the Laplace transform for that is 1/ s.

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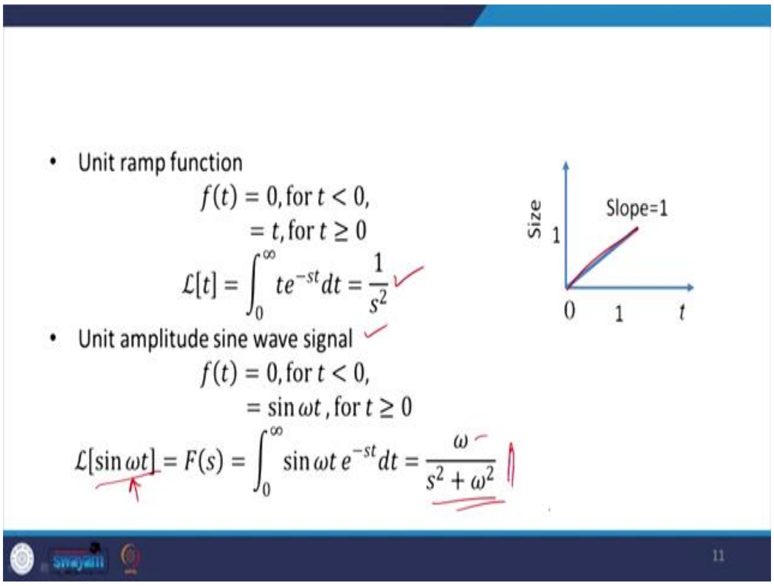

And for unit ramp one, the Laplace transform is $\frac{1}{s^2}$. And for unit amplitude sine wave signal, Laplace transform is

$$L[\sin \omega t] = \frac{\omega}{(s^2 + \omega^2)}$$

where ω is the frequency of the sinusoidal signal.

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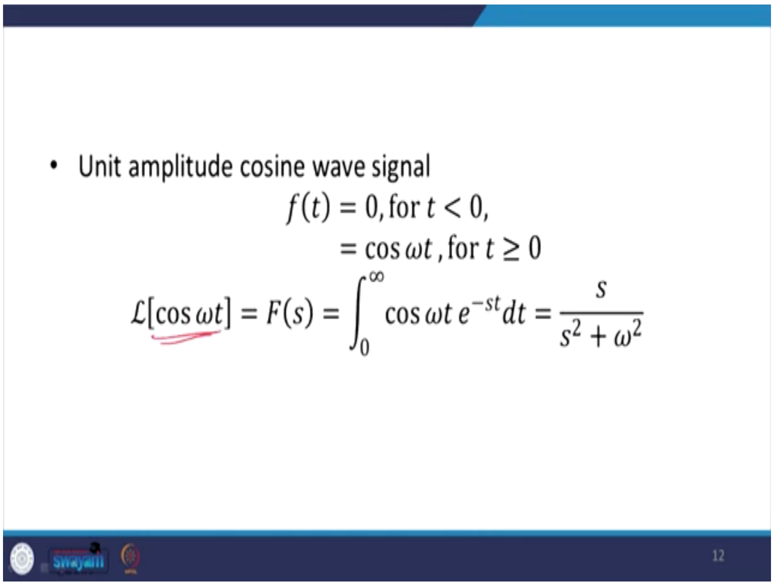

Similarly, the Laplace transform for,

$$L[\cos \omega t] = \frac{s}{(s^2 + \omega^2)}$$

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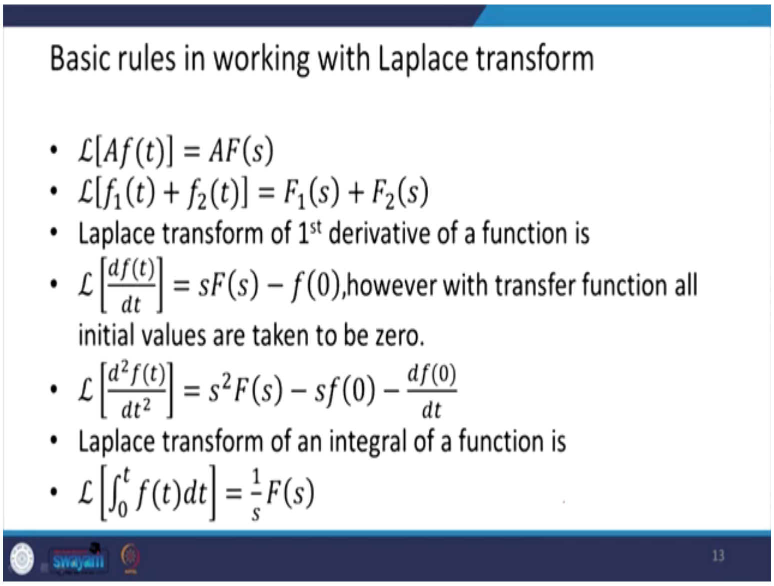

So, there are certain basic rules in working with Laplace transform that are given here.

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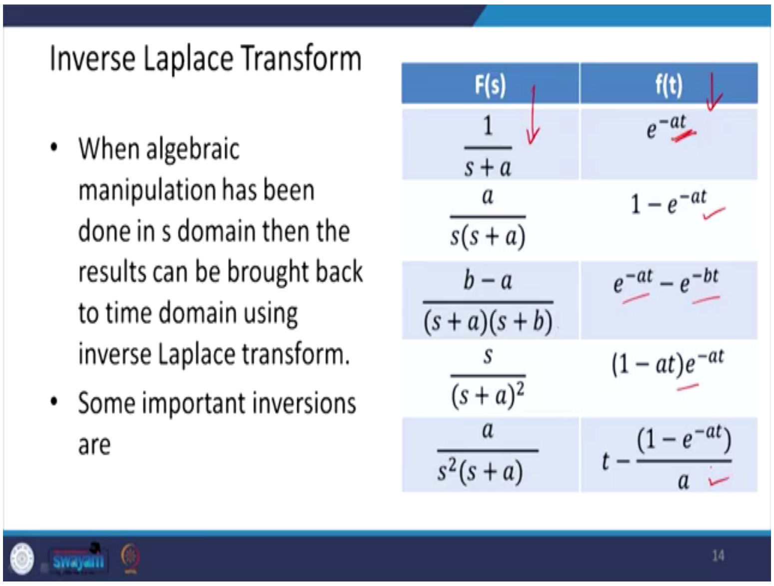

And if we look at the inverse of the Laplace transform. So, if these are the Laplace transform, here is the inverse of that.

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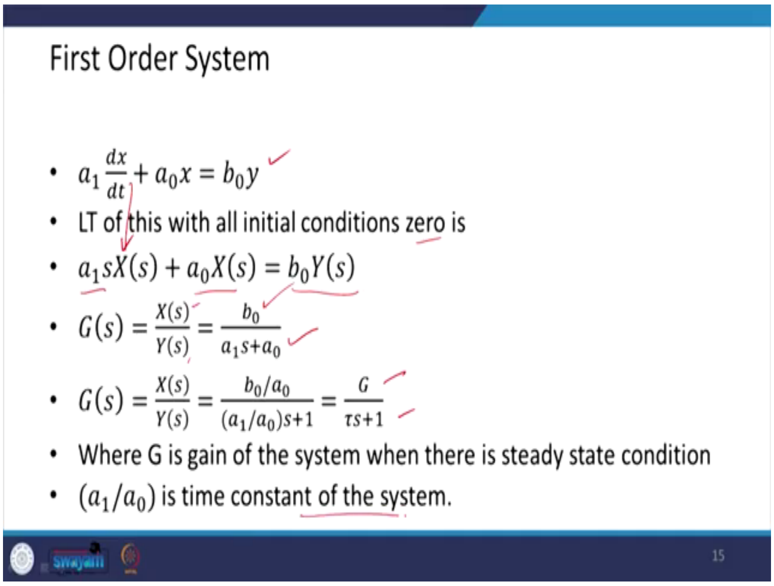

So, now, let us look at the first-order system. So, this is the first-order system, as we have seen in the previous lecture. So, Laplace transforms for this with initial condition zero. I can simply write like a_1. This one I can write as sX(s). This is a_oX(s) is equal to b_oY(s). So, I can write the relationship between output,

$$\frac{X(s)}{Y(s)} = \frac{b_o}{a_1 s + a_o}$$

$$\frac{X(s)}{Y(s)} = \frac{G}{\tau s + 1}$$

where G is the gain of the system when there is a steady-state condition, and $\frac{a_1}{a_o}$ is the time constant of the system.

