Mechatronics
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Lecture - 26
Mechanical System Model

Good morning. I welcome you all to this NPTEL online course on Mechatronics. Today, we are going to discuss the Mechanical System Model. You see that mechatronic components are, or I should mention that a mechatronic system, as I have defined in my earlier introductory slide, it is the electronic control of the mechanical system. And if we want to design a mechatronic system, we need to see the system behavior prior to actual fabricating the system. For that purpose, we need to model the system. So, in today's lecture, next few lectures, I will be discussing the modeling of the various type of systems that are often used in a mechatronic system. So, in this lecture, I will be focusing on the mechanical system model. We will be talking about a translational mechanical system, a rotational mechanical system, as well as a combination of translational and rotational mechanical systems. So, I will be explaining the modeling of this with the help of a few examples. I hope you will enjoy this lecture. Let us look at the mathematical models.

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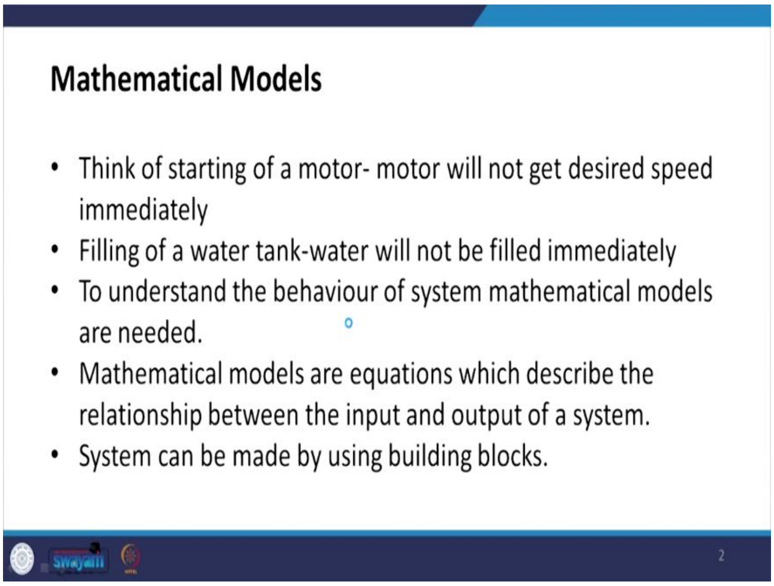

Suppose we think of starting a motor, and we want this motor to go to a certain speed. Naturally, the motor will not get the desired speed immediately. If we want to fill a water tank, let us take another example. If we want to fill a water tank, water will not fill immediately in the tank. So, to understand the behavior of the system how these things happen with time, how the water is getting filled with time, how the water level is changing with time or how the motor speed is changing with time, or how it is varying before it is reaching to the desired speed. So, to understand this behavior of the system model, mathematical models are needed. These mathematical models are the equations that describe the relationship between the input and output of a system. A system can be made by using building blocks. So, what could be these building blocks?

First, we can discuss these building blocks, and then we can identify if I have to model a system, I can identify which element of that system corresponds to which of the building blocks. And then, we can derive the mathematical model of a system.

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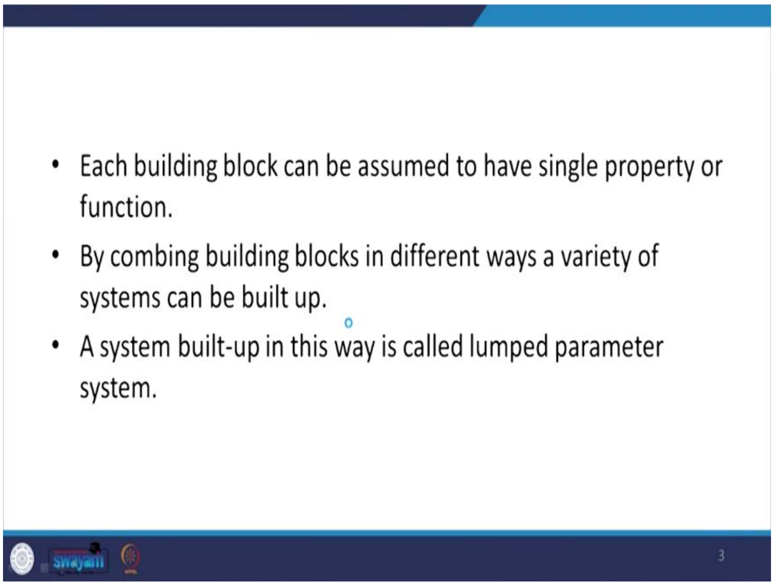

Each building block can be assumed to have single property or function. So, based on that, we have to identify the building block, and by combining building blocks in a different way, a variety of systems can be built. A system built up in this way is what is called the lumped parameter system. The lumped parameter means we are lumping the parameter into one.

