Mechatronics
Prof. Pushparaj Mani Pathak
Department of Mechanical and Industrial Engineering
Indian Institute of Technology, Roorkee

Lecture - 29
Dynamic Response of Systems

I welcome you all to this NPTEL online certification course on Mechatronics. Today, we are going to talk about the Dynamic Response of the systems. In the last three lectures, we have studied the modeling for mechanical systems, electrical systems, hydraulic and pneumatic systems, and in this lecture, I would like to explain the dynamic response of the system and what are the various response measuring parameters, which I am going to discuss at the end of this lecture. The most important function of a model is to predict what output be there for a particular type of input for the models which we have developed in last three lectures. So, the purpose of those models is to predict the output for a particular input.

Now, here we are concerned about what happens in a static situation, that is, when a steady state is reached or how output changes with time when there is the change in input, or how output changes with time when there is the change in the input itself with time. So, these are the various concerns.

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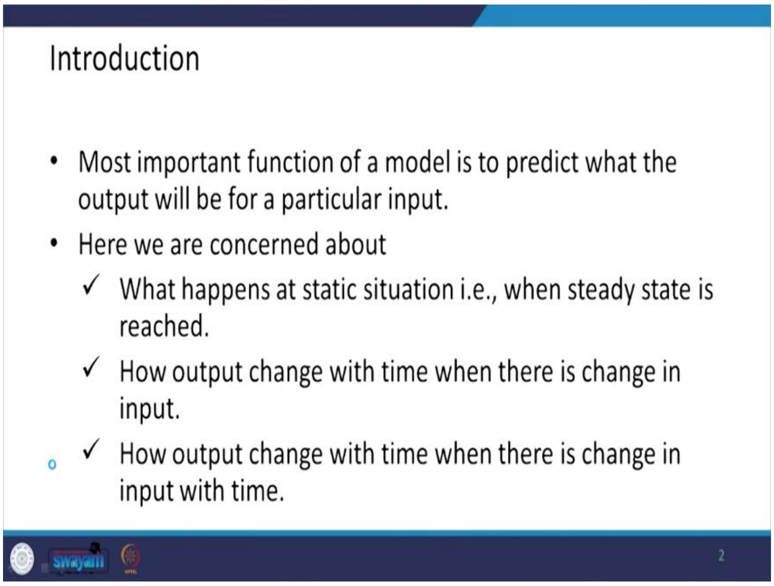

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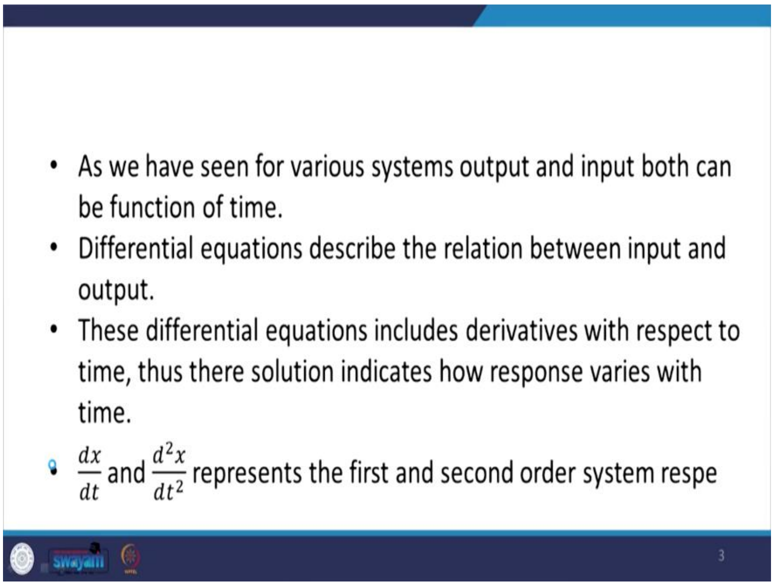

Various system output and input can be both a function of time, and we have seen that the differential equations are used to depict the relationship between the input and output, whether it is a first-order system or it is a second-order system. So, we see that $\frac{dx}{dt}$ represents the first-order system and $\frac{d^2x}{dt^2}$ represents the second-order system. Now, let us see the example of a first-order system.

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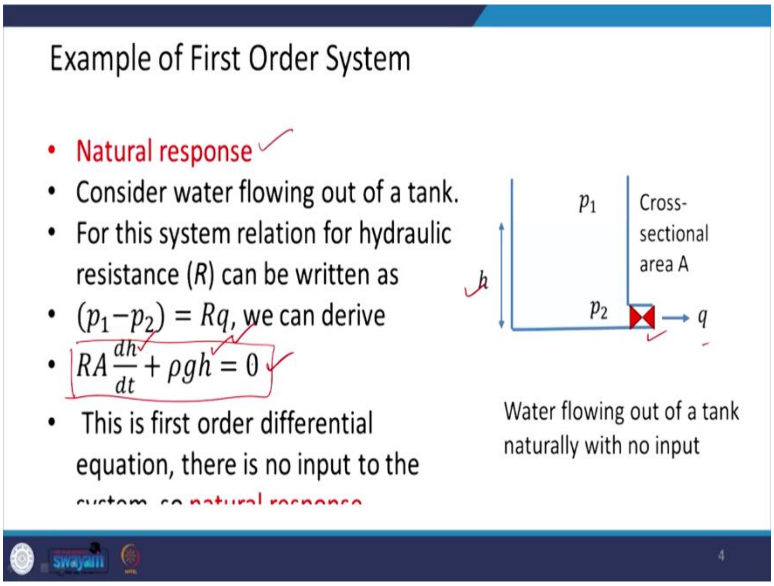

So, as I explained to you in the last lecture, suppose there is a tank, and there is a water-filled up to a height h, and pressure at the top and bottom are p_1 and p_2 respectively and there is a wall through which the discharge takes place, and as p_2 is the pressure hereafter the wall. So, we can write the expression for this as we have derived in the previous lecture. We can write the expression for flow through the wall, and the equation will be,

$$p_1 - p_2 = Rq$$

and this can be written in this particular form. So, this is our first-order differential equation. Here what you can see is that we are getting expression in terms of the height h of the water in the tank. And it is a natural response, so there is no forcing function here. So, we have the right-hand side equal to 0.

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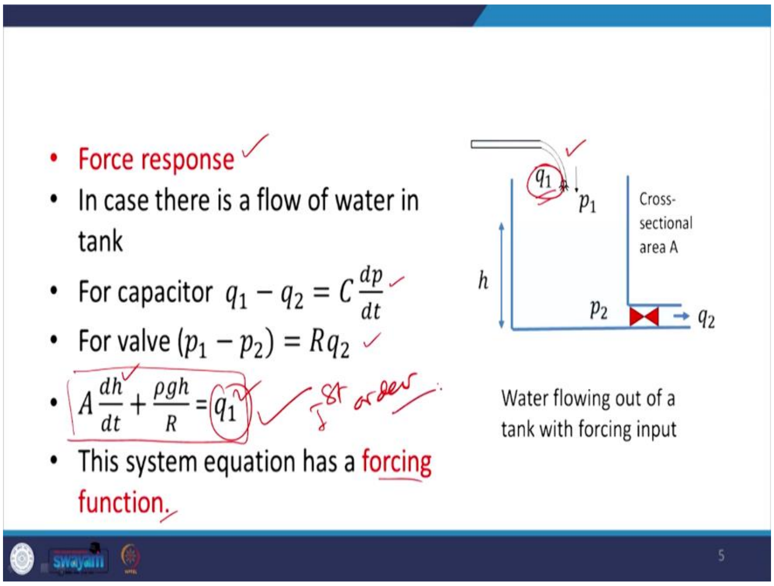

If I talk about force response, it can be considered if the flow is taking place with the help of there is some input over here q_1 and the rest of the things are the same. So, as we have seen in the last lecture using the equation for the capacitor as well as the expression for the resistance through the wall. We can derive the equation like this. And here again, you can see that we have the equation that depicts the variation of h with time, and on the right-hand side, we have q_1 Which is nothing but the input to the tank over here. So, this system equation has got a forcing function.

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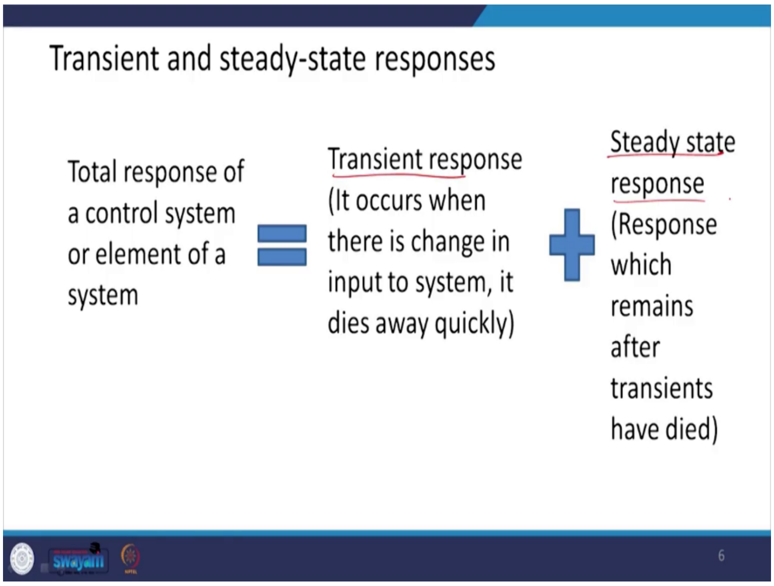

Now, the total response of any system is consists of the transient response as well as the steady-state response. Transient response is the one in which the change occurs when there is a change in the input to the system, and it dies away quickly. And the steady-state response remains, that is, the response that remains after the transients have died out.

