

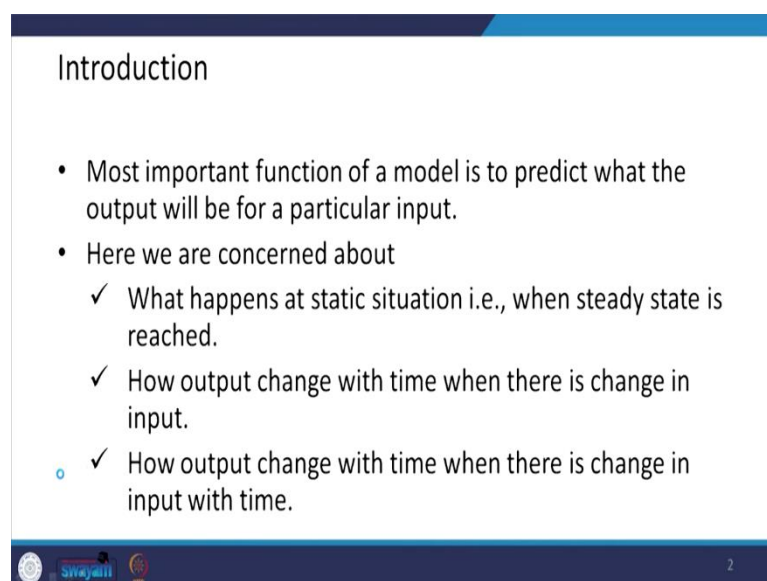
Mechatronics
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Lecture - 29
Dynamic Response of Systems

I welcome you all to this NPTEL online certification course on Mechatronics. Today, we are going to talk about the Dynamic Response of the systems. In the last three lectures, we have studied the modeling for mechanical systems, electrical systems, hydraulic and pneumatic systems, and in this lecture, I would like to explain the dynamic response of the system and what are the various response measuring parameters, which I am going to discuss at the end of this lecture. The most important function of a model is to predict what output be there for a particular type of input for the models which we have developed in last three lectures. So, the purpose of those models is to predict the output for a particular input.

Now, here we are concerned about what happens in a static situation, that is, when a steady state is reached or how output changes with time when there is the change in input, or how output changes with time when there is the change in the input itself with time. So, these are the various concerns.

(Refer Slide Time: 01:59)




Introduction

- Most important function of a model is to predict what the output will be for a particular input.
- Here we are concerned about
 - ✓ What happens at static situation i.e., when steady state is reached.
 - ✓ How output change with time when there is change in input.
 - ✓ How output change with time when there is change in input with time.

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(Refer Slide Time: 02:00)

- As we have seen for various systems output and input both can be function of time.
- Differential equations describe the relation between input and output.
- These differential equations includes derivatives with respect to time, thus there solution indicates how response varies with time.
- $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ represents the first and second order system respe

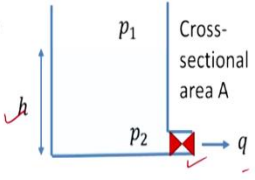


Various system output and input can be both a function of time, and we have seen that the differential equations are used to depict the relationship between the input and output, whether it is a first-order system or it is a second-order system. So, we see that $\frac{dx}{dt}$ represents the first-order system and $\frac{d^2x}{dt^2}$ represents the second-order system. Now, let us see the example of a first-order system.


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Example of First Order System

- **Natural response** ✓
- Consider water flowing out of a tank.
- For this system relation for hydraulic resistance (R) can be written as
- $(p_1 - p_2) = Rq$, we can derive
- $RA \frac{dh}{dt} + \rho gh = 0$ ✓
- This is first order differential equation, there is no input to the system so **natural response**



Water flowing out of a tank naturally with no input



So, as I explained to you in the last lecture, suppose there is a tank, and there is a water-filled up to a height h , and pressure at the top and bottom are p_1 and p_2 respectively and

there is a wall through which the discharge takes place, and as p_2 is the pressure hereafter the wall. So, we can write the expression for this as we have derived in the previous lecture. We can write the expression for flow through the wall, and the equation will be,

$$p_1 - p_2 = Rq$$

and this can be written in this particular form. So, this is our first-order differential equation. Here what you can see is that we are getting expression in terms of the height h of the water in the tank. And it is a natural response, so there is no forcing function here. So, we have the right-hand side equal to 0.

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- Force response ✓
- In case there is a flow of water in tank
- For capacitor $q_1 - q_2 = C \frac{dp}{dt}$ ✓
- For valve $(p_1 - p_2) = Rq_2$ ✓
- $A \frac{dh}{dt} + \frac{\rho gh}{R} = q_1$ ✓ *1st order*
- This system equation has a **forcing function**.

Water flowing out of a tank with forcing input

The diagram shows a tank with a water level height h . An inlet pipe at the top left has a flow rate q_1 and pressure p_1 . The tank has a cross-sectional area A . An outlet pipe at the bottom right has a flow rate q_2 and pressure p_2 . A valve is shown on the outlet pipe.

If I talk about force response, it can be considered if the flow is taking place with the help of there is some input over here q_1 and the rest of the things are the same. So, as we have seen in the last lecture using the equation for the capacitor as well as the expression for the resistance through the wall. We can derive the equation like this. And here again, you can see that we have the equation that depicts the variation of h with time, and on the right-hand side, we have q_1 Which is nothing but the input to the tank over here. So, this system equation has got a forcing function.

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Transient and steady-state responses


Total response of
a control system
or element of a
system

=

Transient response
(It occurs when
there is change in
input to system, it
dies away quickly)

+

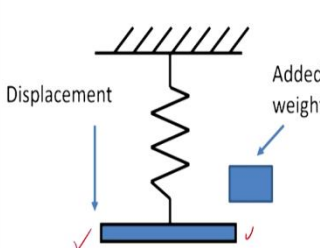
Steady state
response
(Response
which
remains
after
transients
have died)

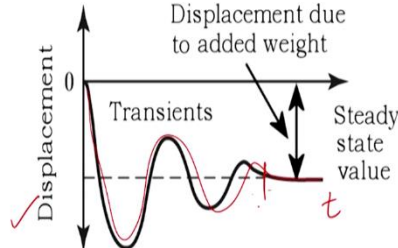

6

Now, the total response of any system is consists of the transient response as well as the steady-state response. Transient response is the one in which the change occurs when there is a change in the input to the system, and it dies away quickly. And the steady-state response remains, that is, the response that remains after the transients have died out.