So, now, let us see the mechanical system building blocks. The model which represents the mechanical system has three basic elements. A spring, damper, and mass are the basic building block of a mechanical system.

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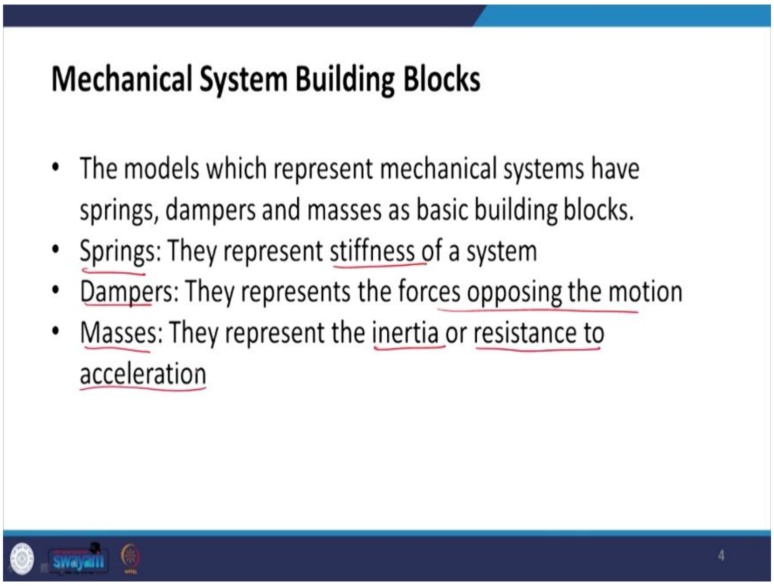

What do these building blocks represent? So, you see, the springs represent the stiffness present in the system, the damper represents the forces opposing the motion of the system, and the masses represent the inertia or resistance to acceleration. When we are modeling any mechanical system, we identify what the things present in that mechanical system that corresponds to spring, damper, and mass are. And this identification will help a lot in modeling a mechanical system model. So, you see, any mechanical system does not be made of spring, damper, and masses. What do I mean by this?

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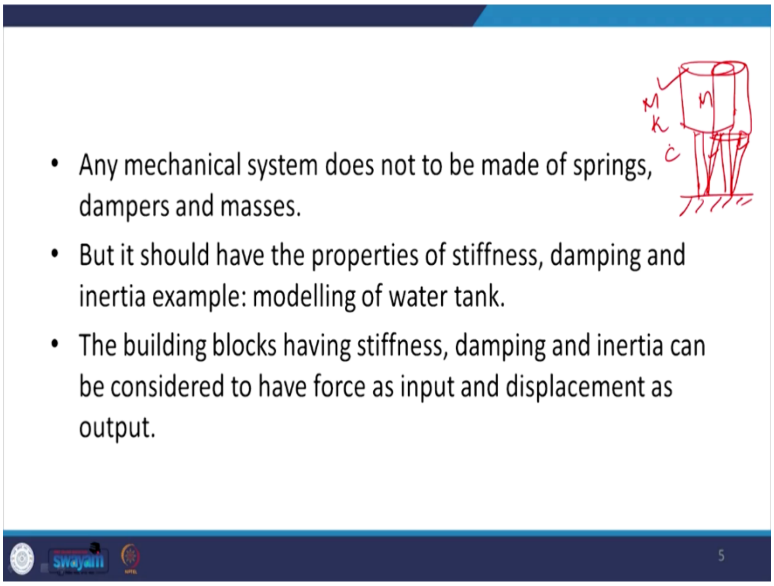

I mean by this that there may not be all these three elements present in that mechanical system to identify, as well as they should, but it should have the properties of stiffness, damping, and inertia. So, in a real system, you will not have these spring, mass, and damper systems, but whatever is present in the system that will have this type of property.