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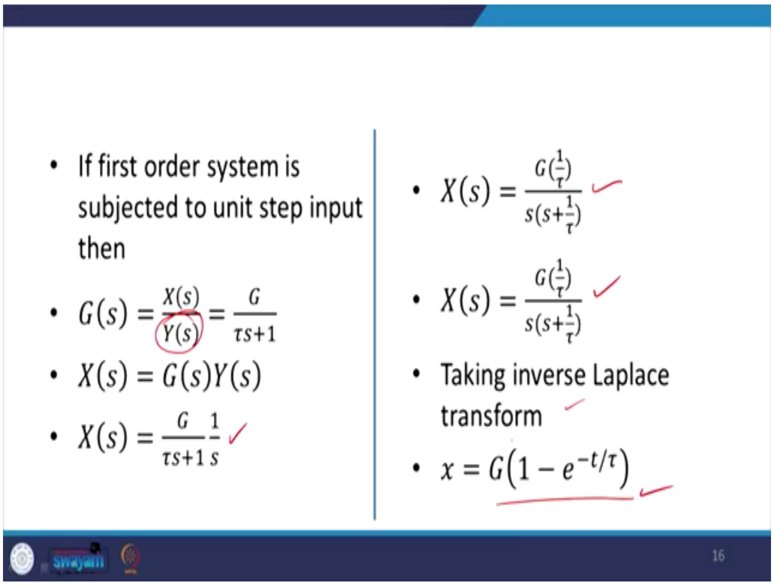

Now, if the first-order system is subjected to a unit step input, then we can take this Y(s) =1/ s. So, I can take write this X(s) = G(s)Y(s). So, this is 1/ s I am doing it I can write it in this form, and if I take the inverse Laplace of this, then this is what we get. So, you see that by taking the inverse Laplace transform, we are coming back to the time domain.

So, here is what we have done we have this expression in which the first-order system dynamic equation is in the form of a first-order differential equation. So, with the help of Laplace transform, we converted it into the algebraic expression, and then after algebraic conversion into algebraic expression, we went back to the time domain by using the inverse Laplace transform.

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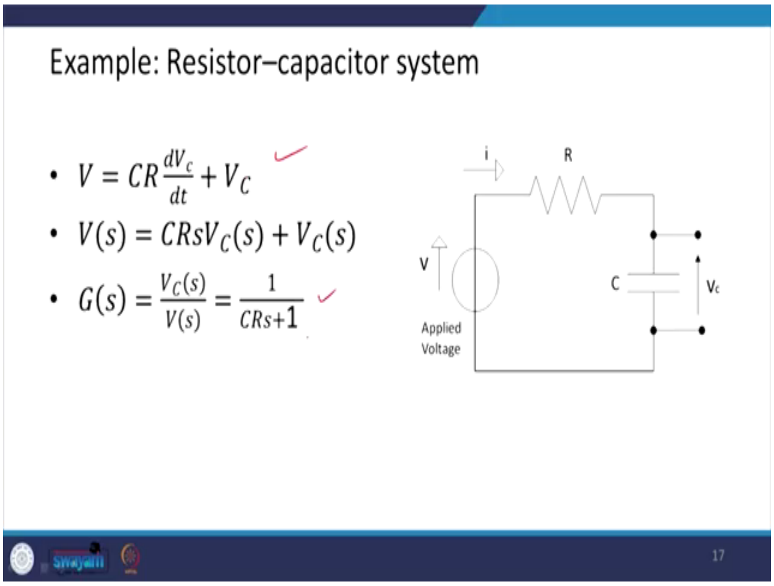

So, again the resistor-capacitor system is an example of the first ordered system. So, we can write the expression for this one, and this could be simplified in this way.

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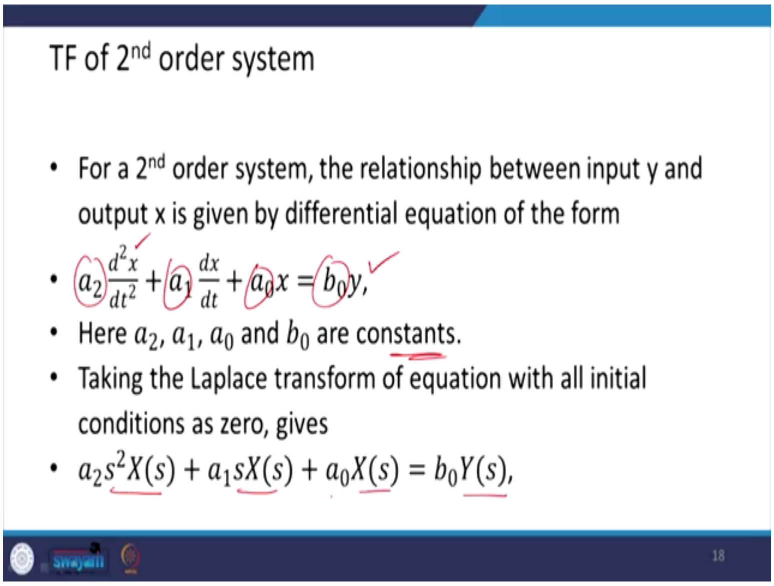

Now, let us look at the second-order system. So, in the case of a second-order system, the relationship between the input and output is given by this one, as we have seen in the previous lecture. Where these a_2, a_1, a_o *and* b_o are constant. Now, if I take the Laplace of this is a second derivative. So, this is $s^2X(s)$ *is* first derivative, so *sX(s)*, and here it will be only *X(s)* and *Y(s)*.

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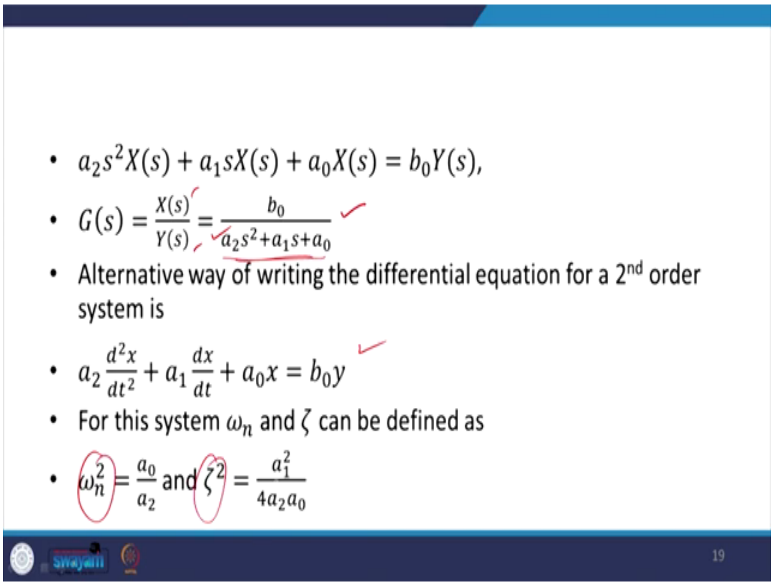

So, now I can find out the relationship between X(s) and Y(s), and it is this one. So, here you can see that we have a quadratic term in s over here. There is an alternate way of writing this differential equation of the second-order system, and that we can do by defining these terms ω_n^2 and ζ^2 like this.

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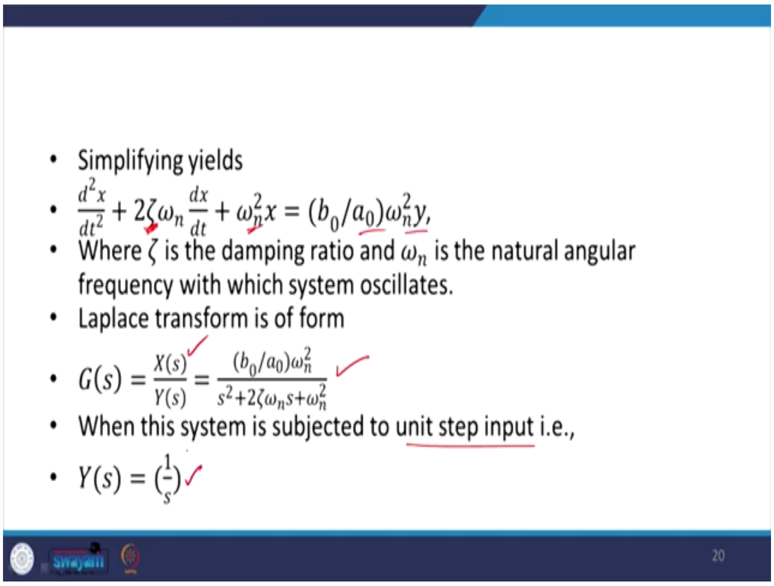

And if I simplify write this expression in terms of ζ and ω_n I can write it like this, and if I take the Laplace, then I can write X(s)/ Y(s) is equal to this one. So, this way, I can write, and if I subject it to a unit step input, then the transfer function for input is 1/ s.

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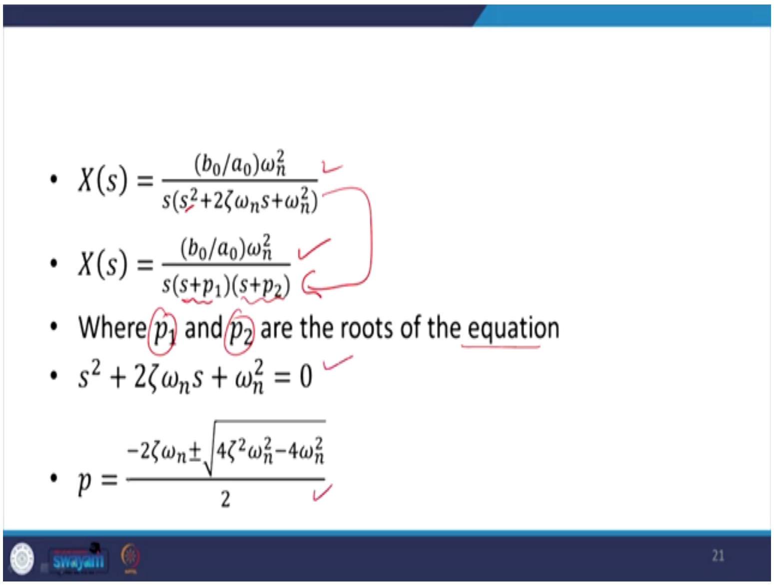

So, I can substitute in this one, and I can see the response for that. So, I have 1/s term over here. So, this is my output in the s domain, and this is my output in the s domain. I can write this always because it is a quadratic equation. So, they are going to be the two poles. So, I can write it in the form of $s + p_1$ and $s + p_2$ and where these p_1 and p_2 are the poles are the roots of the equation. So, I can find out these p_1 and p_2 values by equating this is equal to 0 and then writing the solution of this second-order equation yes.