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Transient and steady-state responses of a spring system





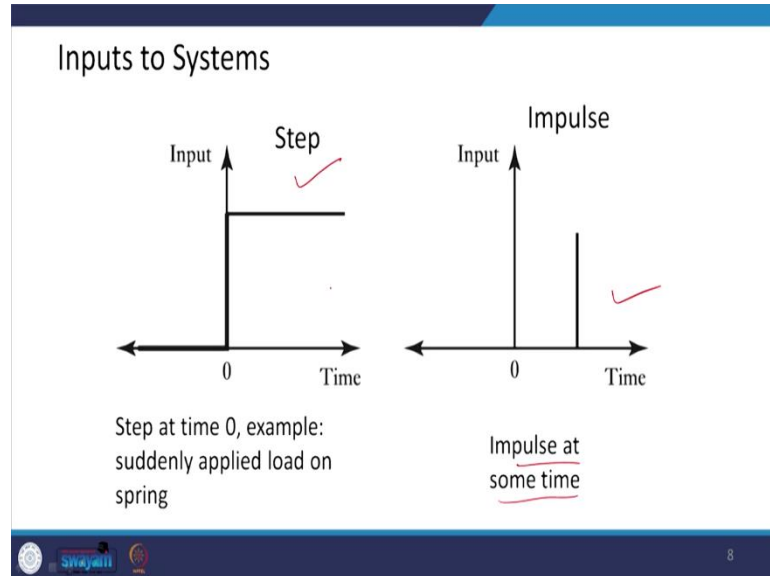
Response of a spring due to sudden addition of weight


7

The example of transient and steady-state response, we can just consider a spring, and there is a hanger at the end of the spring, and if I add a certain weight to that hanger, then the spring will be getting displaced. So, this is how the displacement of the spring of if I plot with time, so this is how it is going to be. So, this is our transient response, and this

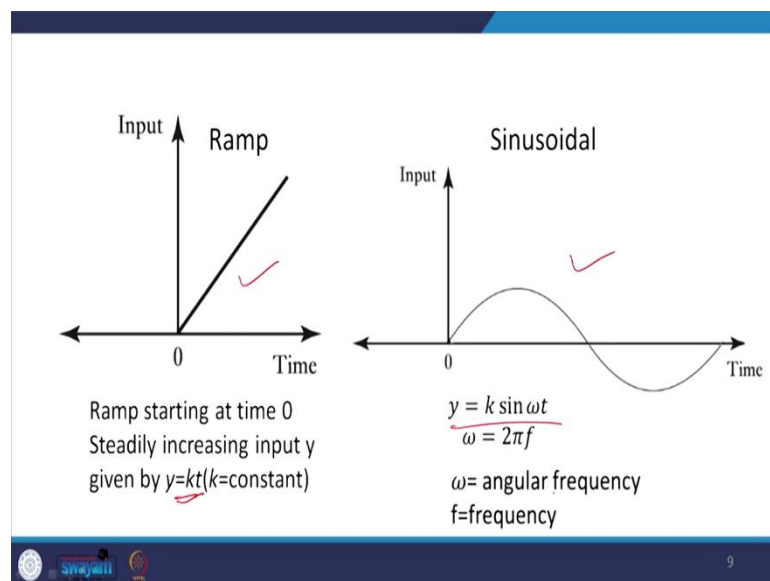
part is the steady-state response. So, this is what I mean by the transient and steady-state response.

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Now, the various input types of input can be used to excite the system. So, it could be a step input, or it could be impulse input, or it could be a ramp input, or the input could be a sinusoidal input.

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So, ramp input is defined as $y = kt$, sinusoidal we can define as $y = k\sin(\omega t)$, and similarly, this impulse at some time and step has a certain value, at time t greater than 0. So, this way, these are the various types of inputs.

(Refer Slide Time: 06:51)

First Order System

Consider first order system

$$a_1 \frac{dx}{dt} + a_0 x = b_0 y$$

Here a_1, a_0, b_0 are constant

The diagram shows a block labeled 'System' with an input arrow from the left labeled 'Input' and 'y(t)' below it, and an output arrow to the right labeled 'Output' and 'x(t)' below it. There are red checkmarks above 'Input' and below 'Output'.

Now, let us take the example of a first-order system. So, first-order system, suppose I have got a system there is some input $y(t)$, and there is some output $x(t)$. So, what we are interested in seeing is that for a given type of input, what my output varies, how my output varies. So, this is the form of the expression for the first-order system, which we have just seen over here.

(Refer Slide Time: 07:40)

Natural response of 1st Order System

- Natural response corresponds to $y(t) = 0$
- $a_1 \frac{dx}{dt} + a_0 x = 0$
- Let solution be $x = Ae^{st}$ (A and s are constants)
- Substituting in above eq.
- $a_1 A s e^{st} + a_0 A e^{st} = 0$
- $a_1 s + a_0 = 0$
- $s = -a_0/a_1$ ✓

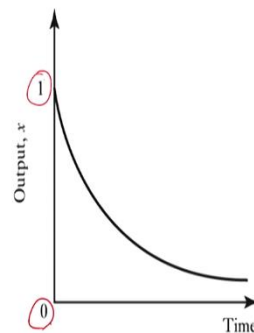


11

Now, if I talk about the natural response of the first-order system so, make the forcing function 0 here if I want to have the natural response. As it is a first-order differential equation, so its solution could be of form $x = Ae^{st}$, where A and s are constants. And this if we substitute it over here then we can get the value of s as this one.

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- The solution was $x = Ae^{st}$ ✓
- This can be written as $x = Ae^{-\left(\frac{a_0}{a_1}\right)t}$
- A can be found by initial condition. Say at $t=0$, $x=1$, so $A=1$, thus output will be given as
- $x = e^{-\left(\frac{a_0}{a_1}\right)t}$ → Natural response



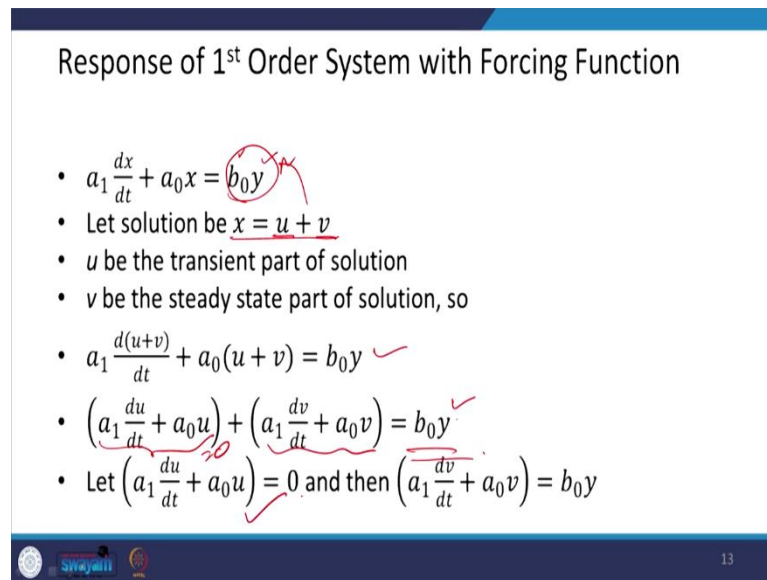
12

Now, if I substitute the value of s, this is the solution of the equation that I get. Here there is an initial A constant that can be evaluated by the initial condition. So, suppose at time $t = 0$ and put $x = 1$, so I will get $A = 1$ from here. So, this is the solution for the expression. So, this is what I said as the natural response. So, this is the natural response for the first-order system.