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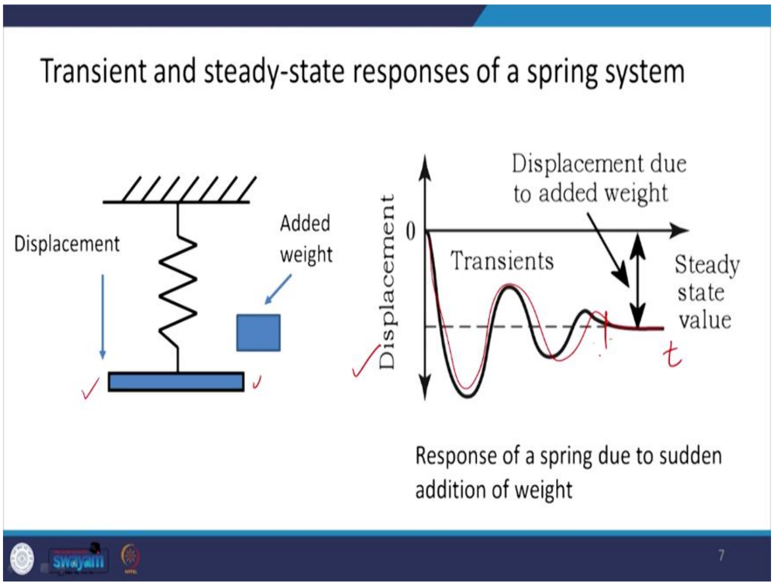

The example of transient and steady-state response, we can just consider a spring, and there is a hanger at the end of the spring, and if I add a certain weight to that hanger, then the spring will be getting displaced. So, this is how the displacement of the spring of if I plot with time, so this is how it is going to be. So, this is our transient response, and this part is the steady-state response. So, this is what I mean by the transient and steady-state response.

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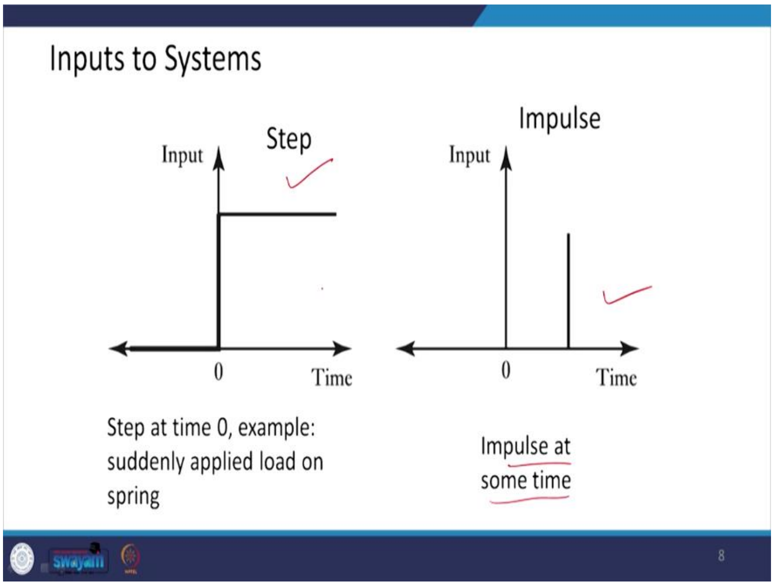

Now, the various input types of input can be used to excite the system. So, it could be a step input, or it could be impulse input, or it could be a ramp input, or the input could be a sinusoidal input.

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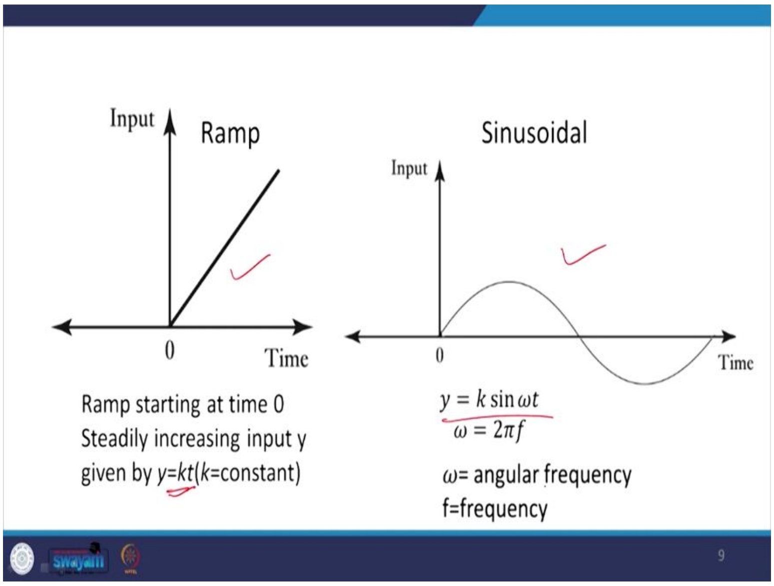

So, ramp input is defined as $y = kt$, sinusoidal we can define as $y = ksin(\omega t)$, and similarly, this impulse at some time and step has a certain value, at time t greater than 0. So, this way, these are the various types of inputs.

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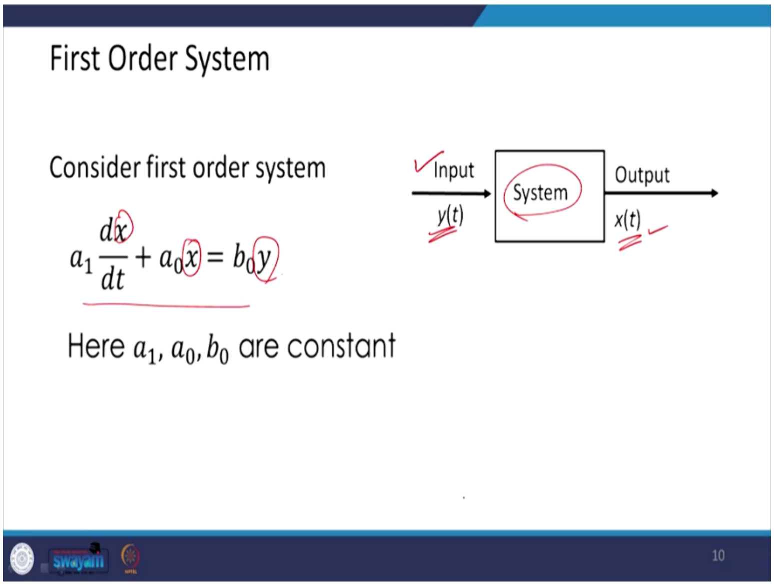

Now, let us take the example of a first-order system. So, first-order system, suppose I have got a system there is some input y(t), and there is some output x(t). So, what we are interested in seeing is that for a given type of input, what my output varies, how my output varies. So, this is the form of the expression for the first-order system, which we have just seen over here.

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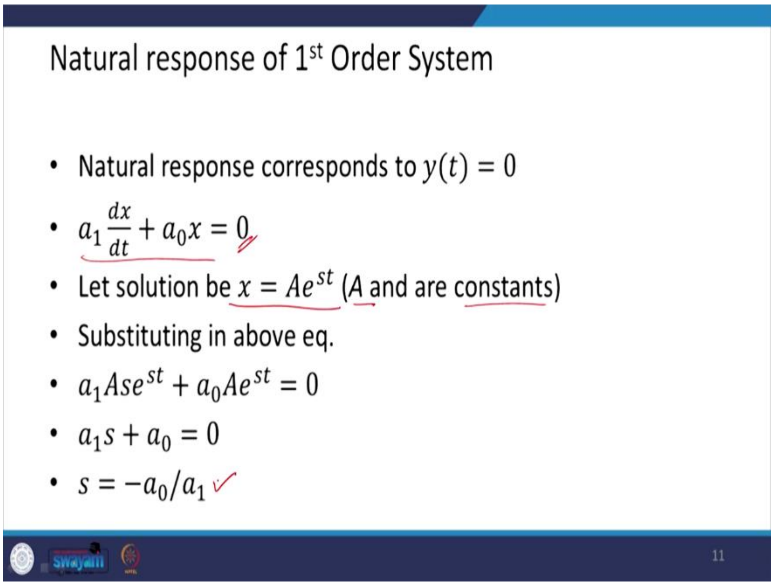

Now, if I talk about the natural response of the first-order system so, make the forcing function 0 here if I want to have the natural response. As it is a first-order differential equation, so its solution could be of form$x = Ae^{st}$, where A and s are constants. And this if we substitute it over here then we can get the value of s as this one.

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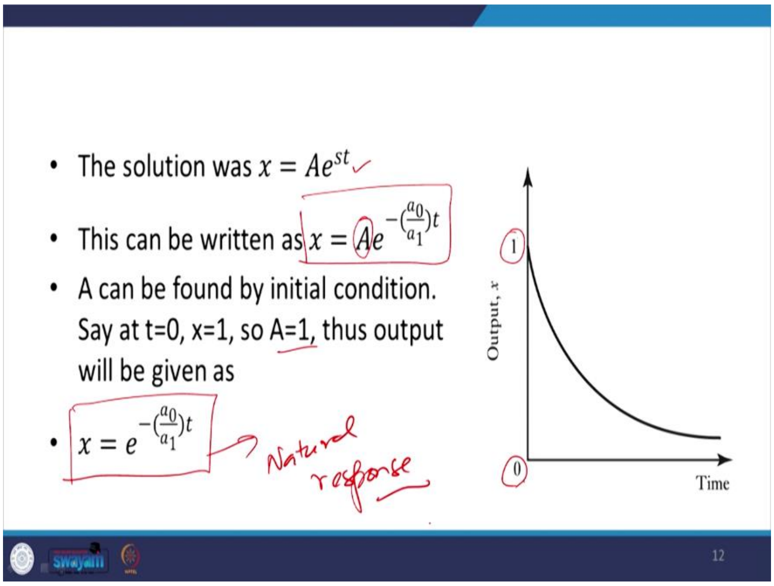

Now, if I substitute the value of *s*, this is the solution of the equation that I get. Here there is an initial *A* constant that can be evaluated by the initial condition. So, suppose at time t = 0 and put x =1, so I will get A = 1 from here. So, this is the solution for the expression. So, this is what I said as the natural response. So, this is the natural response for the first-order system.