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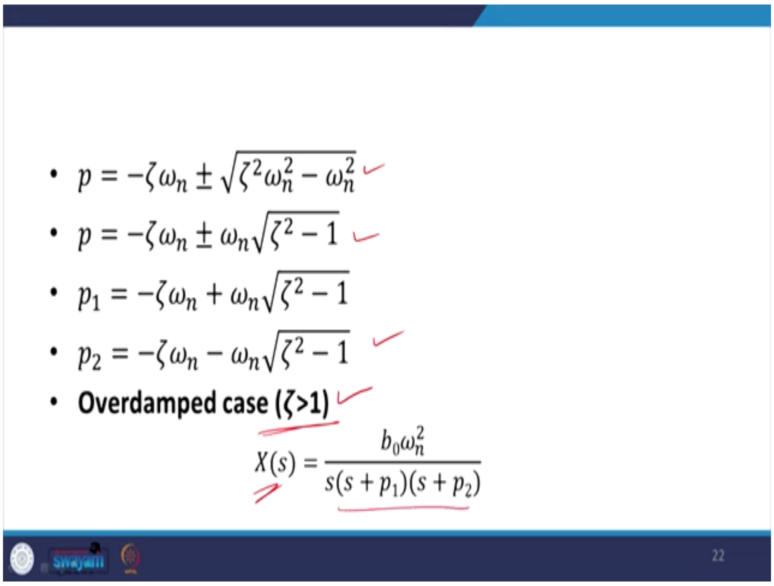

So, this is what is that one, and I can simplify this one this is there, and then I can take write the expression response for $\zeta > 1$ in terms of the p_1 and p_2.

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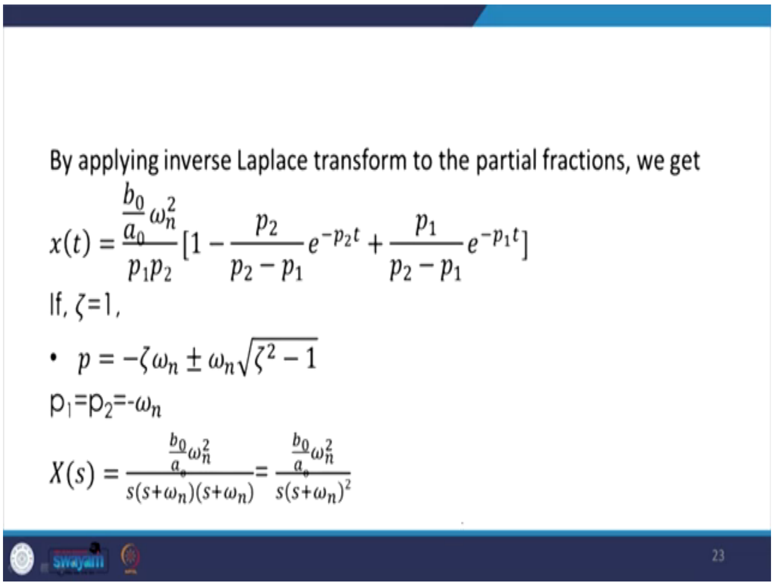

And then, by applying the inverse Laplace transform to the partial fraction, we can get back to the time domain. So, this is my expression in the time domain similarly. If I have $\zeta = 1$, then this is what my expression is going to be. So, these are my roots. Both are going to be equal. So, this is my response in the s domain.

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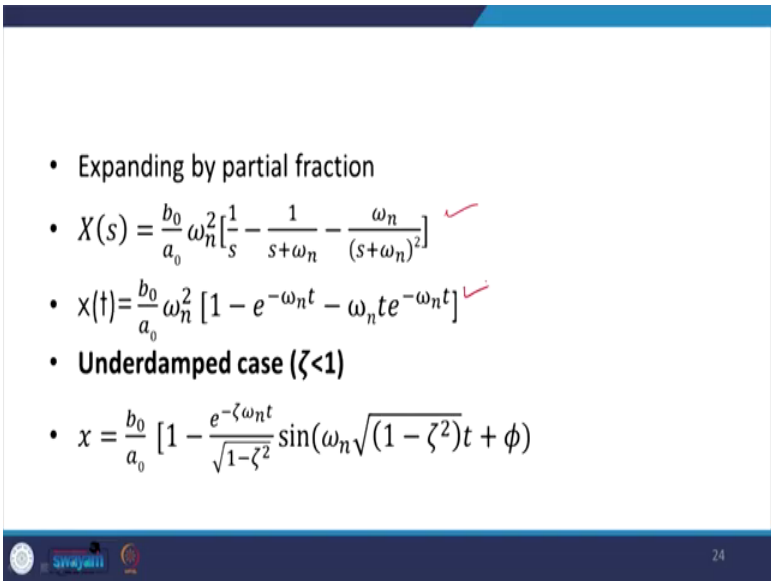

And again, by partial fraction, I can convert it into the time domain. So, this is how we get the response of the system in the time domain using the Laplace transform. Similarly, if it's an underdamped case, then this is my response which I am going to get.

I can explain one example indicating the state of damping of a system having the transfer function this one and subjected to the unit step input function. So, the unit step input function means your Y(s) =1 /s, and G(s) = X(s)/Y(s). So, your X(s) is 1/s, and you can see that this is s+4, and s+ 4.

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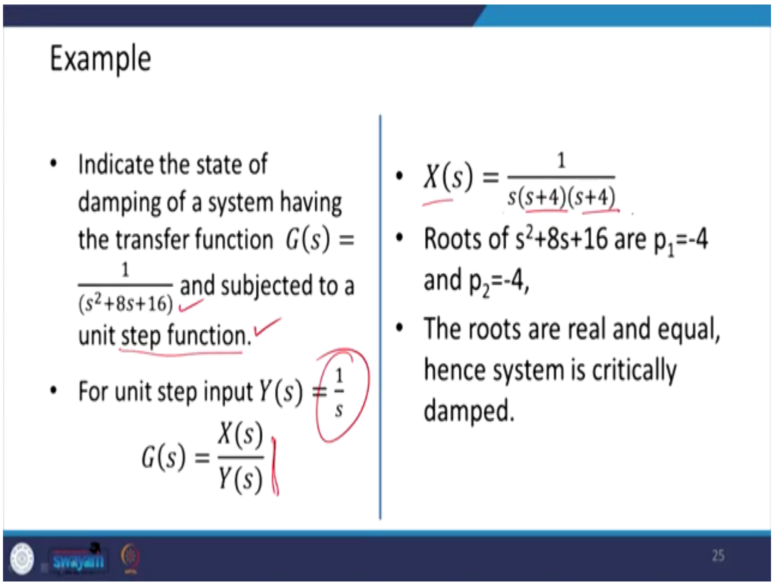

So, in this case, your roots are going to be equal to this one. So, the roots are real and equal, and hence the system is critically damped. So, this way, we can tell the behavior of the system.

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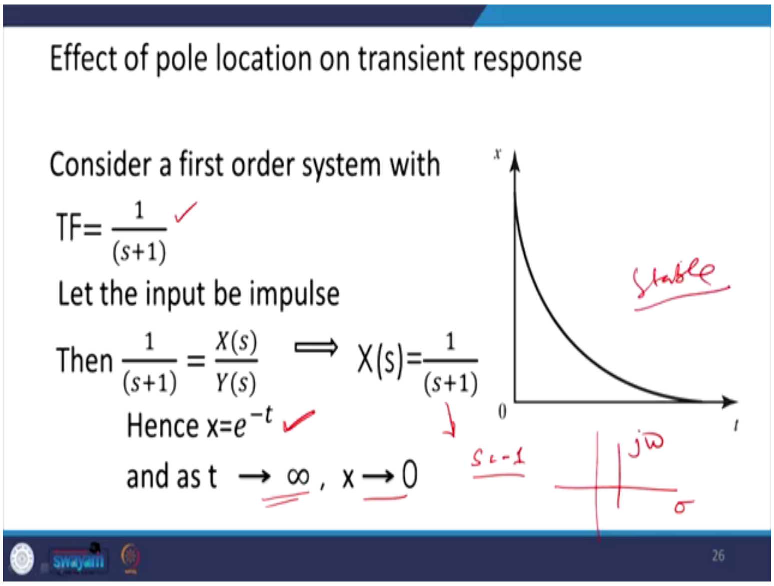

Then let us look at the effect of pole location on the transient response. So, if we consider a first-order system is something like this and if the input is impulse. Then I can write this as 1/(s+1) because Y(s) is going to be one only. So, if I take the inverse Laplace of this, then I will get the time responses to this one, and what does this mean that? This means that as$t \rightarrow \infty, x \rightarrow 0$.

So you see that this is s = -1. So, your root is equal to root is a in the left half of the s plane or the pole is in the left half of the s plane.