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Response of 1st Order System with Forcing Function

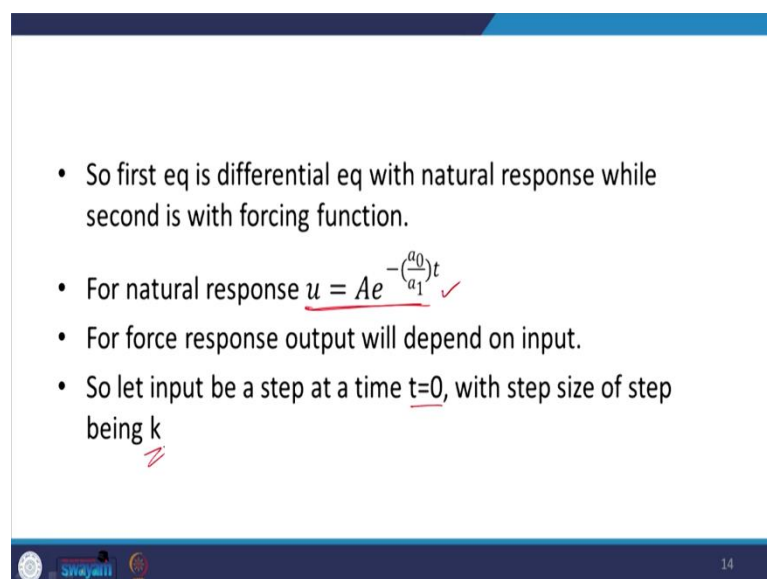
- $a_1 \frac{dx}{dt} + a_0 x = b_0 y$
- Let solution be $x = u + v$
- u be the transient part of solution
- v be the steady state part of solution, so
- $a_1 \frac{d(u+v)}{dt} + a_0(u+v) = b_0 y$
- $\left(a_1 \frac{du}{dt} + a_0 u\right) + \left(a_1 \frac{dv}{dt} + a_0 v\right) = b_0 y$
- Let $\left(a_1 \frac{du}{dt} + a_0 u\right) = 0$ and then $\left(a_1 \frac{dv}{dt} + a_0 v\right) = b_0 y$



Now, let us see the response of the first-order system with the forcing function. So, suppose I have got a forcing function here $b_0 y$, where y is the input, and b_0 is some constant. So, it is a solution. Let us assume that the solution has got a transient part u and steady-state part v . So, I can assume the solution of this form. I can substitute the solution in this expression, and then I can simplify it, so I get this form. So, if I assume this part is equal to 0, then naturally, this part happens to be equal to 1.

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- So first eq is differential eq with natural response while second is with forcing function.
- For natural response $u = Ae^{-\left(\frac{a_0}{a_1}\right)t}$
- For force response output will depend on input.
- So let input be a step at a time $t=0$, with step size of step being k



So, now let us try to find out the solution for the first equation that is with corresponds to 0 at the right-hand side over here. So, again it's the natural response. So, its solution is going to be this one. And for the force response, we can assume a step input, at t, is equal to 0 with a step size of k. If I do that, this is what is my equation.

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- So for $(a_1 \frac{dv}{dt} + a_0 v) = b_0 y$ ✓
- Let solution be $v = B$ (B=constant)
- $a_0 B = b_0 k$ ✓
- $B = \frac{b_0}{a_0} k$ ✓
- So solution is $v = \frac{b_0}{a_0} k$ ✓
- Thus complete solution is $x = u + v$
- $x = Ae^{-\frac{a_0}{a_1}t} + \frac{b_0}{a_0} k$ ✓

The slide also features a small hand-drawn graph of a step function. The vertical axis is labeled 'k' and the horizontal axis is labeled 't'. The function is zero for t < 0 and jumps to a constant value 'k' for t > 0.

As I said, I can assume some solution for that $v = B$, which is a constant. If I substitute it over here, this term will become 0 because I am taking b as a constant. So, from here, I get the value of B. So, I get the solution. So, I can write the complete solution like this. This is the case for the solution of a first-order differential equation with a forcing function.

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• $x = Ae^{-\left(\frac{a_0}{a_1}\right)t} + \frac{b_0}{a_0}k$ ✓

• A can be found using the initial condition, say at $t=0$, $x=0$ ✓

• So $A = -\frac{b_0}{a_0}k$, ✓

• So solution $x = -\frac{b_0}{a_0}ke^{-\left(\frac{a_0}{a_1}\right)t} + \frac{b_0}{a_0}k$ ✓

16

So, I can get the value of A by substituting that $t = 0$, $x = 0$. So, if I put it over here, I can get the value of a like this one. I can substitute it over here.

(Refer Slide Time: 11:21)

• $x = \frac{b_0}{a_0}k(1 - e^{-\left(\frac{a_0}{a_1}\right)t})$ ✓

• So as $t \rightarrow \infty$, exponential tends to zero, thus steady state response is

$x = \frac{b_0}{a_0}k$ ✓

Step input

Resulting output

17

So, this is my final response. So, this is my step input which I said in my first-order differential equation with a forcing function. So, this is my response. So, here as you can see that if I take the value of t is equal to infinity. So, this value is going to be 0. So, I will be getting this value. So, this is what I am getting at $t \rightarrow \infty$ infinity,

$$x = \frac{b_o}{a_o} k$$

(Refer Slide Time: 11:59)

Example: A DC motor

- Relation for an armature controlled motor are
- $V_a - K_3\omega = L_a \frac{di_a}{dt} + i_a R_a$
- $L \frac{d\omega}{dt} = K_4 i_a - R_b \omega$
- We can substitute for i_a in second equation from first in order to get the relation between input V_a and output ω .
- $\frac{LR_a}{k_3 k_4} \frac{d\omega}{dt} + \omega = \frac{1}{k_3} V_a$ Compare with $a_1 \frac{dx}{dt} + a_0 x = b_0 y$

So, the previous example which I have shown was a case of a tank on which water is filled up. We can have another example of the first-order system with a forcing function as that of a DC motor. So, in the case of DC motor, as we have seen in modeling of electrical systems, we have this as the these are the equations. It is the equation for the armature control motor where this is the voltage supply to the armature, and this is the back emf, and so and these are the voltage across the inductance, and this is across the resistance. And this is the equation for the load. So, here we can simplify these equations that are a substitute for i_a from the first one and second one. So, this is what I can get. So, this is also a first-order differential equation of this form similar to what we have seen as the water being filled up in a tank. So, here you can see that output is ω and input is the voltage supplied to the armature. So, we can have a similar treatment way.