For example, If I ask you to model a water tank like this which is put on the pillars, then we can identify in this model what is spring, what is damper, and what is the mass. So, identification of mass is very simple, that is, the mass of the water in the tank, which constitutes the major part. If you want to be more precise, you can take the mass of the concrete, also which constitutes the water tank. Now, what is spring here? You cannot see the spring directly over here, but in this example, you see that the behavior of the pillars is like a spring. For example, if some high wind velocity, high wind forces come, then these pillars will try to deflect, then these pillars will try to deflect. This is the exaggeration I am showing just for you to understand. So, these will be trying to deflect like this. And what is a damper? So, damper could be you see that these are made of the concrete, pillars are there. There is material damping provided by these concrete pillars, so that could constitute the damping. So, in this way, the physical system exists. In this physical system, I can identify what my mass is, what my stiffness is, and what my damping is. The building block having stiffness, damping and inertia can be considered to have force as input and displacement as output. So, let us see that.

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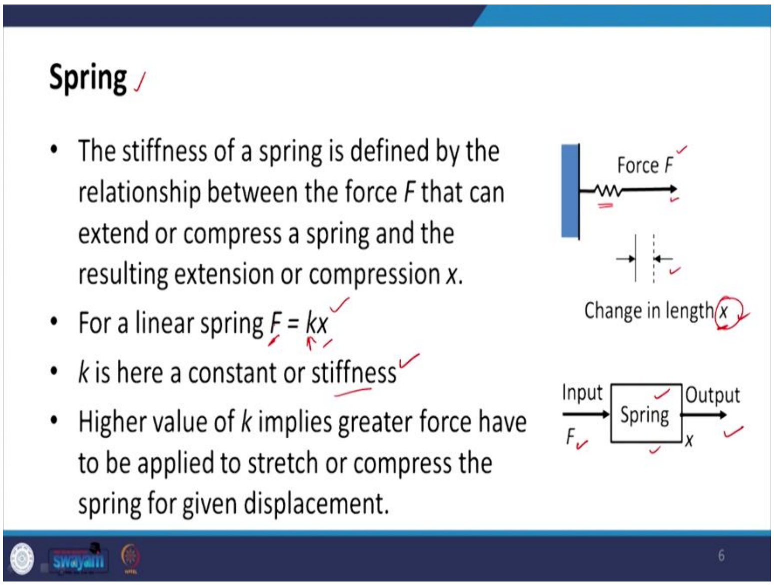

So, first, we begin with one of the building blocks that is spring. So, suppose this is the spring over here, and in this spring, one end a force is applied, and the other end is fixed over here. So, a force $\boldsymbol{F}$ is applied, and so, this is the change in the length which is indicated over here. Now, in this case of the spring, what is my input? Input is my force F. And what is my output? Output is my change in length $\boldsymbol{x}$. So, this is my spring building block. What is the expression? Now, to define the expression, we can go to the definition of the stiffness of the spring. The stiffness of a spring is defined by the relationship between the force $\boldsymbol{F}$ that can extend or compress a spring and the resulting extension or compression $\boldsymbol{x}$. What is that relationship? For a linear spring,

$$F = kx$$

So, for a given spring, the $\boldsymbol{F}$ is proportional to $\boldsymbol{x}$, and here the $\boldsymbol{k}$ is a constant, or that we call the stiffness. Now, a higher value of $\boldsymbol{k}$ implies that you need to apply a larger value of $\boldsymbol{F}$ for the same deformation or for the same displacement of the spring. Next, let us look at another very important element damper, how the physical, how a damper looks physically.

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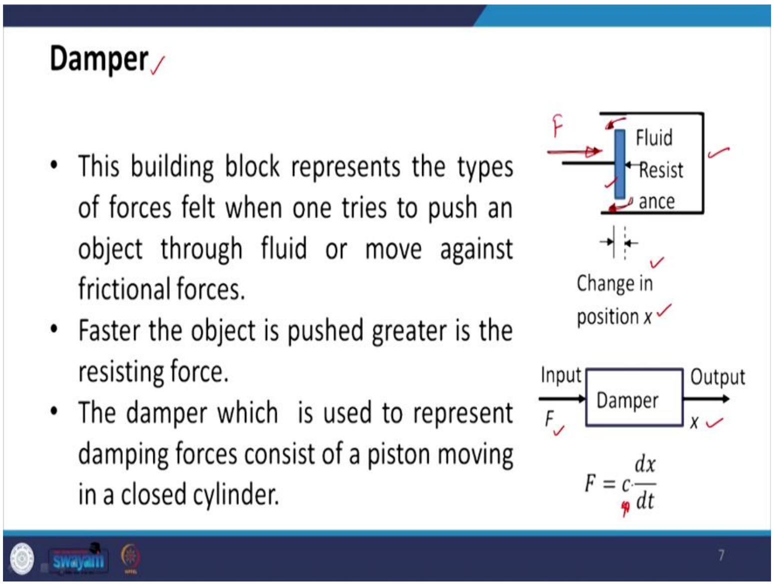

So, the damper has you can see that there is a cylinder and piston arrangement and there is a fluid inside of this. When this piston is moved, then this fluid is passed to allowed to pass through these constricted spaces, and this produces resistance. So, here because of the motion, there is a change in position $\boldsymbol{x}$ here.