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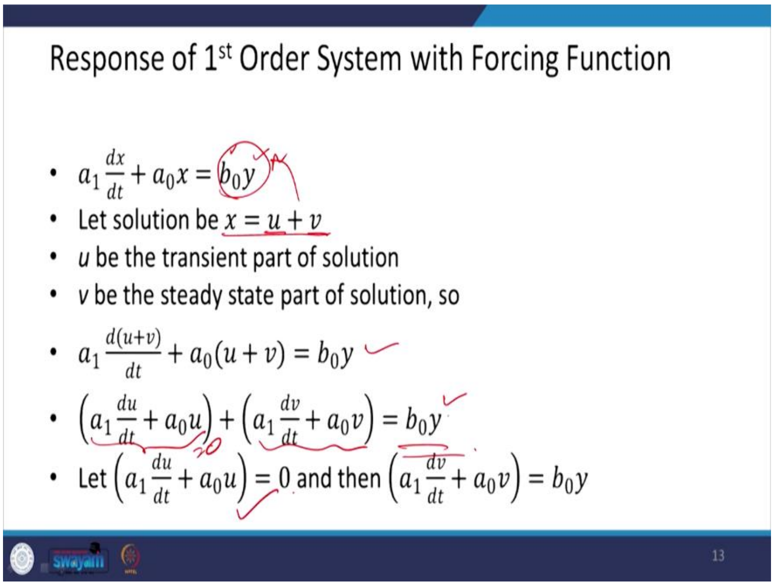

Now, let us see the response of the first-order system with the forcing function. So, suppose I have got a forcing function here$b_o y$, where y is the input, and b_o is some constant. So, it is a solution. Let us assume that the solution has got a transient part u and steady-state part v. So, I can assume the solution of this form. I can substitute the solution in this expression, and then I can simplify it, so I get this form. So, if I assume this part is equal to 0, then naturally, this part happens to be equal to 1.

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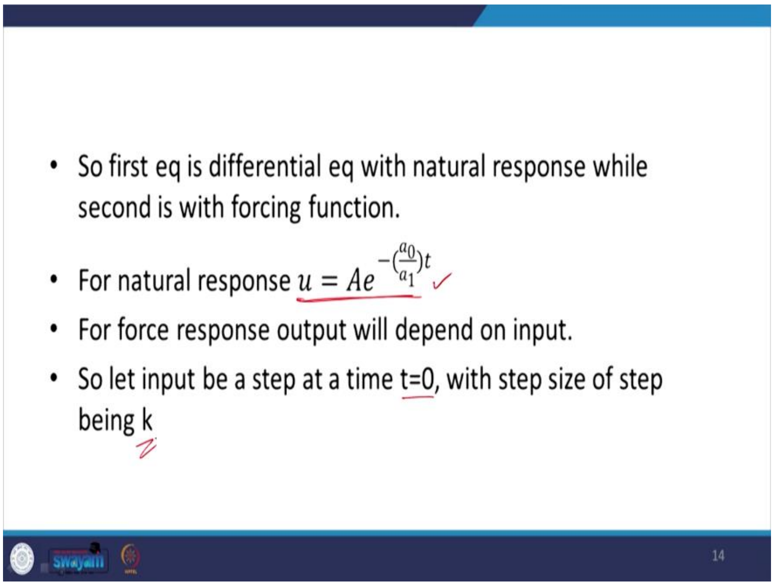

So, now let us try to find out the solution for the first equation that is with corresponds to 0 at the right-hand side over here. So, again it's the natural response. So, its solution is going to be this one. And for the force response, we can assume a step input, at t, is equal to 0 with a step size of k. If I do that, this is what is my equation.

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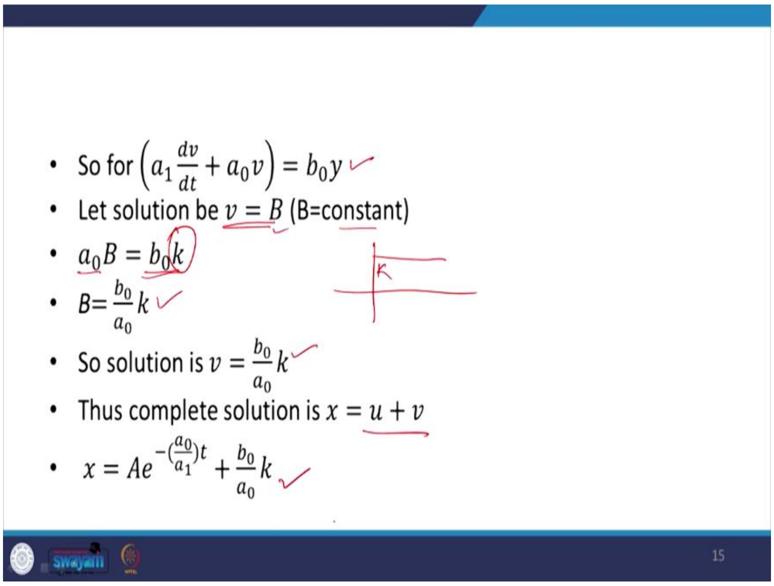

As I said, I can assume some solution for that $v = B$, which is a constant. If I substitute it over here, this term will become 0 because I am taking b as a constant. So, from here, I get the value of B. So, I get the solution. So, I can write the complete solution like this. This is the case for the solution of a first-order differential equation with a forcing function.

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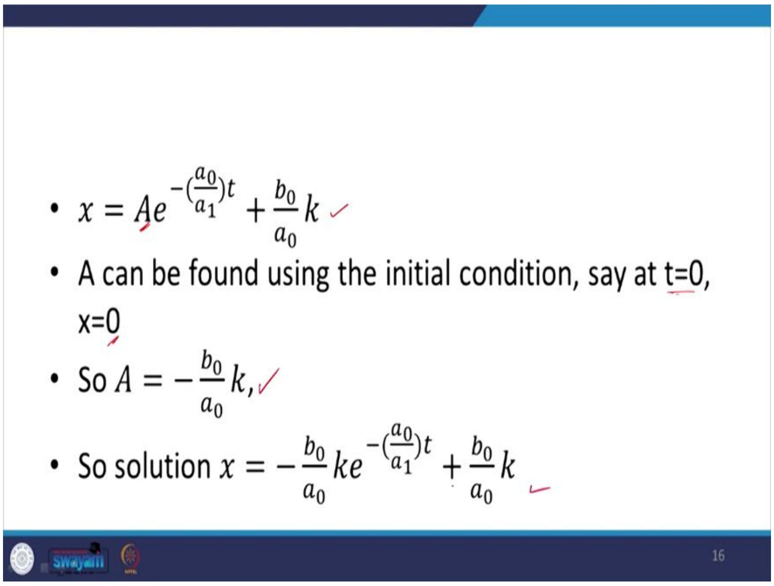

So, I can get the value of A by substituting that t = 0, x = 0. So, if I put it over here, I can get the value of a like this one. I can substitute it over here.

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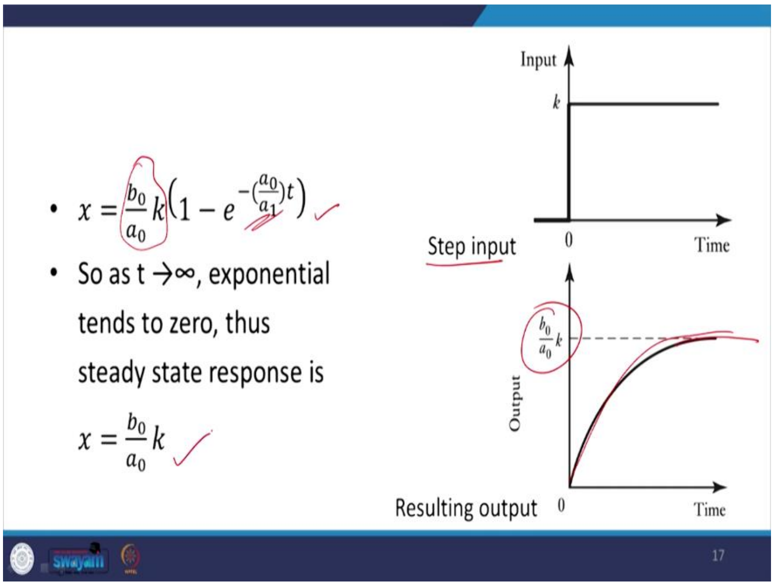

So, this is my final response. So, this is my step input which I said in my first-order differential equation with a forcing function. So, this is my response. So, here as you can see that if I take the value of *t* is equal to infinity. So, this value is going to be 0. So, I will be getting this value. So, this is what I am getting at $t \to \infty$ infinity,

$$x = \frac{b_o}{a_o} k$$

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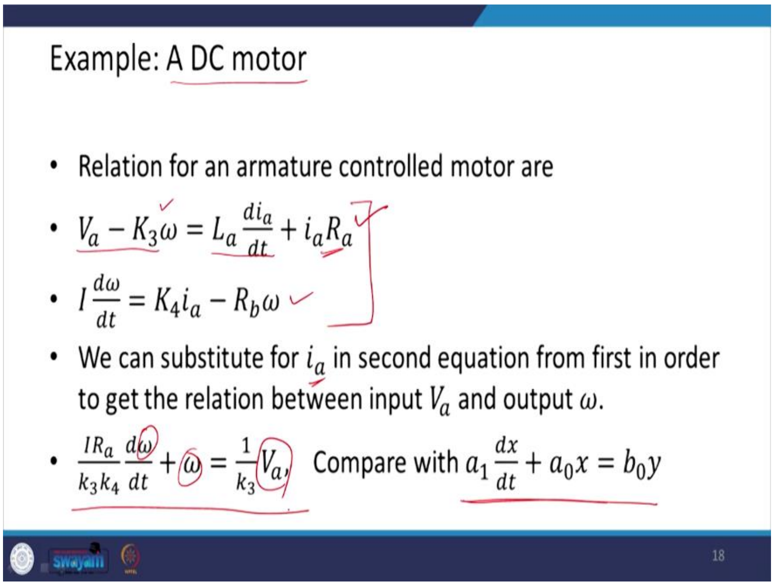

So, the previous example which I have shown was a case of a tank on which water is filled up. We can have another example of the first-order system with a forcing function as that of a DC motor. So, in the case of DC motor, as we have seen in modeling of electrical systems, we have this as the these are the equations. It is the equation for the armature control motor where this is the voltage supply to the armature, and this is the back emf, and so and these are the voltage across the inductance, and this is across the resistance. And this is the equation for the load. So, here we can simplify these equations that are a substitute for i_a from the first one and second one. So, this is what I can get. So, this is also a first-order differential equation of this form similar to what we have seen as the water being filled up in a tank. So, here you can see that output is ω and input is the voltage supplied to the armature. So, we can have a similar treatment way.