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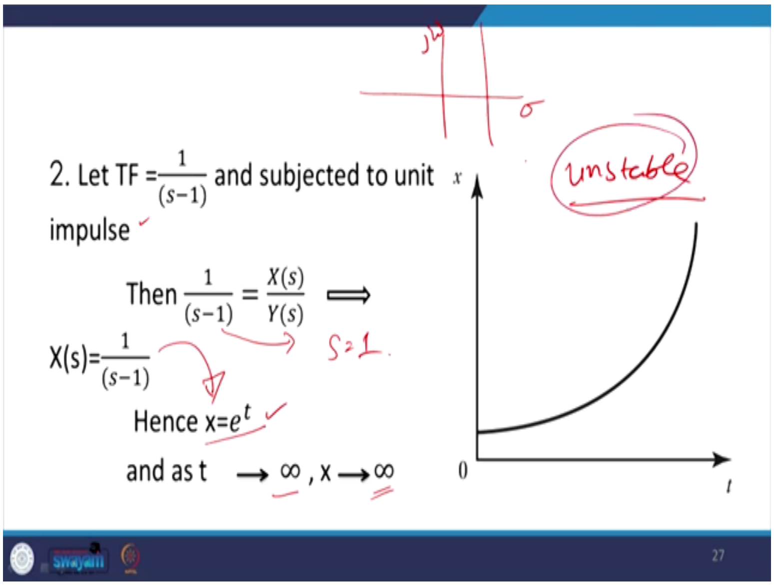

And if your transfer function is 1/ (s-1) and subjected to unit impulse, then again this one, and if I write the take the inverse Laplace of this and go in the time domain, this is$x = e^t$. And this means that as $t \to \infty$., your response will be going $x \to \infty$. So, this is an example of an unstable system. So we can conclude that if your root is here, in this case, you see where is the pole s is equal to 1. It means that if your pole is on the right half of the s plane, your system is going to be unstable, and if your pole is on the left half of the s plane, your system is going to be stable. So, this is what we can conclude from that.

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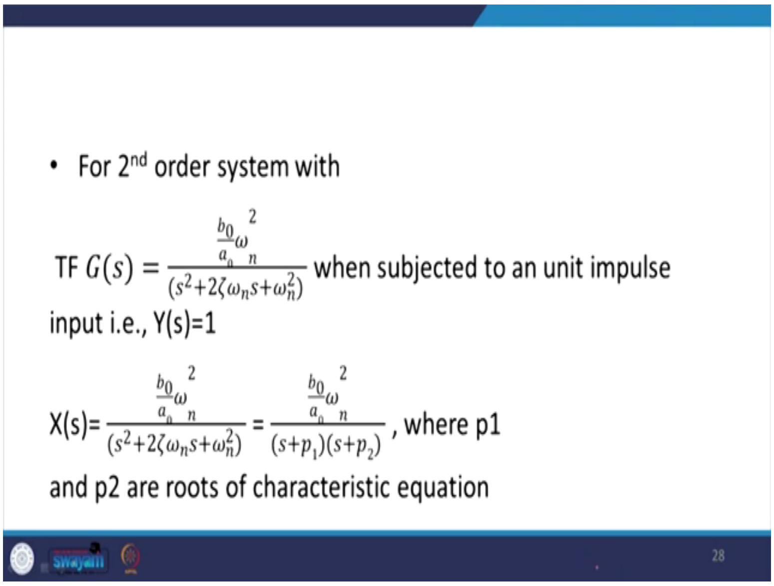

For a second-order system. Similarly, we know the response is this one. When subjected to a unit impulse, I can see that Y(s)=1. I can write X s as this one, or I can write it in terms of the two roots, where p_1 and p_2 are the roots of this characteristic equation.

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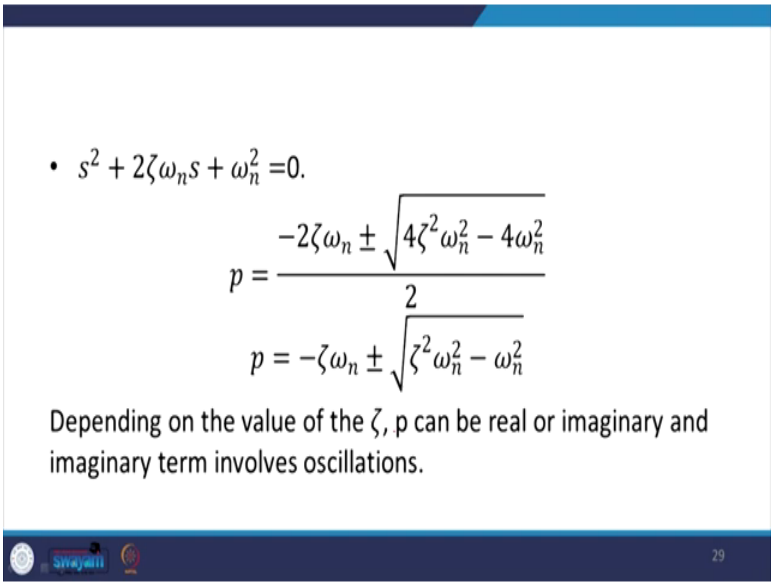

So, again here I can write these roots like this one as we have seen, and depending on the value of ζ, p can be real or imaginary terms that involves oscillation. So, that way, we can predict the behavior.

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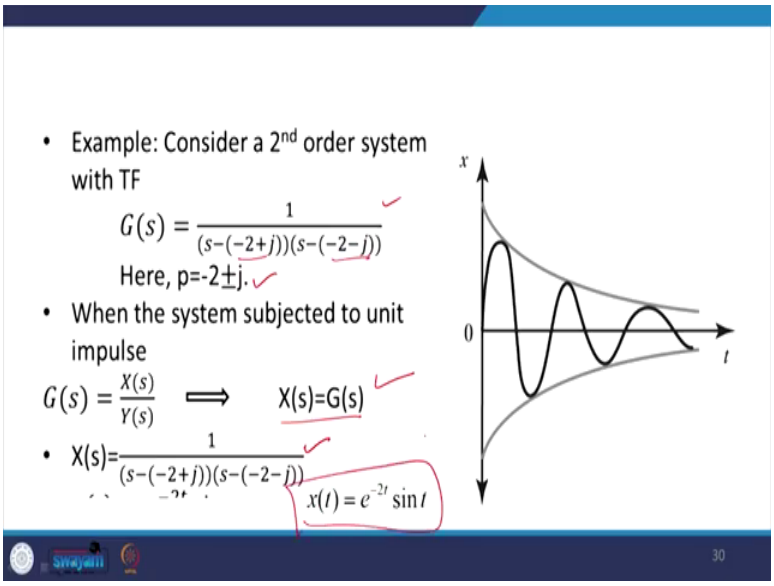

So, if I consider a second-order system something like this, so, its roots are you can see that -2+j and -2-j over here. And when this is subjected to unit impulse, your X(s) = G(s). And here, if I take the inverse Laplace of this, this is what I am going to get.

And so, in this case, we can see that there is an exponentially decaying envelope on this one on the *sin* function over here. And this way, oscillations are going to gradually decrease, and your system is going to be stable.

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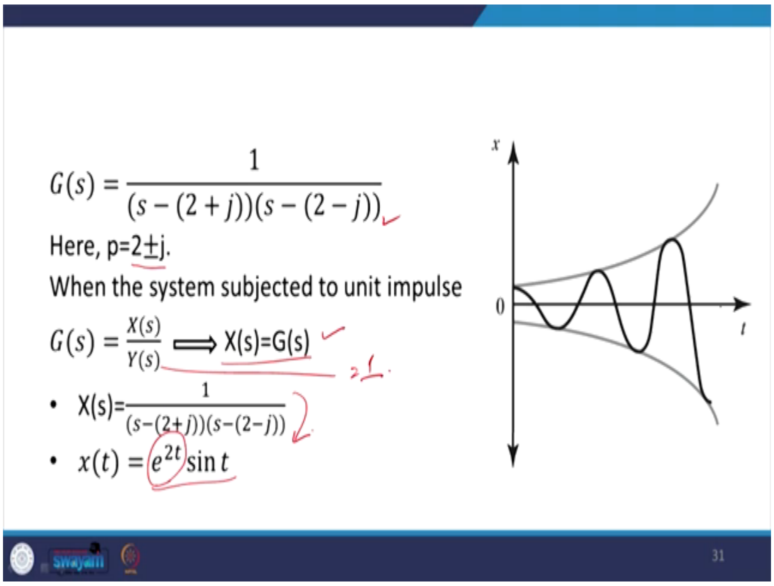

And if you have a root like this. So, this one, then your poles are $2 + j$. So, when this system is subjected to unit impulse again, you are going to get X(s) = G(s) because unit impulse Y(s) = 1 and if I take the inverse Laplace of this is I am going to get the response. So, here you can see that we have an exponentially increasing term over here, and this is enveloping the sinusoidal functions. So, that makes the system unstable.