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The Time Constant

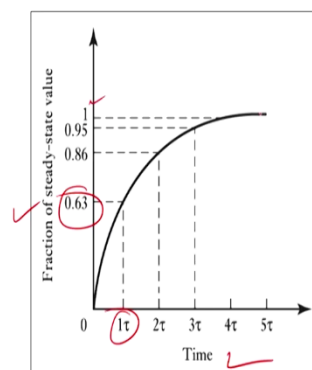
- $x = \frac{b_0}{a_0} k (1 - e^{-\frac{a_0}{a_1} t})$ ✓
- $x = (\text{Steady state value}) (1 - e^{-\frac{a_0}{a_1} t})$ ✓
- When time $t = \frac{a_1}{a_0}$, output has risen 0.63 times the steady state value.
- This time is called the time constant $\tau = \frac{a_1}{a_0}$ ✓
- So the response of the first order system for a step input is $x = (\text{Steady state value}) (1 - e^{-\frac{t}{\tau}})$ ✓

Now, if I look at this expression for the solution for the differential equation with a forcing function that is the first-order differential equation, this is my expression, and we have seen that this is the steady-state value. So, this is a steady-state value in this part. Now, here you see that if I write $t = \frac{a_1}{a_0}$, then these terms will be getting canceled, and we will have that is the value of x . And that will be 0.63 times the steady-state value. And this time is called the time constant. So, this is $\tau = \frac{a_1}{a_0}$ that is known as the time constant, and if I write it in terms of that, then this is my response to the first-order system for the step input.

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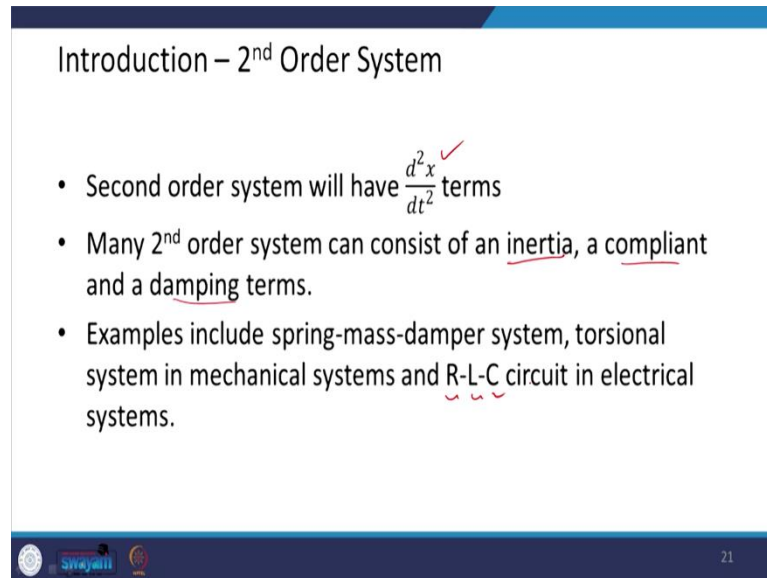
Response of a first-order system to a step input

Time t	Fraction of steady-state output
0	0
1τ ✓	<u>0.63</u>
2τ	0.86
3τ	0.95
4τ	0.98
5τ	0.99
∞	1 ✓



So, $\tau=1$ or 1τ , this is 0.63. So, the fraction of steady-state value if I plot over time, so this is 0.63 at 1τ , and this way I can have the response, so you can see that at time infinity I get the steady-state value that is 1 over here.

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Introduction – 2nd Order System

- Second order system will have $\frac{d^2x}{dt^2}$ terms ✓
- Many 2nd order system can consist of an inertia, a compliant and a damping terms.
- Examples include spring-mass-damper system, torsional system in mechanical systems and R-L-C circuit in electrical systems.

Next, let us talk about the second-order system. So, the second-order system, as we have seen that they have $\frac{d^2x}{dt^2}$. And the second-order system can consist of inertia, compliance, and damping for a mechanical system, and we have also seen on several occasions during this course the electrical system can have the resistor, inductor, as well as capacitor. They also represent the second-order system.

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Second Order System

- Relationship between the input force F and output of a displacement x is
- $m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = F$

Spring-dashpot-mass system

So, let us take the second-order system, a simple spring-mass damper system over here. So, this spring mass damper system is excited by an external force F . So, in this case, my input is F , and the output is this displacement x over here. So, I can write the equation of motion for this system by drawing the free body diagram, and it is this,

$$m \frac{d^2x}{dt^2} + \frac{cdx}{dt} + kx = F$$

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- The variation of x with time depends on amount of damping present in the system.
- If force is applied as step input then
- If no damping is present then mass will freely oscillate.
- Damping causes oscillations to die away until steady displacement of mass is obtained.
- If damping is high there will be no oscillations which means displacement of mass will slowly increase with time and moves towards steady displacement position.

So, the variation of x with time depends on the amount of damping present in the system, and the force is applied as a step input, then. If there is no damping present in the system,

then the mass will keep on oscillating, and the damping causes the oscillations to die away unless the steady displacement of the mass is obtained. If damping is high, then there will not be oscillations and which means that the displacement of mass will slowly increase with time and move towards the steady displacement position. Now, first, let us consider the second-order differential equation with no damping.

(Refer Slide Time: 17:04)

2nd order differential equation with no damping

- $m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = 0$
- If $c = 0$ (no damping present)
- We will have continuous oscillations
- So assume the solution to be .
- $x = A \sin \omega_n t$ (A = amplitude of oscillations, ω_n be angular frequency of free undamped oscillations)

So, if there is no damping and there is no input excitation, I take here 0, and I have my input force 0, and I can take this $c = 0$. So, we will have because damping is not there, so we will have the continuous oscillations, and we can assume the solution of this form,

$$x = A \sin \omega_n t$$

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- $x = A \sin \omega_n t$ ✓
- $\frac{dx}{dt} = A\omega_n \cos \omega_n t$
- $\frac{d^2x}{dt^2} = -A\omega_n^2 \sin \omega_n t$
- $\frac{d^2x}{dt^2} = -\omega_n^2 x$ ✓
- $\frac{d^2x}{dt^2} + \omega_n^2 x = 0$ ✓

If I take the first derivative, so

$$\frac{dx}{dt} = A\omega_n \cos \omega_n t$$

And if I take the second derivative, then this is,

$$\frac{d^2x}{dt^2} = -A\omega_n^2 \sin \omega_n t$$

Now, you see this is $A \sin \omega_n t$ is the same as that of x . So, I can write this as $-\omega_n^2 x$ Or I can write it in this form.