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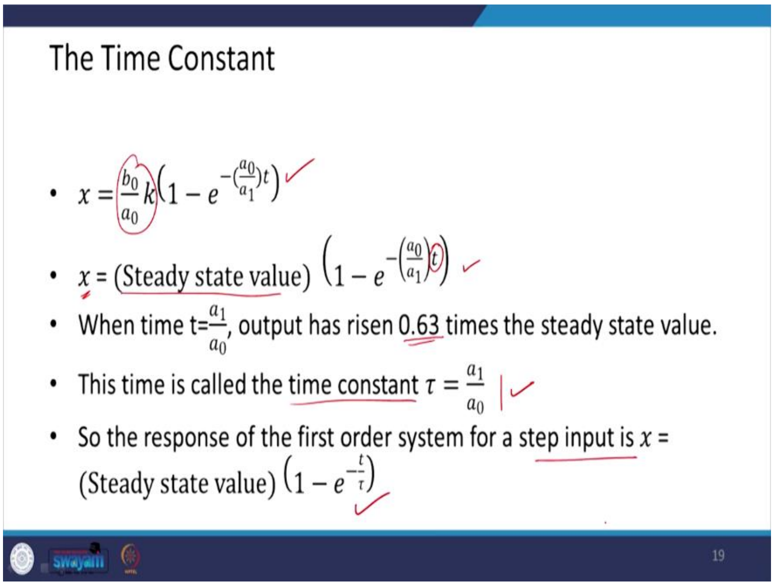

Now, if I look at this expression for the solution for the differential equation with a forcing function that is the first-order differential equation, this is my expression, and we have seen that this is the steady-state value. So, this is a steady-state value in this part. Now, here you see that if I write $t = \frac{a_1}{a_o}$, then these terms will be getting canceled, and we will have that is the value of x. And that will be 0.63 times the steady-state value. And this time is called the time constant. So, this is $\tau = \frac{a_1}{a_o}$ that is known as the time constant, and if I write it in terms of that, then this is my response to the first-order system for the step input.

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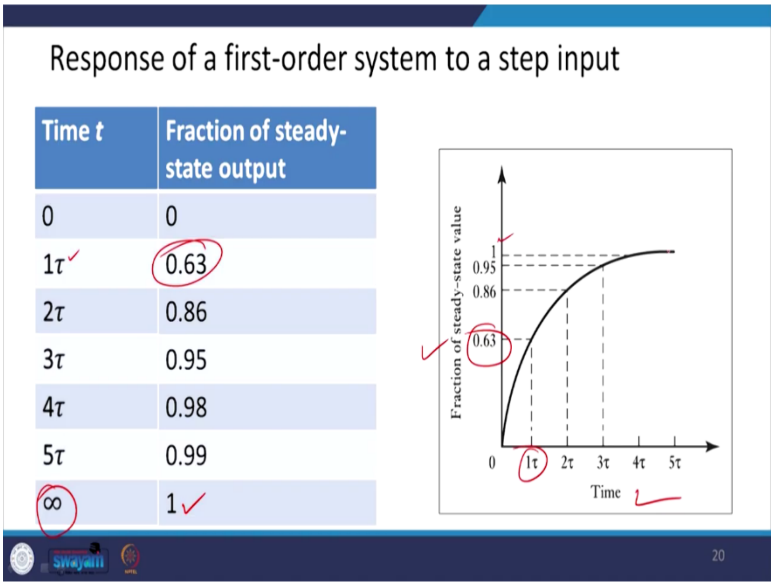

So, τ=1 or 1 τ, this is 0.63. So, the fraction of steady-state value if I plot over time, so this is 0.63 at 1 τ, and this way I can have the response, so you can see that at time infinity I get the steady-state value that is 1 over here.

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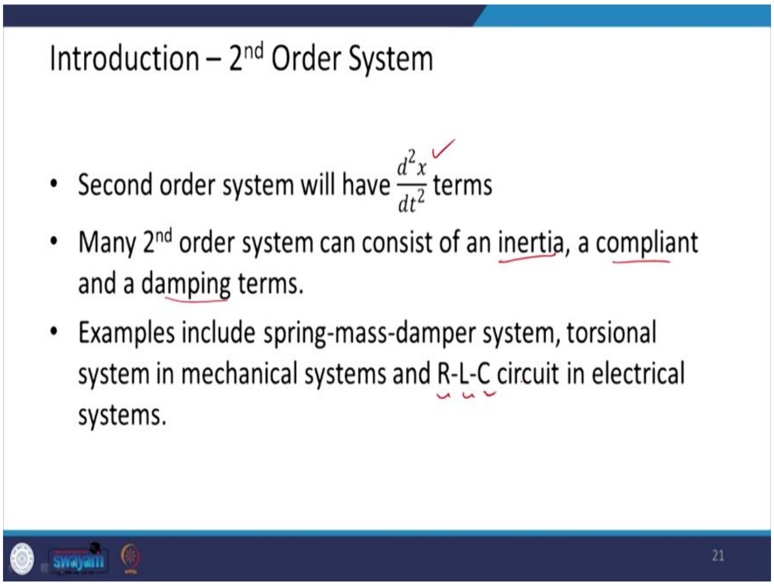

Next, let us talk about the second-order system. So, the second-order system, as we have seen that they have $\frac{d^2x}{dt^2}$. And the second-order system can consist of inertia, compliance, and damping for a mechanical system, and we have also seen on several occasions during this course the electrical system can have the resistor, inductor, as well as capacitor. They also represent the second-order system.

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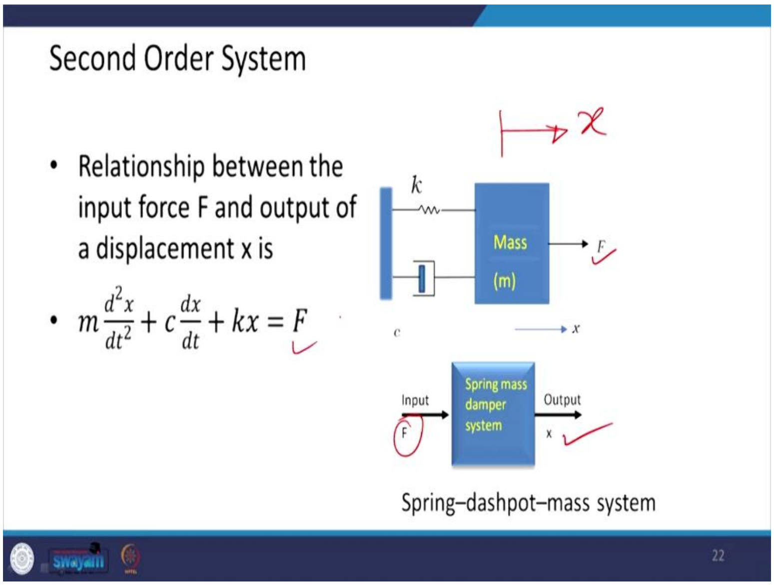

So, let us take the second-order system, a simple spring-mass damper system over here. So, this spring mass damper system is excited by an external force F. So, in this case, my input is F, and the output is this displacement x over here. So, I can write the equation of motion for this system by drawing the free body diagram, and it is this,

$$m\frac{d^2x}{dt^2} + \frac{cdx}{dt} + kx = F$$

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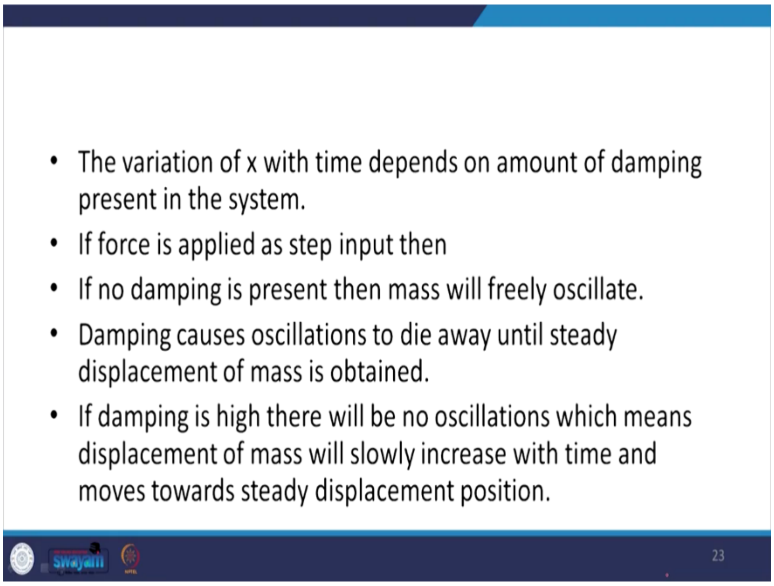

So, the variation of x with time depends on the amount of damping present in the system, and the force is applied as a step input, then. If there is no damping present in the system, then the mass will keep on oscillating, and the damping causes the oscillations to die away unless the steady displacement of the mass is obtained. If damping is high, then there will not be oscillations and which means that the displacement of mass will slowly increase with time and move towards the steady displacement position. Now, first, let us consider the second-order differential equation with no damping.