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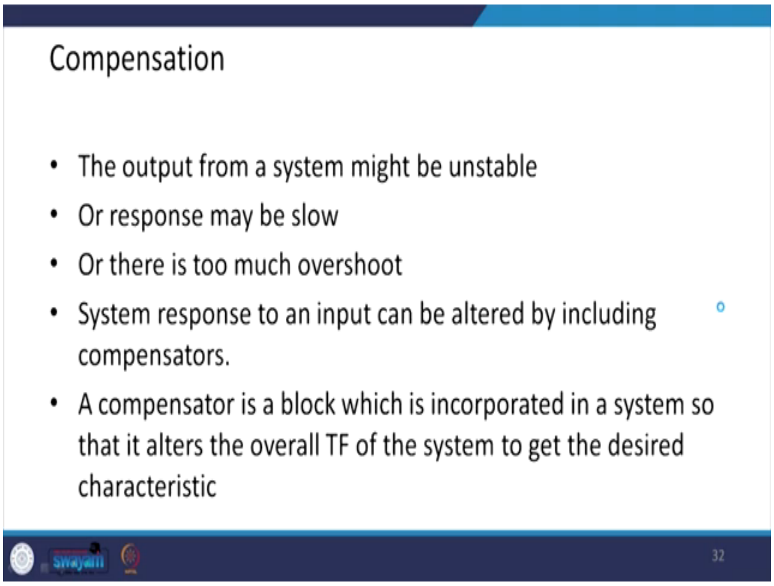

Now we have seen the response of the system for input. Now, as I said, the output from a system might be unstable, the response may be very slow, or there may be too much of an overshoot. So, such system behavior can be altered by including the compensator. So, a compensator is a block that is incorporated into a system so that it alters the overall transfer function of the system to get the desired characteristic.

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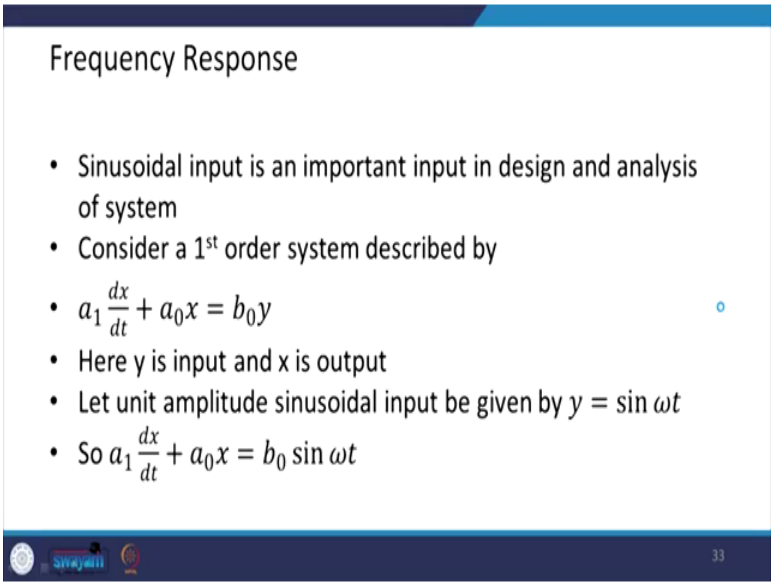

So, after seeing that let us try to learn the frequency response. So, till now, I had talked about the step input to a system, but our input to the system could be sinusoidal as well. And the sinusoidal inputs are an important input in the design and analysis of systems. So, if the first order system which we have considered over here. If I have a sinusoidal input like $y = sin\omega t$, this is how my system expression is going to be.

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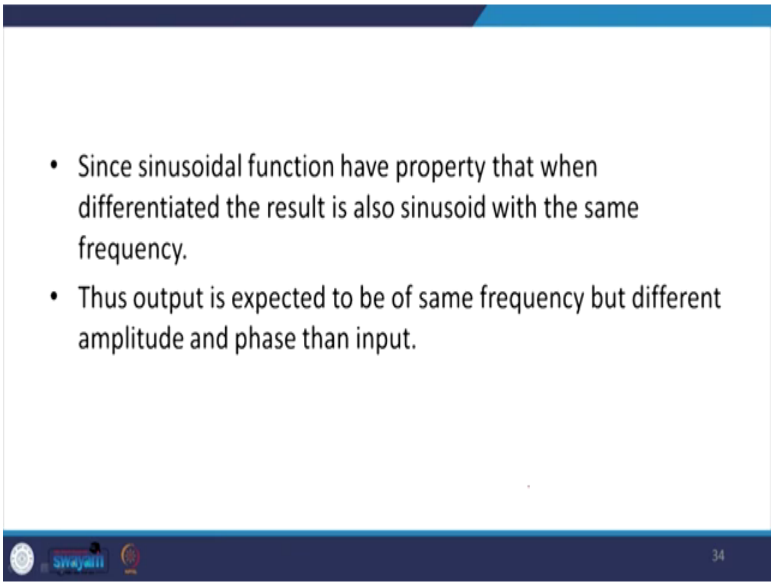

Now, since the sinusoidal function have the property that when differentiated, the result is also sinusoidal with the same frequency. Thus the output is expected to be of the same frequency but may be of different amplitude and phase than that of the input. So, there, the frequency response is generalized with the help of a phasor, and the sinusoidal signals are better represented by a phasor.

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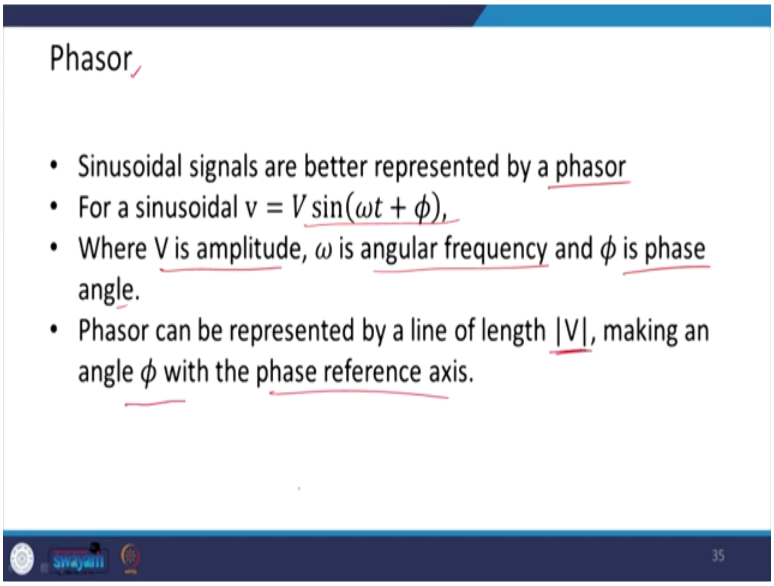

So, for a sinusoidal $v = V\sin(\omega t + \phi)$ V is the amplitude and ω is the angular frequency, and ϕ is the phase angle. So, the phasor can be represented by a line of length magnitude of V, making an angle phi with the phase reference axis.

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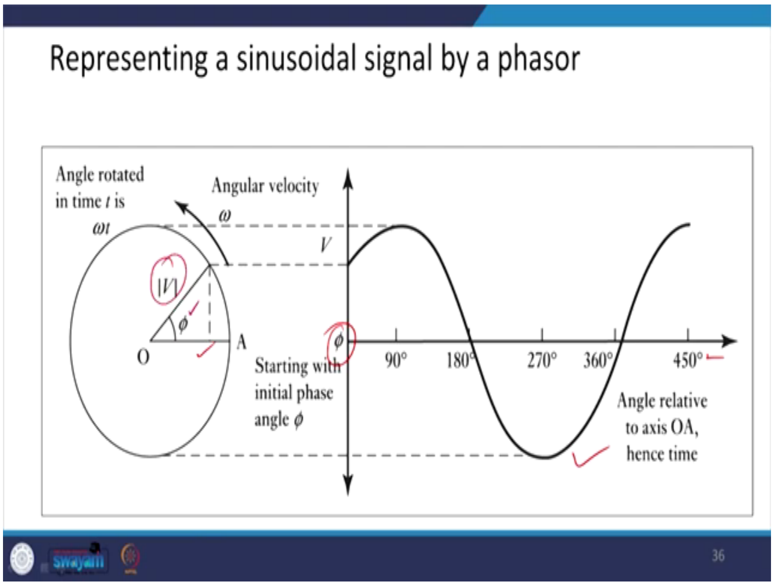
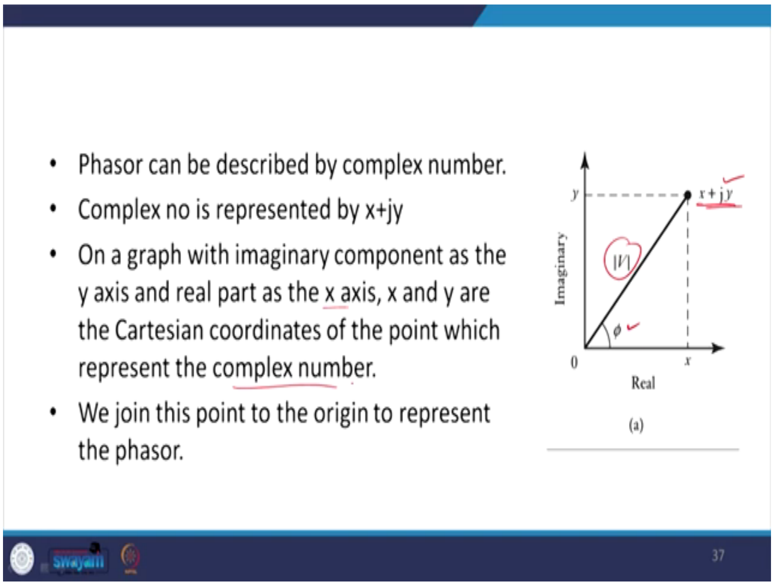


So, here you can see that you have a sinusoidal signal over here. And starting with initial phase angle ϕ and you have the angle related to x is OA hence time this one. So, this is how it can be represented by a phasor of magnitude V and the angle ϕ.