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- $\frac{d^2x}{dt^2} + \omega_n^2 x = 0$ ✓ (1)
- In absence of c and F system equation is
- $m \frac{d^2x}{dt^2} + kx = 0$ ✓
- $\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0$ ✓ (2)
- Comparing (1) and (2)
- $\omega_n^2 = \frac{k}{m}$ ✓
- $x = A \sin \sqrt{\frac{k}{m}} t$ is the solution of differential equation. ✓ *no damping*

Now, if I compare this equation with the equation without damping, that is this one, then you can see that the, $\omega_n^2 = \frac{k}{m}$. So, I can write the solution as,

$$x = A \sin \sqrt{\frac{k}{m}} t$$

and this is the solution of the differential equation if there is no damping.

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2nd order differential equation with damping

- Motion of the mass is described by
- $m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = F$
- Let the solution consists of transient response and force response
- $x = x_n + x_f$
- $m \frac{d^2(x_n+x_f)}{dt^2} + c \frac{d(x_n+x_f)}{dt} + k(x_n+x_f) = F$

Now, what if there is damping? So, if damping is present, then this term is not going to be equal to 0, and we have the excitation force also F being over here. So, again here, I can take the solution to be composed of two forms is the natural response and the force response, and as we have done earlier, we can substitute this solution in this expression, and we can segregate the two parts over here.

(Refer Slide Time: 18:55)

Slide 28 contains the following content:

- $m \frac{d^2(x_n+x_f)}{dt^2} + c \frac{d(x_n+x_f)}{dt} + k(x_n+x_f) = F$
- $\left(m \frac{d^2x_n}{dt^2} + c \frac{dx_n}{dt} + kx_n \right) + \left(m \frac{d^2x_f}{dt^2} + c \frac{dx_f}{dt} + kx_f \right) = F$
- If $m \frac{d^2x_n}{dt^2} + c \frac{dx_n}{dt} + kx_n = 0$;
- We must have
- $m \frac{d^2x_f}{dt^2} + c \frac{dx_f}{dt} + kx_f = F$ ✓

At the bottom of the slide, there are logos for Swayam and a page number 28.

So, this part we can equate to 0, which will be giving the natural response, and this part will be going to equal to this one, and that will be giving us the force response.

(Refer Slide Time: 19:11)

Slide 29 is titled "Solution of transient equation" and contains the following content:

- The equation is
- $m \frac{d^2x_n}{dt^2} + c \frac{dx_n}{dt} + kx_n = 0$; ✓
- Let the solution be
- $x_n = Ae^{st}$ (A and s are constants)
- Substituting in above equation
- $mAs^2e^{st} + cAse^{st} + kAe^{st} = 0$
- $Ae^{st}(ms^2+cs+k) = 0$
- $Ae^{st} \neq 0$, as it will result in $x_n = 0$

At the bottom of the slide, there are logos for Swayam and a page number 29.

So, if we look at the solution for the transient equation that is for the natural response, so here I have I can take, $x_n = Ae^{st}$ as the solution, and if I substitute it back over here. So I get this one, and this cannot be equal to 0 because if I make it is equal to 0, this itself will become 0, so I make the other equal to 0. So, this is my auxiliary equation. So, from here, I can get the value of s. It is a quadratic equation. So, its solution is going to be this form.

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- So $ms^2 + cs + k = 0$ (Auxiliary equation)
- So $s = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$
- $s = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$
- $s = -\frac{c}{2m} \pm \sqrt{\frac{k}{m} \left(\frac{c^2}{4mk}\right) - \frac{k}{m}}$
- But $\omega_n^2 = \frac{k}{m}$ and if we define $\zeta^2 = \frac{c^2}{4mk}$ then $\zeta = \frac{c}{2\sqrt{mk}}$

And if I define this, $\omega_n^2 = \frac{k}{m}$ as we have seen for the natural response and I define, $\zeta^2 = \frac{c^2}{4mk}$, then $\zeta = \frac{c}{2\sqrt{mk}}$. So, I can substitute that over here, and I can get the root of this form.

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- So s can be given as
- $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$
- If $\zeta > 1$ (two different real roots)
- $s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$
- $s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$
- So the general solution is
- $x_n = Ae^{s_1 t} + Be^{s_2 t}$
- System is said to be overdamped.

Now, here there are 3 cases, we are going to have this if ζ is going to be more than 1, then these are the two solutions which I am going to get s_1 and s_2 , so the general solution is going to be this one. And we see that the system is overdamped.

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- If $\zeta=1$ (two equal roots)
- $s_1 = s_2 = -\omega_n$ ✓
- $x_n = (At + B)e^{-\omega_n t}$ ✓
- System is said to be critical damped ✓
- If $\zeta < 1$ (two roots are complex)
- $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$
- So let $\omega_d = \omega_n\sqrt{1-\zeta^2}$
- $s = -\zeta\omega_n \pm j\omega_d$ ✓

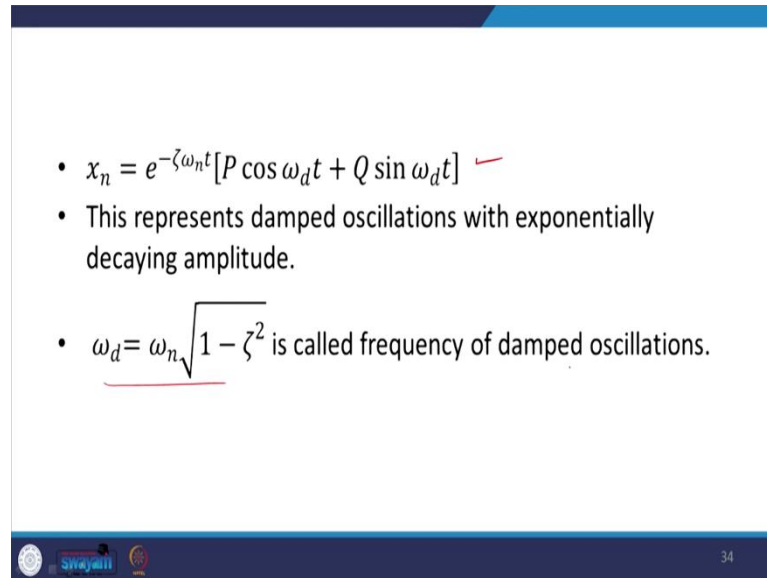
And if the $\zeta = 1$, this is going to be the solution, these are going to be the roots both equal, and this is going to be the solution, and the system is said to be critically damped. And if $\zeta < 1$, then the two roots that are complex are going to be there. We define this as the damped frequency, and the two roots can be represented like this. So, this is going to be the general solution.