So, the output is x here, input is F here the force which is being applied, so this is your F, this is your x, and the relationship for a damper is,

$$F = c\frac{dx}{dt}$$

Where this c is the damping coefficient, and this building block, damper building block, represents the type of forces felt when one tries to push an object through a fluid or move against the frictional forces. The faster the object is supposed greater is going to be the resistive force. And the damper, which is used to represent damping force, consists of a piston moving in a closed cylinder, as I explained to you. Now, when the piston is moved, the fluid on the other side tries to flow through the flow or pass the friction, as I explained, and this flow produces the resistive motion.

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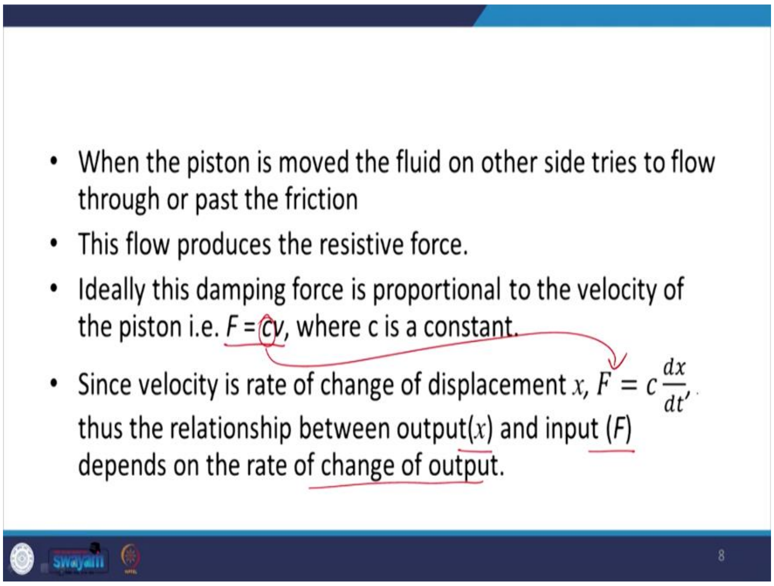

And ideally, the damping force is proportional to the velocity of the piston, and you have (F=cv) here, where c is a constant. And you see the velocity is what? Velocity is the rate of change of displacement. So, I can write this as,

$$F = c\frac{dx}{dt}$$

So, thus the relationship between the output x and the input F depends on the rate of change of the output. Next, let us look at another building block that is mass.

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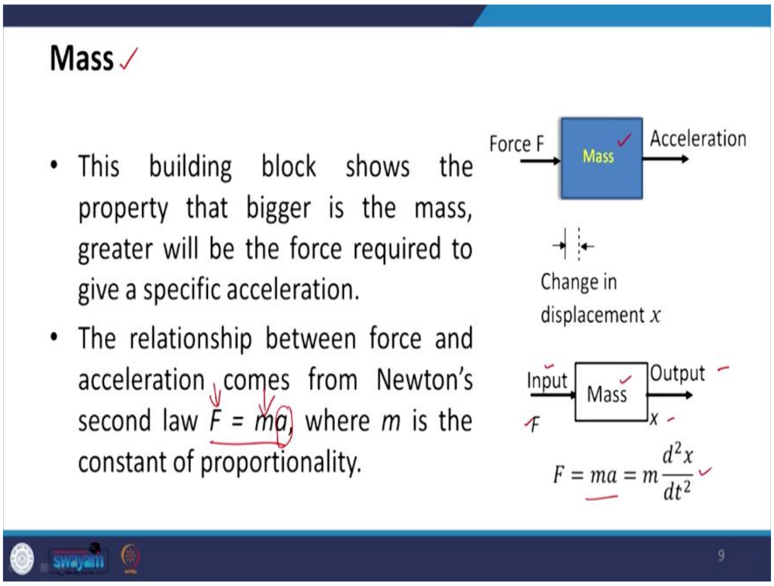

So, you have a mass over here, and you see what does this block? This building block shows the property that the bigger is the mass greater will be the force required to give a specific acceleration. So, that is there. So, the relationship is F = ma, as we know from Newton's law. So, if your **a** is constant, that is, you want to give a specific acceleration, so if your mass increases, you need a greater amount of force. This is what is meant by this. So, you have the mass here, the input is F, and output is x here, and the relationship is,

$$F = ma = \frac{d^2x}{dt^2}$$

Now, let us look at energy and power. You see that the spring and mass store energy. The energy stored by spring we call potential energy and energy stored by mass we call it the kinetic energy, whereas the damper dissipates energy. So, the energy is required, you know to stretch a spring, accelerate a mass and move the piston inside the damper.