(Refer Slide Time: 23:10) ϕ



So, as I was saying, so, phasor can be described by a complex number, and the complex number is represented by x+jy. So, here you have a phasor of magnitude V and at an angle ϕ. This I can describe by complex number x+jy. So, here on the graph with imaginary component as the y axis and the real part at the x-axis x and y are the Cartesian coordinate of the point which represents the complex number. We joined this point to the origin to represent the phasor.

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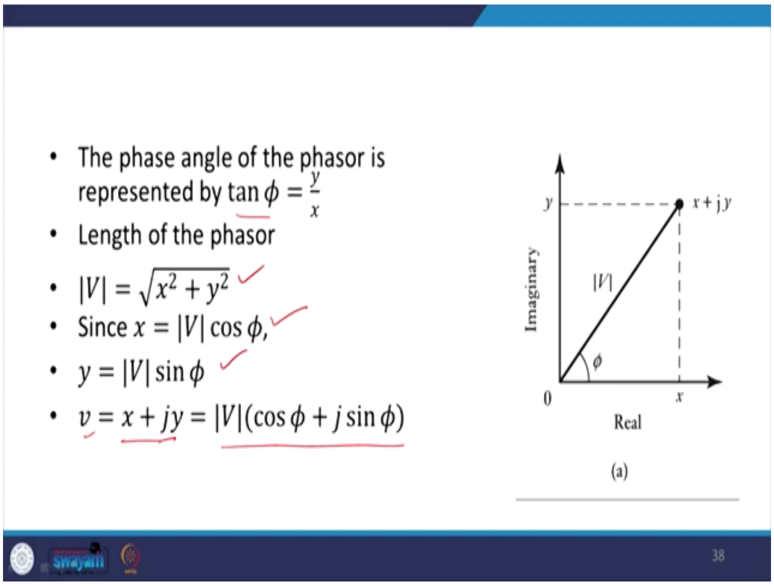

Now, the phase angle of the phasor is represented by the tanϕ = y/x, and the length of the phasor is,

$$|V| = \sqrt{x^2 + y^2}$$

And since x =$|V|$ cosϕ and y = $|V|$ sin ϕ, I can write this v = x+jy, and this is how it can be written. So, this is how I can represent a phasor as a complex number. So, here you can see the phasor. So, if a complex representation of phasor at here.

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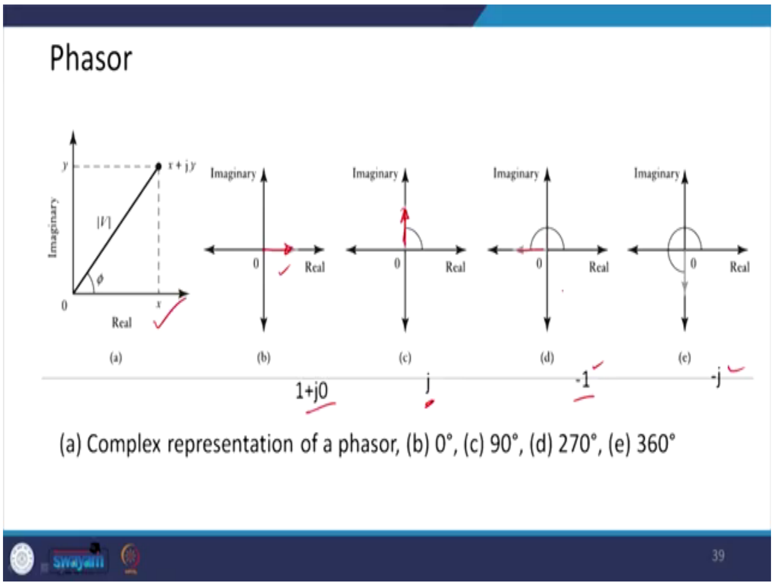

So, if it is 0 degrees, then this is 1+ j0 in the complex one if it is over here. So, this is only j real component is not there. And if it is over here, then we have a real component. An imaginary is 0. So, this is -1, and here it has got only imaginary, so - j. So, here you can see that if the angle is 0 or so, this is this one.

So, when we are turning it by 90^0, we are getting this one. So, we are moving from the real to the imaginary if we turn by it 90^0 or imaginary to real so that way. So, the phasor equation, as we have seen x = sinωt. So, the first derivative is ωcos ωt, and this I can write as ωsin (ωt+90^0)

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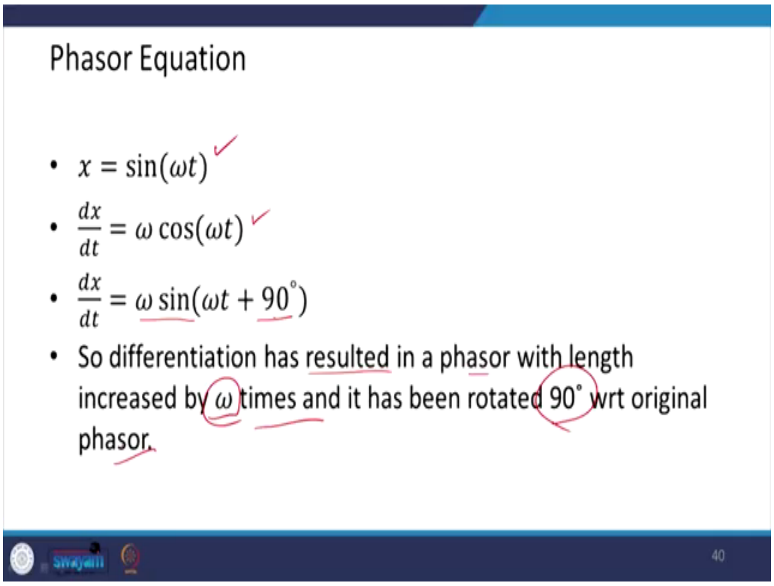

So, what has happened is that the differentiation has resulted in a phasor with length increased by ω times, and it has been rotated 90^0 with respect to the original phasor.

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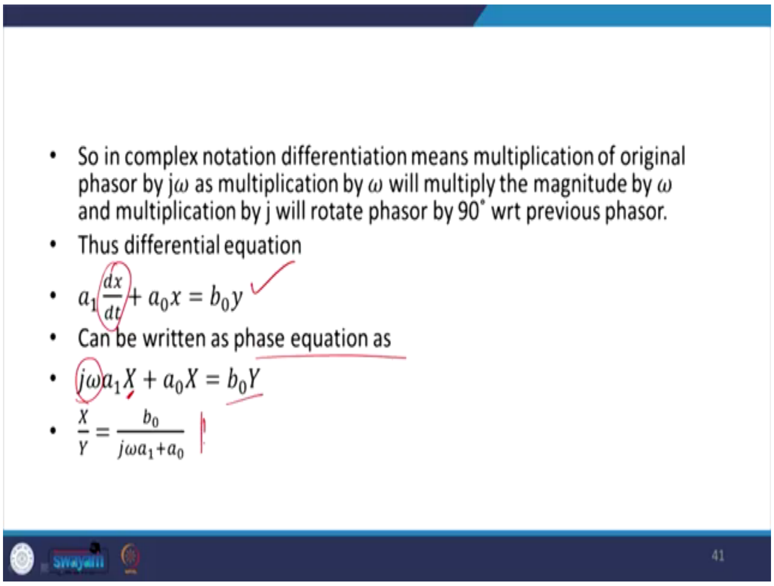

So, in complex notation, differentiation means the multiplication of the original phasor by ω as multiplication by ω will multiply the magnitude by ω. And multiplication by j will produce the rotate phasor by 90^0 with respect to the previous phasor. Thus this differential equation I can write as a phasor equation. So, this differentiation dx/dt I am replacing by j ω . So,

$$j\,\omega a_1 X + a_o X = b_o Y$$

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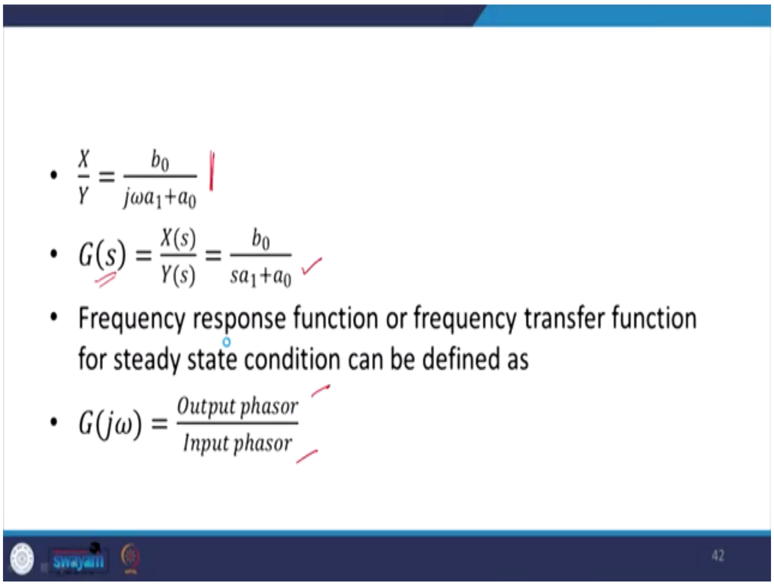