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- So the general solution is
- $x_n = Ae^{(-\zeta\omega_n + j\omega_d)t} + Be^{(-\zeta\omega_n - j\omega_d)t}$
- $x_n = e^{-\zeta\omega_n t} (Ae^{j\omega_d t} + Be^{-j\omega_d t})$ ✓
- $e^{j\omega_d t} = \cos \omega_d t + j \sin \omega_d t$
- $e^{-j\omega_d t} = \cos \omega_d t - j \sin \omega_d t$
- Substituting
- $x_n = e^{-\zeta\omega_n t} [(A + B) \cos \omega_d t + j(A - B) \sin \omega_d t]$
- $x_n = e^{-\zeta\omega_n t} [P \cos \omega_d t + Q \sin \omega_d t]$ ✓

If I just substitute, I can further simplify this solution by representing the exponential function as cosine and sin functions. And this is going to be my natural response in that particular case.

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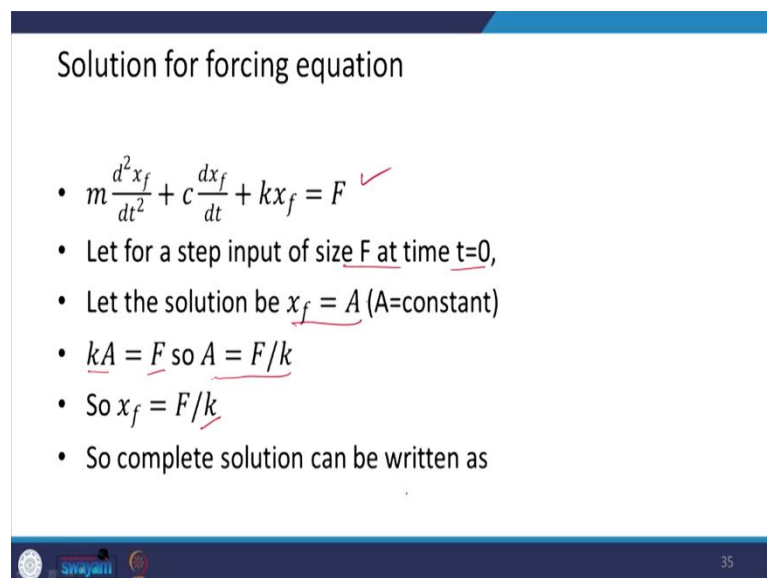
Slide 34 contains the following content:

- $x_n = e^{-\zeta\omega_n t} [P \cos \omega_d t + Q \sin \omega_d t]$ ✓
- This represents damped oscillations with exponentially decaying amplitude.
- $\omega_d = \omega_n \sqrt{1 - \zeta^2}$ is called frequency of damped oscillations.

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So, this is the natural response, and this one is called the damped frequency of oscillation, as I have explained to you.

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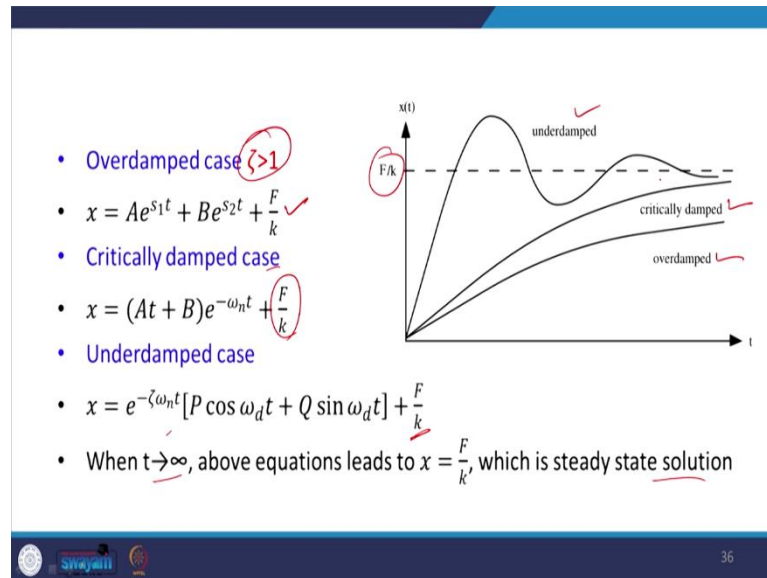
Slide 35 is titled 'Solution for forcing equation' and contains the following content:

- $m \frac{d^2 x_f}{dt^2} + c \frac{dx_f}{dt} + kx_f = F$ ✓
- Let for a step input of size F at time t=0,
- Let the solution be $x_f = A$ (A=constant)
- $kA = F$ so $A = F/k$
- So $x_f = F/k$
- So complete solution can be written as

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Now, the solution for the forcing equation if you look at this one. So, as we have seen for a step of size F at time $t = 0$, we can take the solution $x_f = A$. So, $kA = F$, so the $A = F/k$, and $x_f = \frac{F}{k}$.

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So, my complete solution I can write for all the 3 cases that are for $\zeta > 1$. I edit up here overdamped case, for the critically damped case, I edit up here, and for the underdamped case, I edit it over here.

So, when $t \rightarrow \infty$ above equation leads to $x = F/k$, which is the steady-state solution over here. So, here you can see that this is the overdamped case, critically damped case, and this is the underdamped case, and this is F/k is the steady-state solution.

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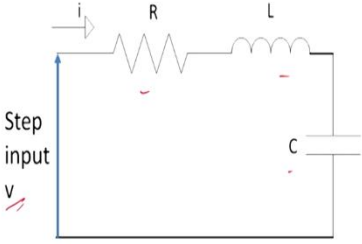
Example: R-L-C System

$V = V_R + V_L + V_C$ ✓ *KVL*

$V = iR + L \frac{di}{dt} + V_C$ ✓

$i = C \frac{dV_C}{dt}$ ✓

$V = iR + L \frac{di}{dt} + V_C$ ✓



Step input v

37

Another example, as I was telling you in the electrical system, it is a simple RLC circuit which is an example of the second-order system. We can, for a given step input I can write the Kirchhoff's Voltage Law, which is a KVL, and I can write this voltage across resistor, inductor, capacitor, and iR , Ldi/dt and V_C respectively. I can write it like this. So, this way, I can write an expression like this one.