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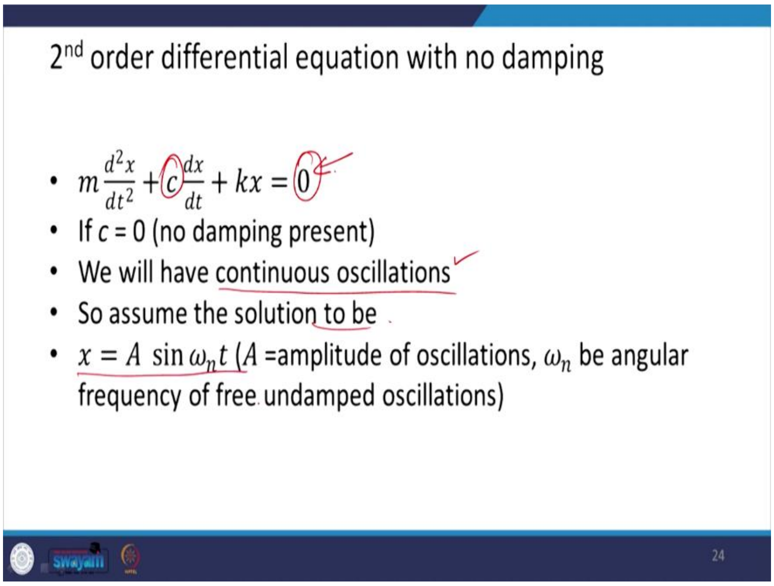

So, if there is no damping and there is no input excitation, I take here 0, and I have my input force 0, and I can take this c = 0. So, we will have because damping is not there, so we will have the continuous oscillations, and we can assume the solution of this form,

$$x = A\ sin\omega_n t$$

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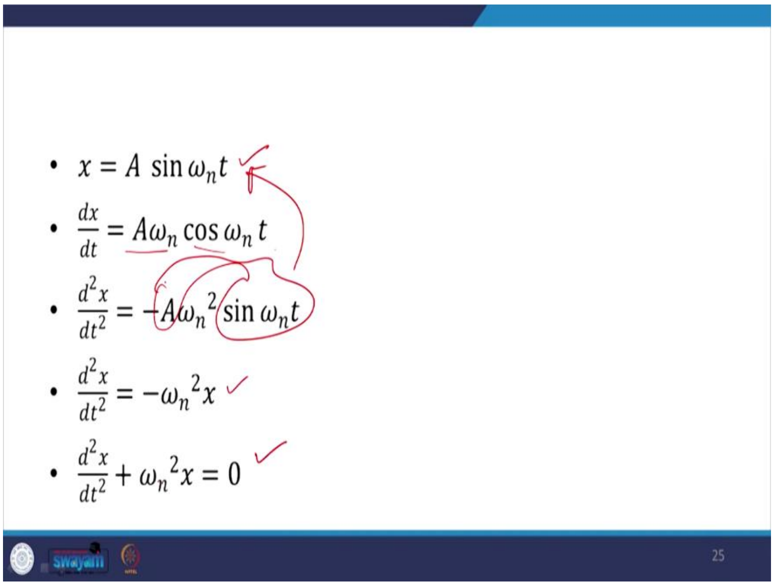

If I take the first derivative, so

$$\frac{dx}{dt} = A\omega_n\, cos\omega_n t$$

And if I take the second derivative, then this is,

$$\frac{d^2x}{dt^2} = -A\omega_n^2\, sin\omega_n t$$

Now, you see this is $A\, sin\omega_n t$ is the same as that of x. So, I can write this as $-\omega_n^2 x$ Or I can write it in this form.

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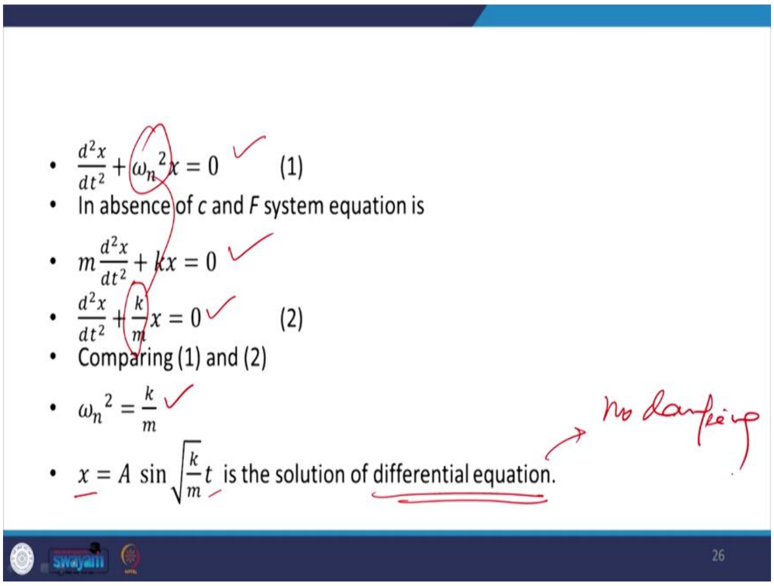

Now, if I compare this equation with the equation without damping, that is this one, then you can see that the, $\omega_n^2 = \frac{k}{m}$. So, I can write the solution as,

$$x = A\, sin \sqrt{\frac{k}{m}} t$$

and this is the solution of the differential equation if there is no damping.

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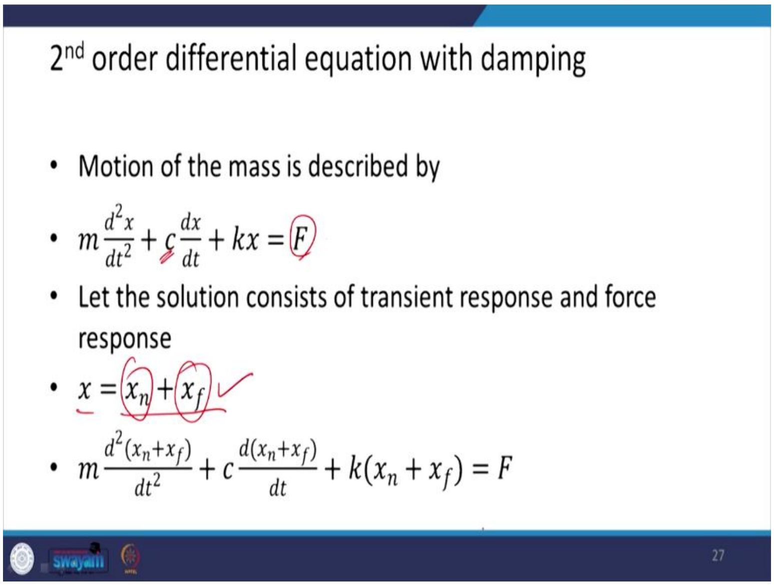

Now, what if there is damping? So, if damping is present, then this term is not going to be equal to 0, and we have the excitation force also F being over here. So, again here, I can take the solution to be composed of two forms is the natural response and the force response, and as we have done earlier, we can substitute this solution in this expression, and we can segregate the two parts over here.

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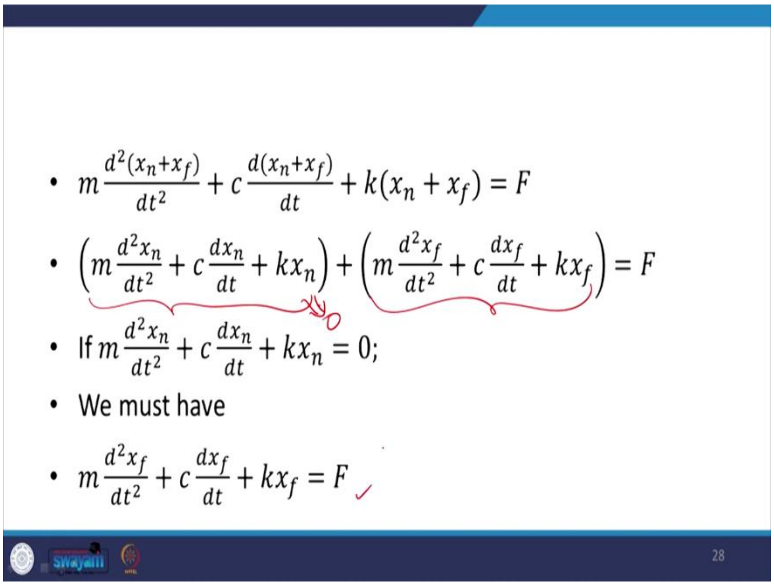

So, this part we can equate to 0, which will be giving the natural response, and this part will be going to equal to this one, and that will be giving us the force response.

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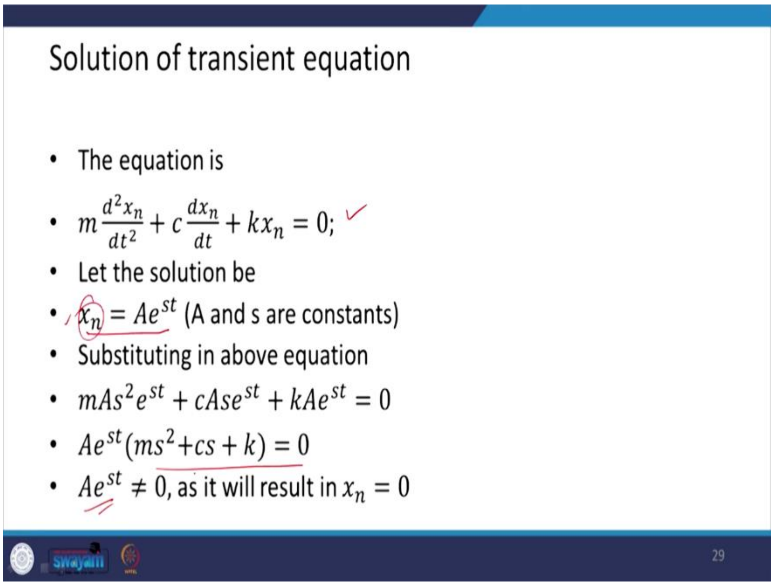

So, if we look at the solution for the transient equation that is for the natural response, so here I have I can take, $x_n = Ae^{st}$ as the solution, and if I substitute it back over here. So I get this one, and this cannot be equal to 0 because if I make it is equal to 0, this itself will become 0, so I make the other equal to 0. So, this is my auxiliary equation. So, from here, I can get the value of s. It is a quadratic equation. So, its solution is going to be this form.

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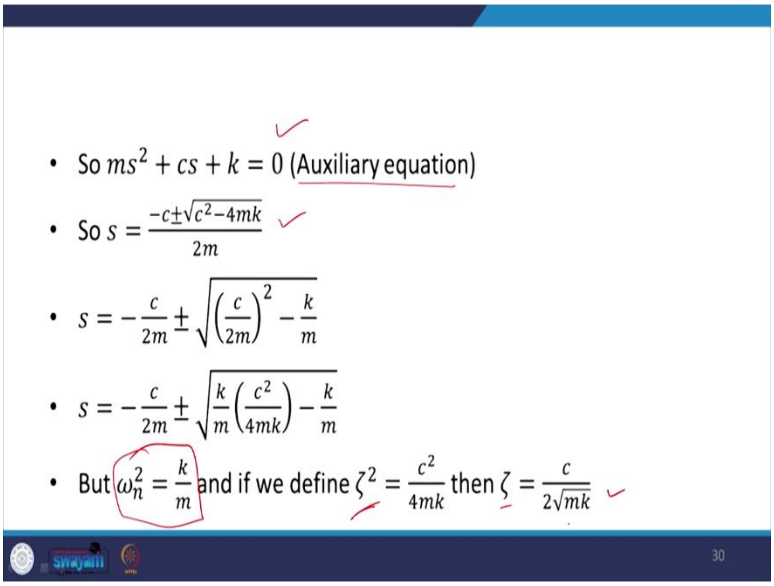

And if I define this, $\omega_n^2 = \frac{k}{m}$ as we have seen for the natural response and I define, $\zeta^2 = \frac{c^2}{4mk}$, then$\zeta = \frac{c}{2\sqrt{mk}}$. So, I can substitute that over here, and I can get the root of this form.