So, the transfer function was,

$$\frac{X}{Y} = \frac{b_o}{j\omega a_1 + a_o}$$

So, what has happened is the frequency response function or frequency transfer function for a steady-state condition can be defined as,

$$G(j\omega) = \frac{Output\ phasor}{Input\ phasor}$$

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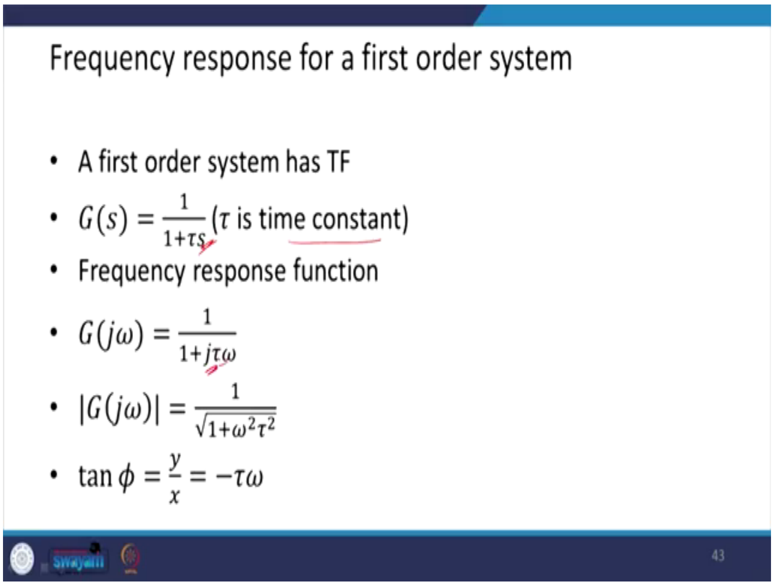

So, the frequency response of the first order system if I look at. So, this,

$$G(s) = \frac{1}{1 + \tau s}$$

So, in the frequency response here, you can see that what is done is this s is being replaced by this $j\omega$. So, here what we do is that this is in the Laplace domain. If I want to go into the frequency response, I just replace this s/ $j\omega$ over here, and then I find out the magnitude I find out the phase of it.

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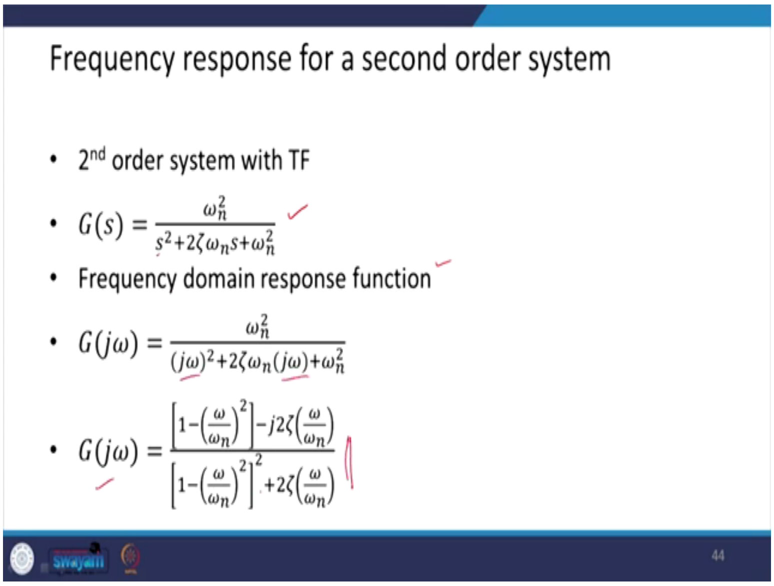

And for the second-order function system, this is the transfer function. So, again frequency domain response function I can write here by replacing s with j omega, I can write it, and then I can write it off this form in the complex form.

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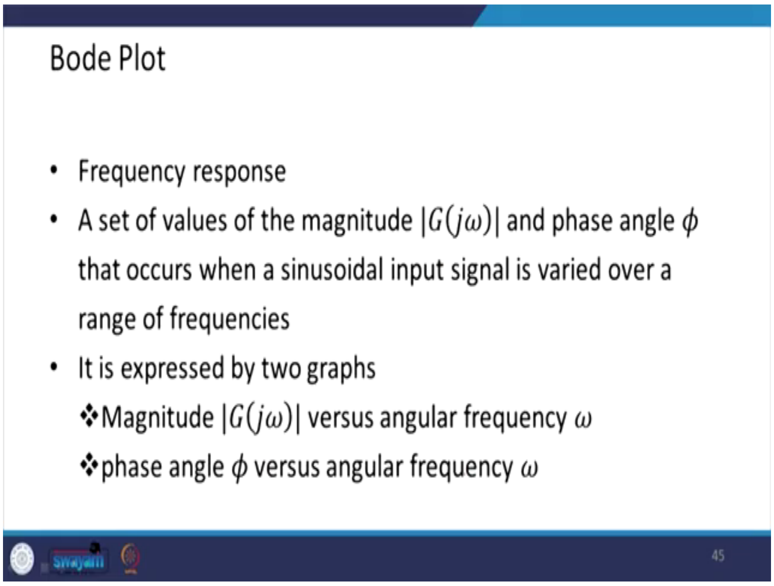

Bode plot is used for the frequency response. So, a set of values of the magnitude $|G(j\omega)|$ phase angle ϕ occur when a sinusoidal input signal is varied over a range of frequencies. And it is expressed by two graphs that is one is the magnitude versus angular frequency, and another is the phase angle versus the angular frequency. So, magnitude and angular frequency are plotted using the logarithmic scale, and such a plot is called the bode plot.

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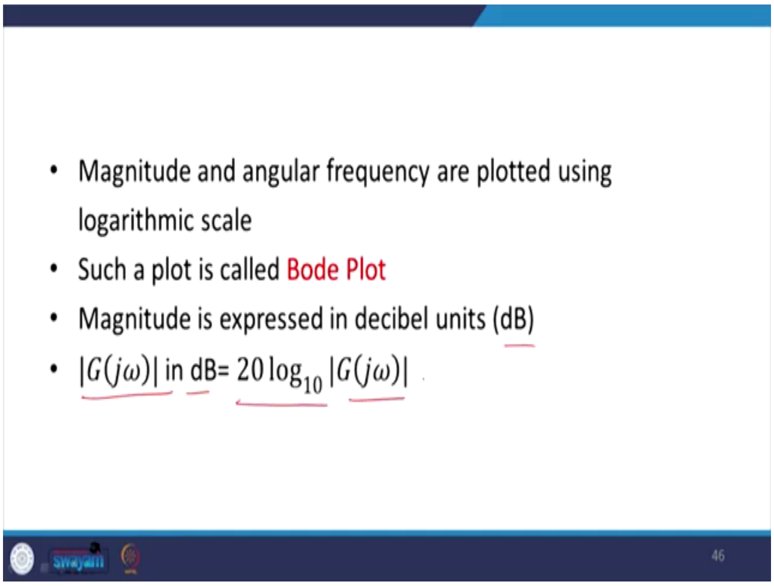

And the magnitude is expressed in the decibel unit. So, this,

$$|G(j\omega)| \text{ in dB} = 20\log_{10}|G(j\omega)|$$

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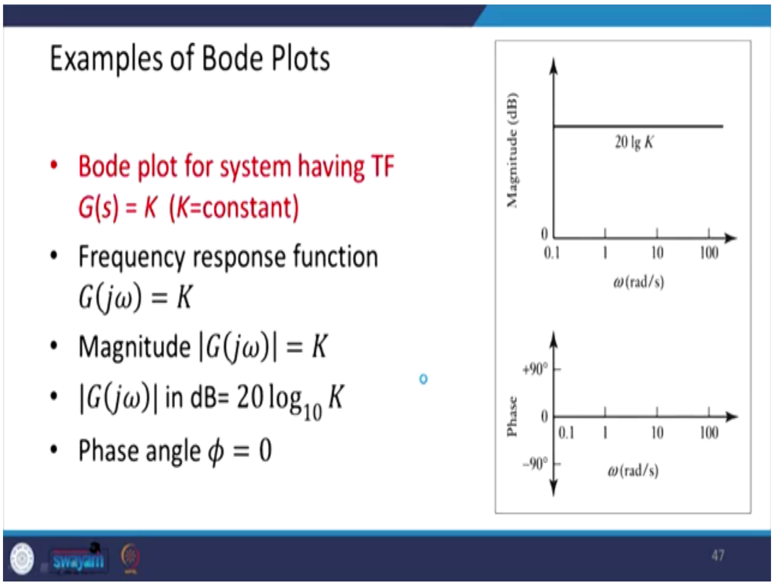

So, let us take an example of the bode plot. Suppose I have got the bode plot of a system having transfer function K = constant. So, the frequency response $G(j\omega)$ = K only, and the magnitude of that will be K only. So, the magnitude in terms of decibel will be $20\log_{10}K$ And the phase angle is going to be 0. So, this is the bode plot for this one G(s) = K.