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Performance Measures

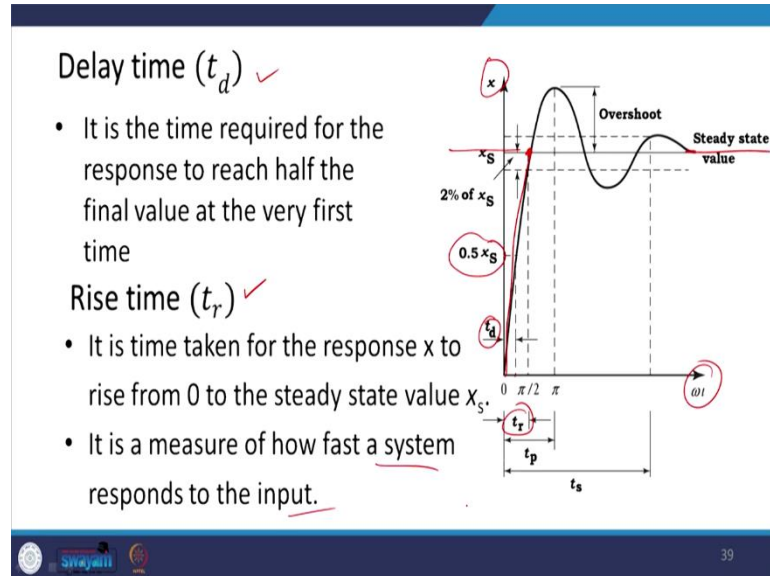
- There are some parameters by which we can specify the performance of an underdamped second order system to a step input.
- These parameters are
- Delay time (t_d) ✓
- Rise time (t_r) ✓
- Peak time (t_p) ✓
- Maximum overshoot (M_p) ✓
- Settling time (t_s) ✓

38

Then, there are various performance measures parameters, and these parameters are how the system is responding. Those parameters can be measured with the help of delay time,

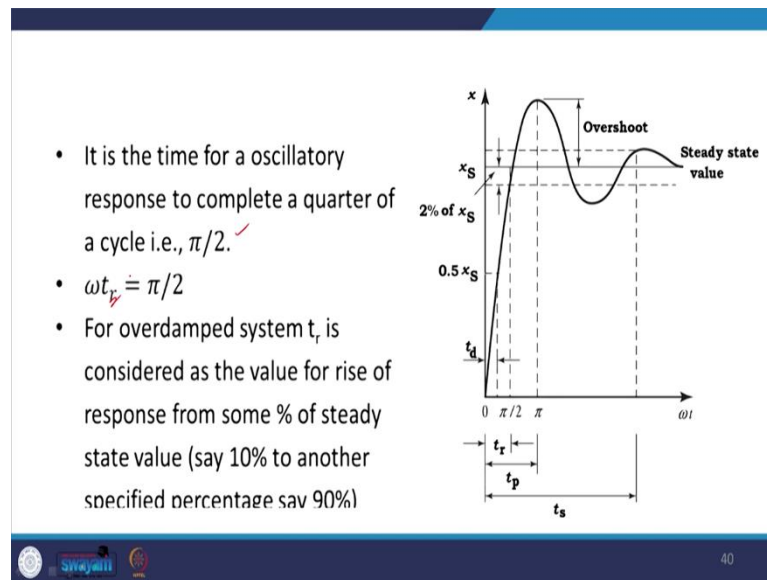
rise time, peak time, maximum overshoot, and settling time. So, with the help of these parameters, we can measure the performance of the dynamic system.

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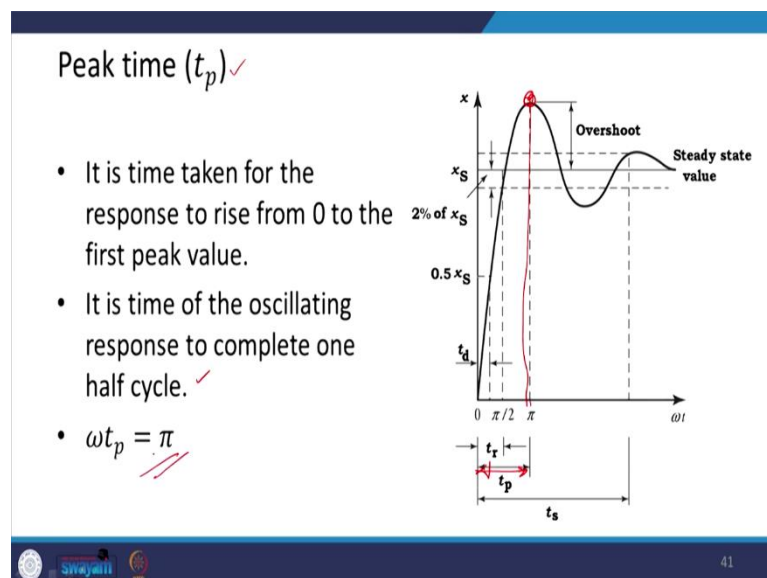
So, what is the delay time? So, if I plot x vs ωt over here, it is the time required for the response to reach half of the final value at the very first time. So, this is my steady-state value that is the final value, so 0.5 of x_s it reaches over here. So, this time is what I am calling the delay time. The rise time is the time taken for the response to rise from 0 to the steady-state value. So, this is a steady-state value. So, this is the rise time that is moving from here to, moving from here to here whatever time is taking that is the rise time. And its measure of how fast a system responds to the input.

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And it is the time for oscillatory response to complete a quarter of a cycle. So, this $\omega t_r = \frac{\pi}{2}$. So, this way we can find out the rise time. And for overdamped system t_r is considered as the value for the rise of response from some percentage of the steady-state value, so 10 percent to another specified percentage 90 percent.