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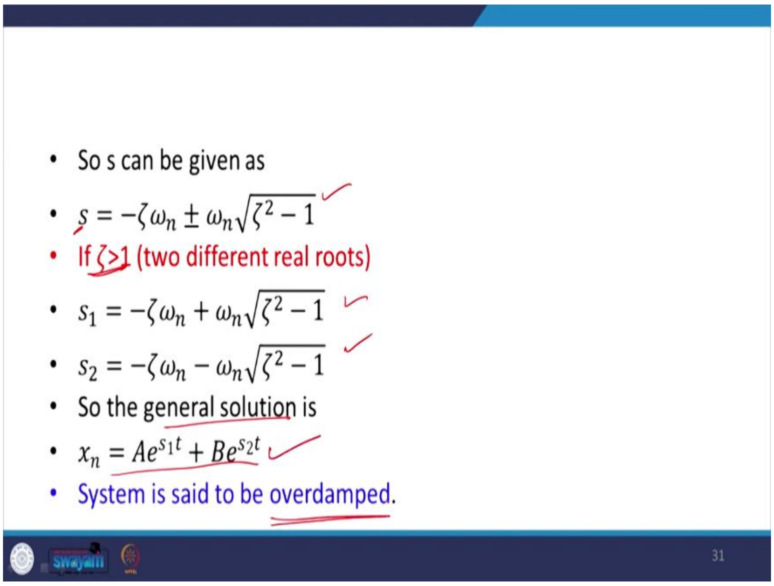

Now, here there are 3 cases, we are going to have this if $\zeta$is going to be more than 1, then these are the two solutions which I am going to get s_1 and s_2, so the general solution is going to be this one. And we see that the system is overdamped.

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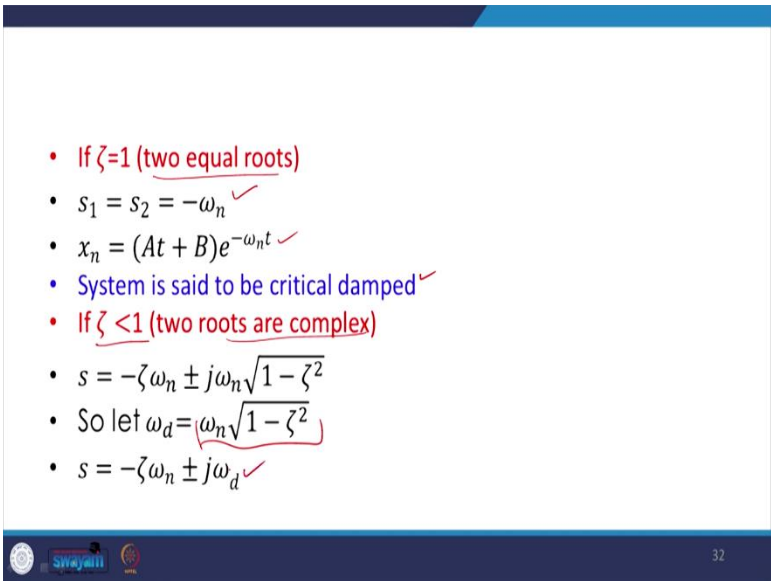

And if the $\zeta = 1$, this is going to be the solution, these are going to be the roots both equal, and this is going to be the solution, and the system is said to be critically damped. And if $\zeta < 1$, then the two roots that are complex are going to be there. We define this as the damped frequency, and the two roots can be represented like this. So, this is going to be the general solution.

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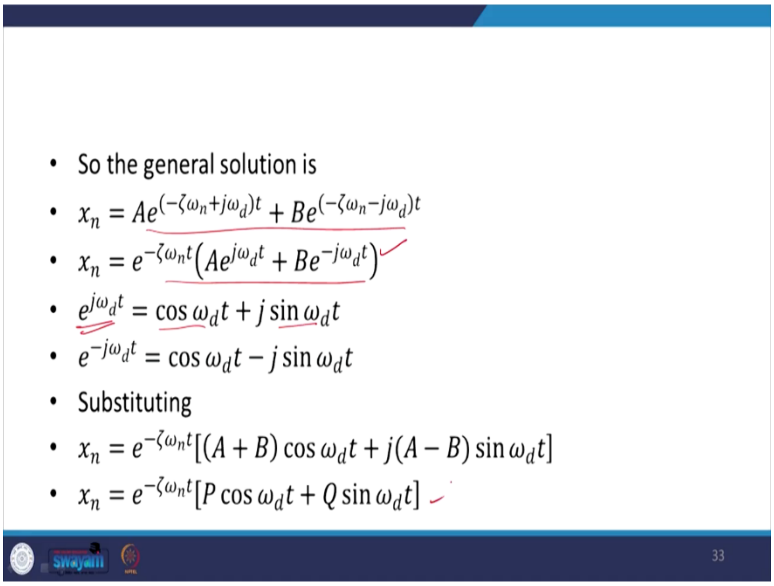

If I just substitute, I can further simplify this solution by representing the exponential function as cosine and sin functions. And this is going to be my natural response in that particular case.

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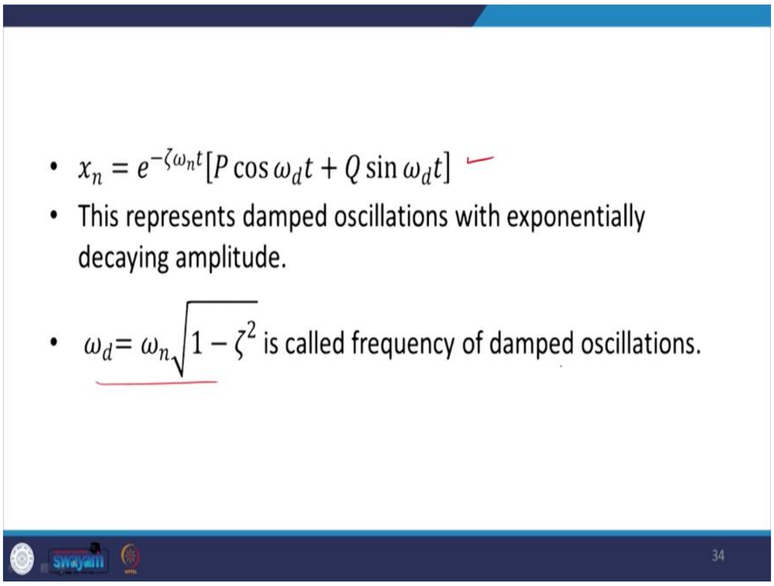

So, this is the natural response, and this one is called the damped frequency of oscillation, as I have explained to you.

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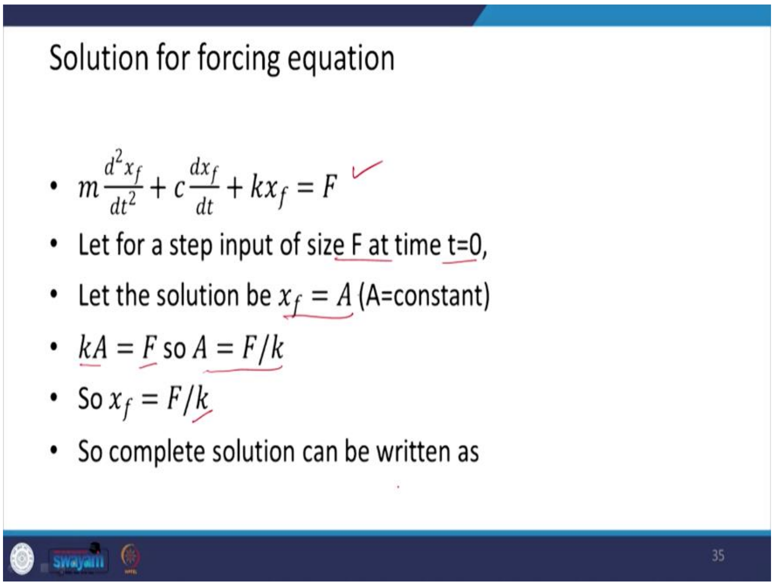

Now, the solution for the forcing equation if you look at this one. So, as we have seen for a step of size F at time t = 0, we can take the solution $x_f = A$. So, kA = F, so the A = F/k, and $x_f = \frac{F}{k}$.

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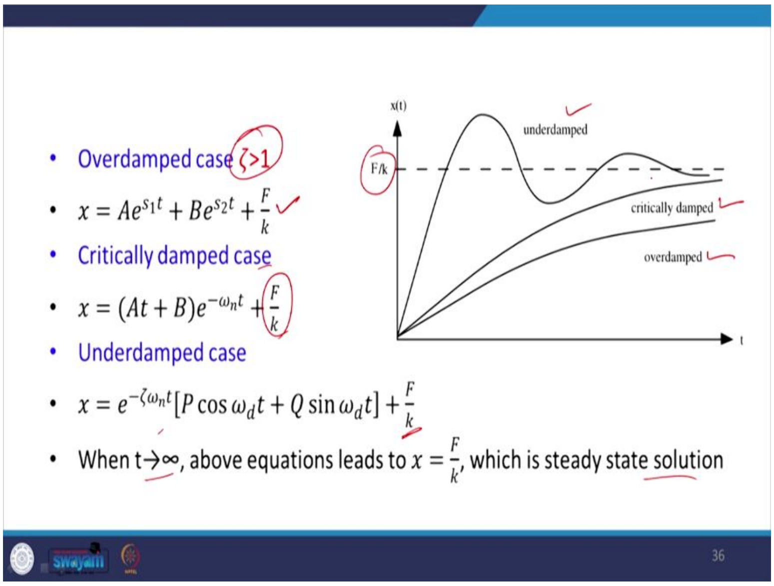

So, my complete solution I can write for all the 3 cases that are for $\zeta > 1$. I edit up here overdamped case, for the critically damped case, I edit up here, and for the underdamped case, I edit it over here.