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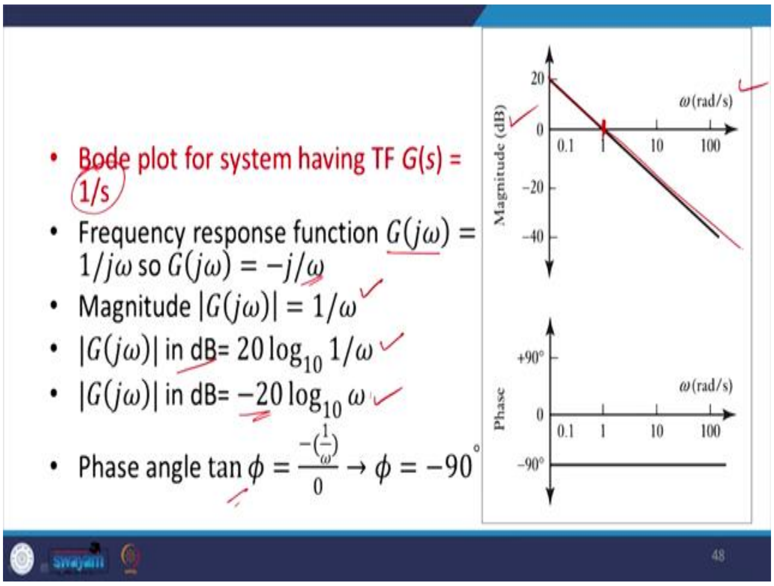

Now, if my G(s) =1 / s, then again, I can find out $G(j\omega) = \frac{1}{j\omega}$, and I can write it in the complex form like this. And its magnitude will be $1/\omega$, or I can write it like this.

So, if I plot it with ω, the magnitude this is what I am going to get, and you that at omega is equal to 1, this is going to be equal to 0. So, this plot is intersecting over here. Similarly, I can find out phase as,

$$\tan\phi = -\frac{\frac{1}{\omega}}{0} \to -90^0$$

so I can plot it like this.

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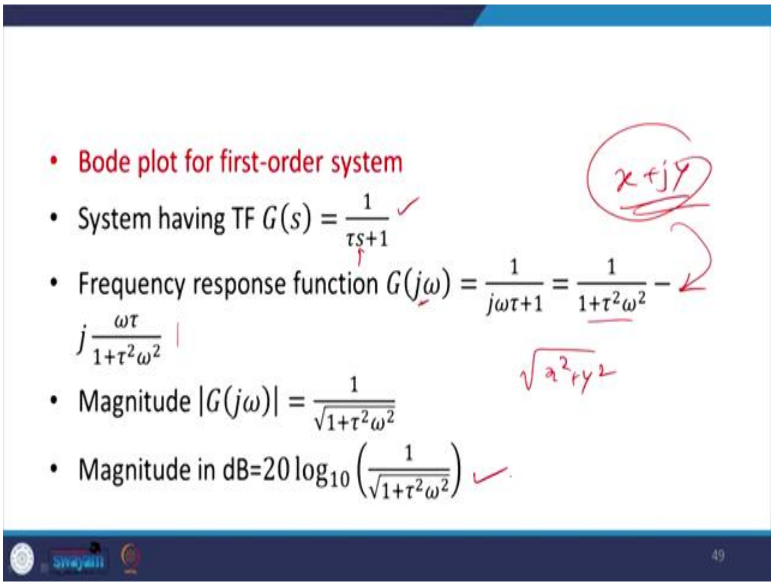

And the bode plot for the first order system if I want G(s) $=\frac{1}{\tau s+1}$. So, I replace this s by j ω over here, and I write it in the form of the complex notation that is x + j y. So, I write it in this form over here, and then I can find out the magnitude as, $\sqrt{x^2 + y^2}$, and this can be written in the decimal form like this.

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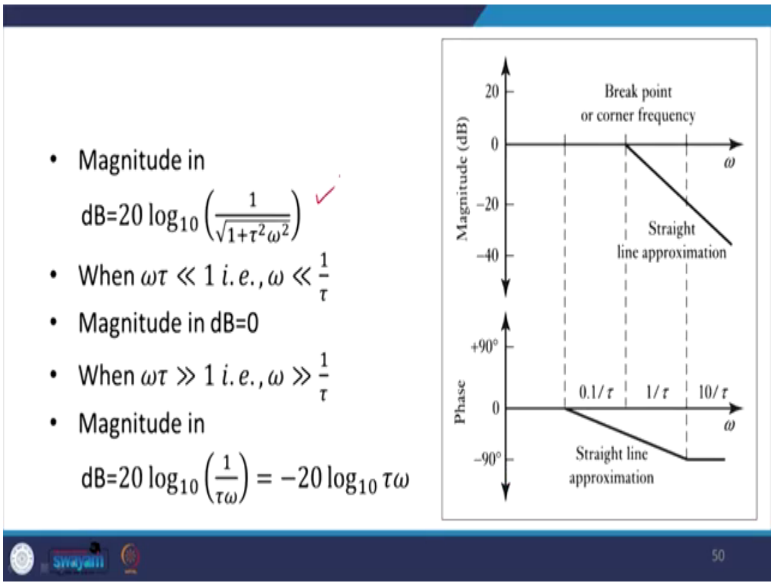

And so the magnitude is this one. Now here you see if this $\omega\tau$ << 1 or what we can see that ω <<$\frac{1}{\tau}$. Then what happens? I can neglect this term. So, this becomes $\log_{10} 1$. So, this magnitude is going to be 0. So, this is what I am going to get for less than 1 value this magnitude is going to be equal to 0.

If $\omega\tau$ >>1, ω >>$\frac{1}{\tau}$. So, in this case, the magnitude will be turning out to be like this, which is a straight line approximately. So, this is my plot this frequency breakpoint or the corner frequency it is called.

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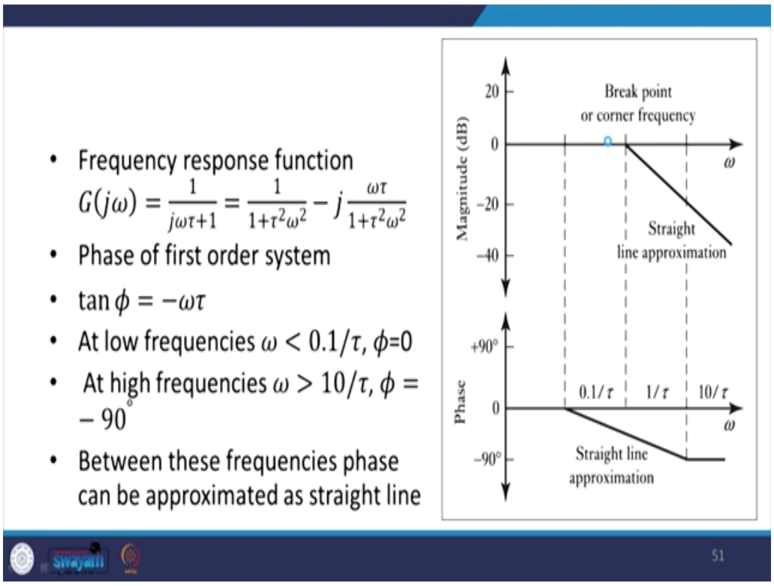

And similarly, I can have this one the phase also. So, this is the frequency response function. So, for the phase for this is a $\tan^{-1}\frac{y}{x}$ if I do that. So, it will be$-\omega\tau$, and we can see that as the $\omega < 0.1/\tau$. This value is going to be 0, and at a higher frequency, this is going to be-90^0. So this is what we are going to get an in-between this is for high value, this is for low value, and in between, we can approximate with a straight line.

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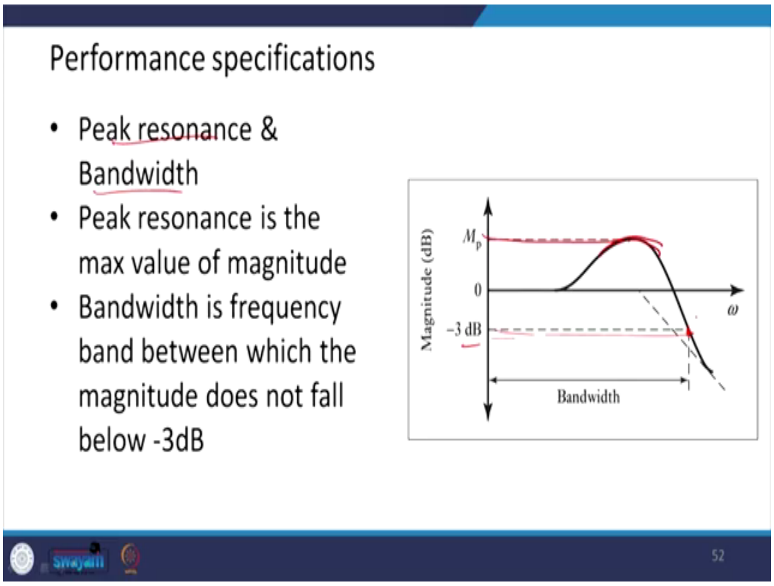

So, the performance specification for the system can be given by the two parameters, is the peak resonance and the bandwidth. The peak resonance is the maximum value of the magnitude. So, the maximum value of the magnitude over here is the peak resonance, and bandwidth is a frequency band between which the magnitude does not fall below -3 dB.

So if this is the -3 dB, so, this is going to be the bandwidth. And this concept is used in the case of the census the bandwidth of the census. So, these are the references for your further reading.

Thank you.