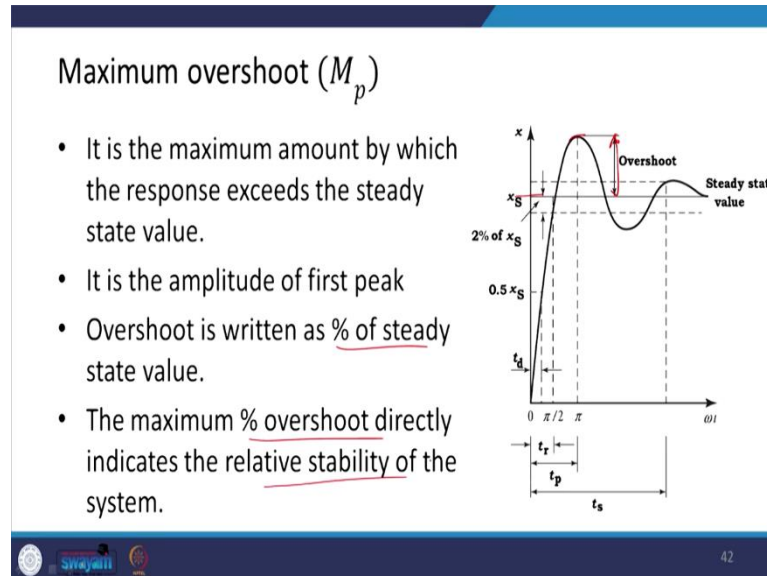
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Then another parameter is the peak time, and it is the time taken for the response to rising from 0 to the first peak value. So, here you can see that the first peak value is coming over

here. So, it is this time. This is your peak time. And it is the time of the oscillation response to complete one half of the cycle. So, this is how it is written as π over here.

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Then, you have the maximum overshoot. So, this is from the steady-state value. Whatever value response, how much is the maximum value from the steady-state that is what is called the overshoot. So, it is the maximum amount by which the response exceeds the steady-state value. And it is amplified, the amplitude of the first peak. And this overshoot is written as a percentage of the steady-state value. The maximum percentage overshoot directly indicates the relative stability of the system. And this maximum overshoot can be derived to be like this in terms of the steady-state value. And in terms of the maximum percentage overshoot, it could be derived like this.

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- Maximum overshoot = $x_S e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)}$
- % Maximum Overshoot = $e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)} \times 100$
- Percentage peak overshoot

Damping ratio	Percentage overshoot
0.2	52.7
0.4	25.4
0.6	9.5
0.8	1.5

I am not working on the derivations over here. You can refer to the reference books which I will be telling you towards the end of this lecture. So, the percentage of the peak overshoot is going to be as you can function of the damping ratio over here. So, for the damping ratio, 0.2 overshoot is 52.7, and so on. So, as you can see, as we are increasing the damping ratio, the overshoot is decreasing. So, that is one of the observations which I wanted you to observe.

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Subsidence Ratio or Decrement ✓

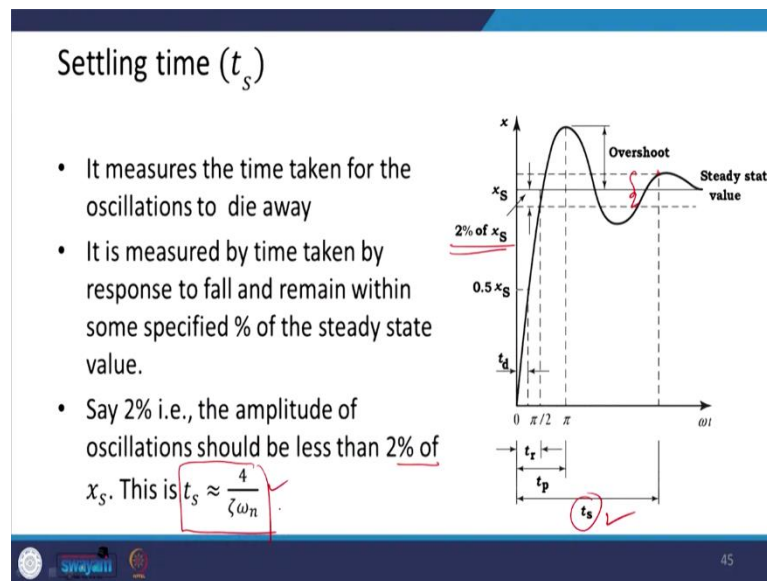
- This provides the information about how fast the oscillations decay
- This is defined as ratio of second overshoot to the first overshoot.
- The first overshoot occurs at $\omega t = \pi$ ✓
- The second overshoot occurs at $\omega t = 2\pi$ ✓

- 1st overshoot = $x_S e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)}$ ✓
- 2nd overshoot = $x_S e^{-\left(\frac{\zeta(2\pi)}{\sqrt{1-\zeta^2}}\right)}$ ✓
- So decrement = $\frac{\text{2nd overshoot}}{\text{1st overshoot}} = e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)}$ ✓

Then, the subsidence ratio or the decrement is again another major performance parameter. This provides information about how fast the oscillations are decaying, and it is defined as the ratio of the second overshoot to the first overshoot. So, the first overshoot occurs at $\omega t = \pi$, the second overshoot will be occurring at $\omega t = 2\pi$.

So, the first overshoot can be written like this expression. The second overshoot expression can be written like this. So, the decrement expression can be derived in this way.

(Refer Slide Time: 28:03)



Then, the settling time. You see that for a system reaching the exact steady-state value may not be possible. So, when do we say that the system has reached the steady-state value? So, there is a certain percentage of the steady-state value of 2 percentage. If your system reaches two percent of the steady-state value, within that limit, if it is there, we that the system has got settled. So, the settling time this t_s for you can see that this zone is two percent of the x_s . Wherever this has been achieved, that time is called the settling time, and it is measured by the time taken by the response to fall and remain within some specified percentage of the steady-state value. And this, for the 2 percent of the settling of x_s , this value can be worked out as, $t_s \approx \frac{4}{\zeta\omega_n}$.

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Number of Oscillations

- It is given by settling time/periodic time.
- If settling time t_s corresponds to 2% of the steady state value,

$$\bullet \text{ No of oscillations} = \frac{\frac{4}{\zeta\omega_n}}{\frac{2\pi}{\omega}} = \frac{4}{\zeta\omega_n} \times \frac{\omega}{2\pi} = \frac{4}{\zeta\omega_n} \times \frac{\omega_n\sqrt{1-\zeta^2}}{2\pi}$$

$$\bullet \text{ No of oscillations} = \frac{2\sqrt{1-\zeta^2}}{\pi\zeta}$$

Then another performance parameter is the number of oscillations, and this is given by the settling time divided by the periodic time. So, the settling time of 2% settling time divided by periodic time, if I substitute it over here, this is how to get the expression for the number of oscillations which is again the function of the damping ratio.

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Reference

- W. Bolton, Mechatronics: Electronic Control Systems in Mechanical and Electrical Engineering (6th Edition), Pearson, 2015
- K. Ogata, Modern Control Engineering, PHI, 2000
- R. Merzouki, A. K. Samantaray, P. M. Pathak, B. Ould Bouamama, Intelligent Mechatronic Systems: Modeling, Control and Diagnosis, ISBN 978-1-4471-4627-8, 2013, Springer, London
- D.G. Alciatore and Michael B. Hiestand, Introduction to Mechatronics, Tata Mc Graw Hill, 2012.

So, these are the further references you can look at if you want to read it further.

Thank you very much.