So, when $t \to \infty$ above equation leads to x =F/ k, which is the steady-state solution over here. So, here you can see that this is the overdamped case, critically damped case, and this is the underdamped case, and this is F /k is the steady-state solution.

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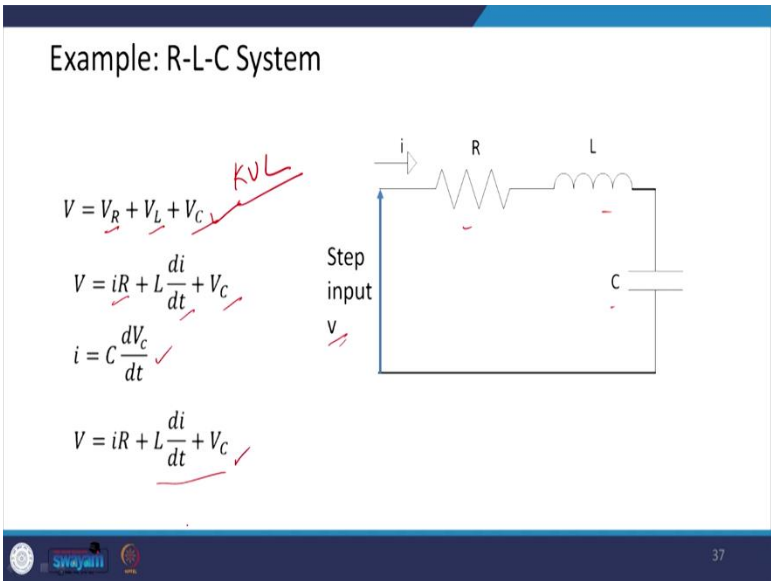

Another example, as I was telling you in the electrical system, it is a simple RLC circuit which is an example of the second-order system. We can, for a given step input I can write the Kirchhoff's Voltage Law, which is a KVL, and I can write this voltage across resistor, inductor, capacitor, and *iR*, Ldi/dt and V_c respectively. I can write it like this. So, this way, I can write an expression like this one.

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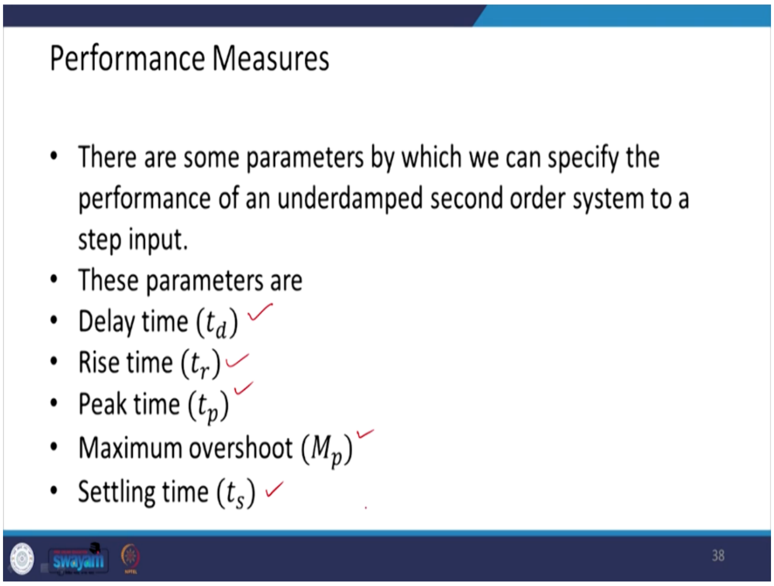

Then, there are various performance measures parameters, and these parameters are how the system is responding. Those parameters can be measured with the help of delay time, rise time, peak time, maximum overshoot, and settling time. So, with the help of these parameters, we can measure the performance of the dynamic system.

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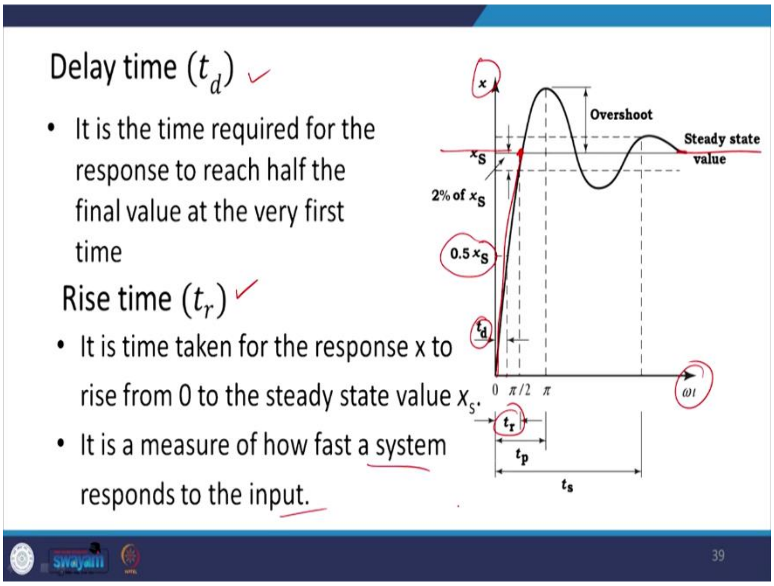

So, what is the delay time? So, if I plot x vs ωtover here, it is the time required for the response to reach half of the final value at the very first time. So, this is my steady-state value that is the final value, so 0.5 of x_s it reaches over here. So, this time is what I am calling the delay time. The rise time is the time taken for the response to rise from 0 to the steady-state value. So, this is a steady-state value. So, this is the rise time that is moving from here to, moving from here to here whatever time is taking that is the rise time. And its measure of how fast a system responds to the input.

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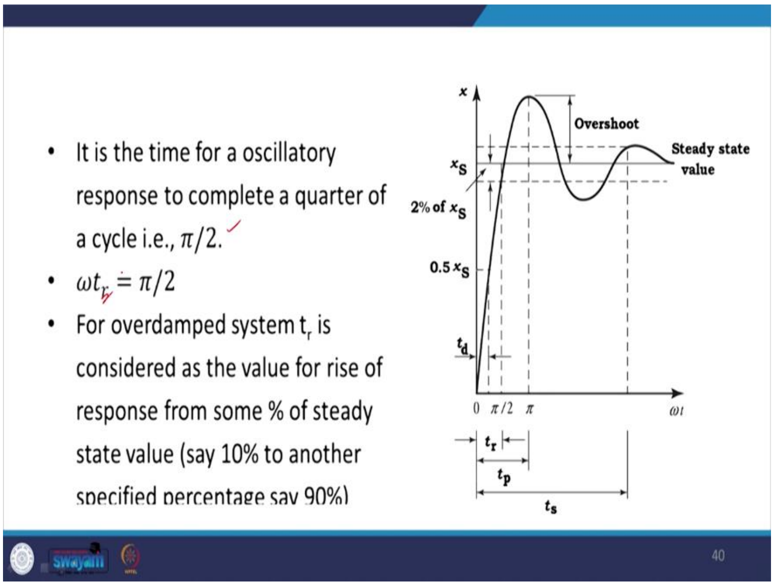

And it is the time for oscillatory response to complete a quarter of a cycle. So, this $\omega t_r = \frac{\pi}{2}$. So, this way we can find out the rise time. And for overtime system t_r is considered as the value for the rise of response from some percentage of the steady-state value, so 10 percent to another specified percentage 90 percent.

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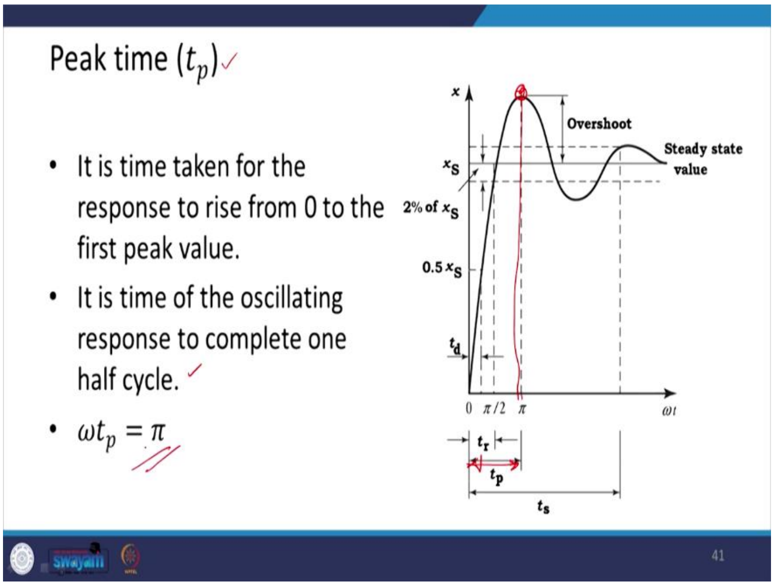

Then another parameter is the peak time, and it is the time taken for the response to rising from 0 to the first peak value. So, here you can see that the first peak value is coming over here. So, it is this time. This is your peak time. And it is the time of the oscillation response to complete one half of the cycle. So, this is how it is written as π over here.

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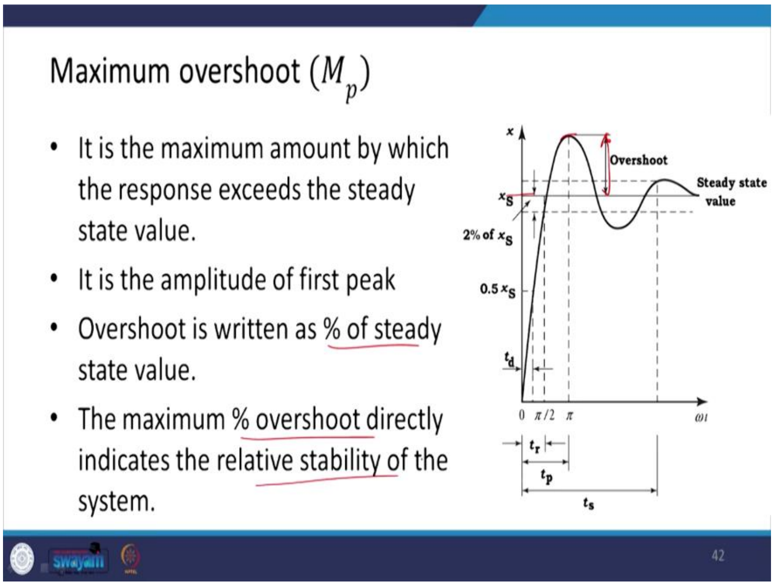

Then, you have the maximum overshoot. So, this is from the steady-state value. Whatever value response, how much is the maximum value from the steady-state that is what is called the overshoot. So, it is the maximum amount by which the response exceeds the steady-state value. And it is amplified, the amplitude of the first peak. And this overshoot is written as a percentage of the steady-state value. The maximum percentage overshoot directly indicates the relative stability of the system. And this maximum overshoot can be derived to be like this in terms of the steady-state value. And in terms of the maximum percentage overshoot, it could be derived like this.

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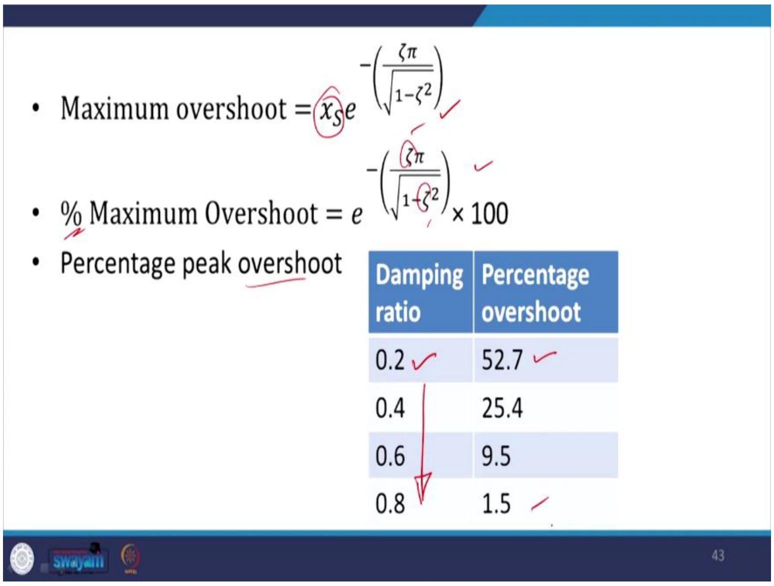

I am not working on the derivations over here. You can refer to the reference books which I will be telling you towards the end of this lecture. So, the percentage of the peak overshoot is going to be as you can function of the damping ratio over here. So, for the damping ratio, 0.2 overshoot is 52.7, and so on. So, as you can see, as we are increasing the damping ratio, the overshoot is decreasing. So, that is one of the observations which I wanted you to observe.

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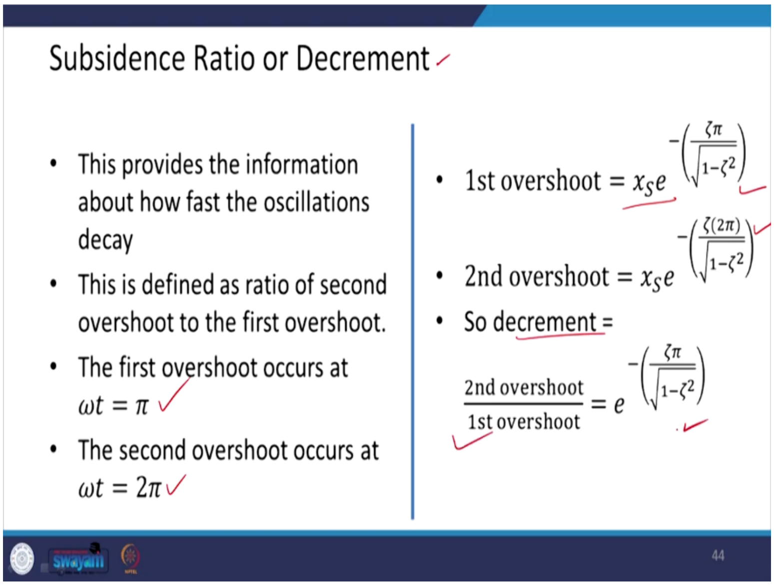

Then, the subsidence ratio or the decrement is again another major performance parameter. This provides information about how fast the oscillations are decaying, and it is defined as the ratio of the second overshoot to the first overshoot. So, the first overshoot occurs at $\omega t = \pi$, the second overshoot will be occurring at omega t is equal to $\omega t = 2\pi$.

So, the first overshoot can be written like this expression. The second overshoot expression can be written like this. So, the decrement expression can be derived in this way.

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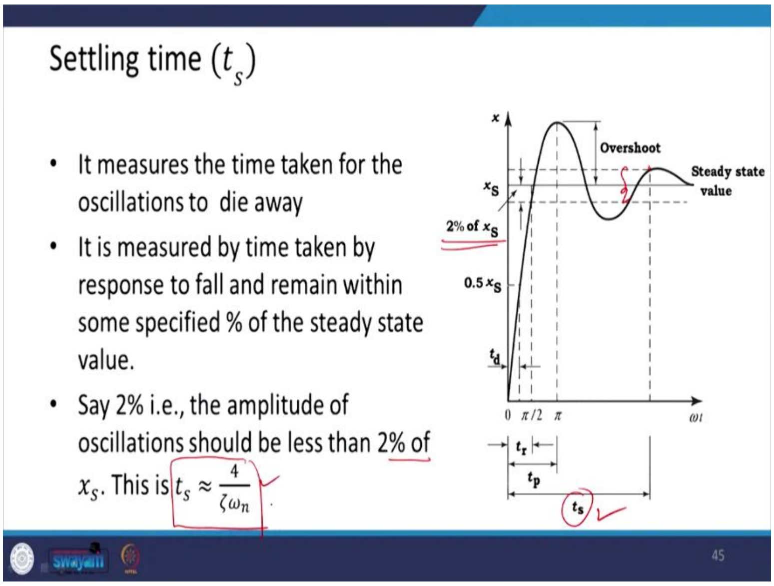

Then, the settling time. You see that for a system reaching the exact steady-state value may not be possible. So, when do we say that the system has reached the steady-state value? So, there is a certain percentage of the steady-state value of 2 percentage. If your system reaches two percent of the steady-state value, within that limit, if it is there, we that the system has got settled. So, the settling time this t_s for you can see that this zone is two percent of the x_s. Wherever this has been achieved, that time is called the settling time, and it is measured by the time taken by the response to fall and remain within some specified percentage of the steady-state value. And this, for the 2 percent of the settling of x_s, this value can be worked out as, $t_s \approx \frac{4}{\zeta w_n}$.

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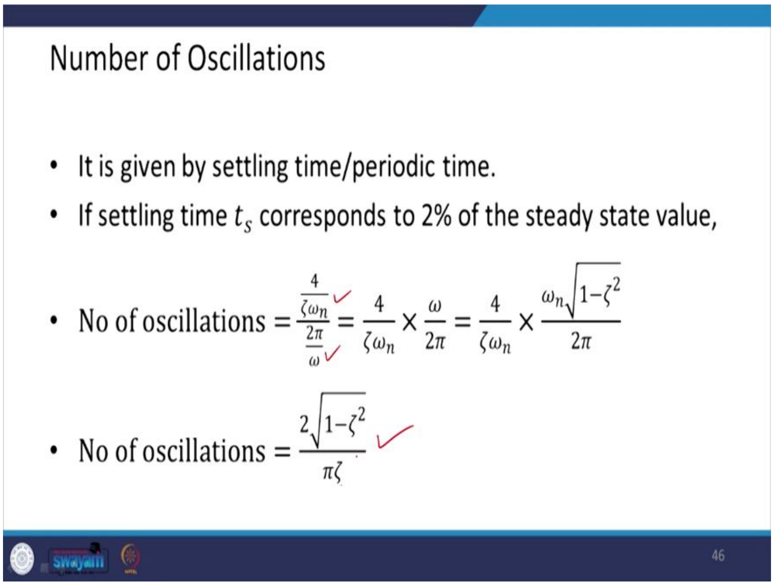

Then another performance parameter is the number of oscillations, and this is given by the settling time divided by the periodic time. So, the settling time of 2% settling time divided by periodic time, if I substitute it over here, this is how to get the expression for the number of oscillations which is again the function of the damping ratio.

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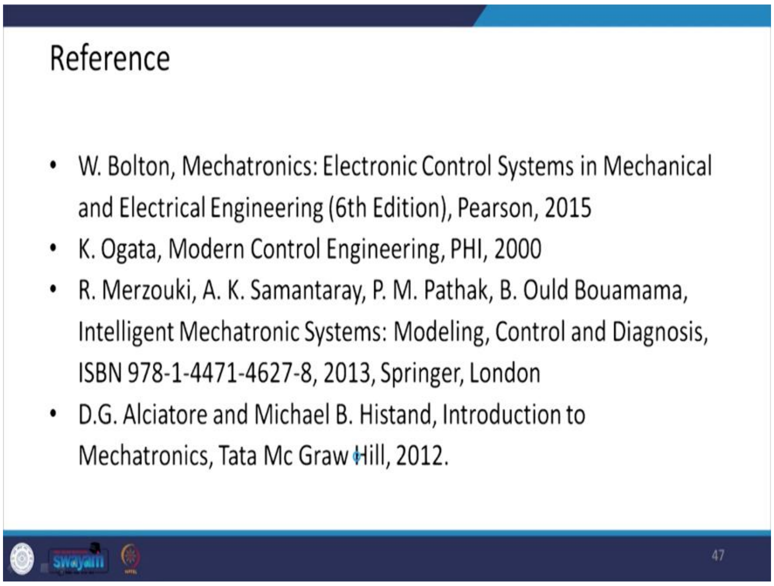

So, these are the further references you can look at if you want to read it further.

Thank you very much.