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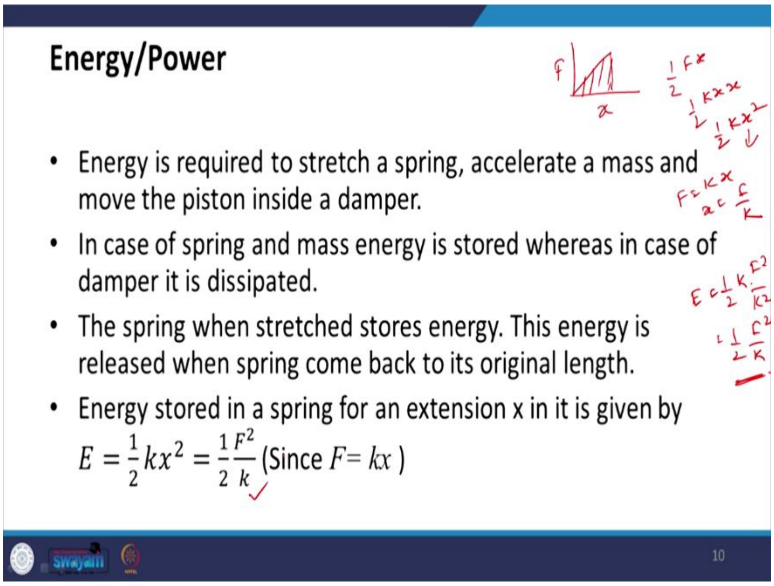

Now, as I said, for spring and mass, the energy is stored, whereas, in the case of a damper, it is then dissipated to the environment. And the spring, when you stress it stores energy, and this energy is released when spring comes back to the original length. And you know energy is stored in the spring is given by spring for a given extension, x is

$$E = \frac{1}{2}kx^2 = \frac{1}{2}\frac{F^2}{k}$$

So, for a spring, the force and displacement relationship is this one. So, energy is stored is given by this area. So, this is half F into x. And this is what half F is your kx. x, and if I write x as, you see, for a spring, F = kx, so if I write x as F/ k.This is what you are going to get.

Energy is stored in mass when it is moving with a velocity v. As I said, it is called kinetic energy, and this energy is released when mass stops moving. And energy is stored the kinetic energy of a mass is given by half mv square, and you see dampers dissipate energy.

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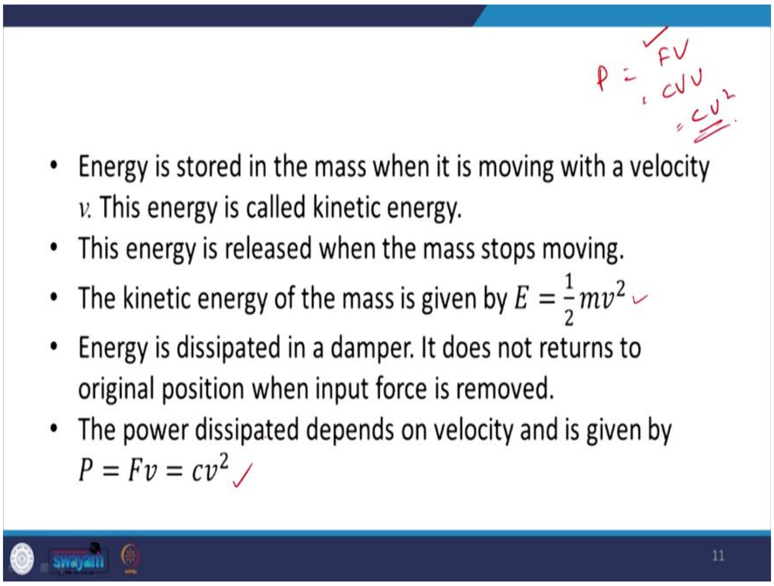

So, energy is dissipated in the damper. It does not return to the original position when the input force is removed. Why does it not return to its original position? Because it has dissipated the energy. And the power dissipated depends on the velocity, and this is given by this power is given by force into the velocity. And this force is a damping force.

$$P = Fv = cv^2$$

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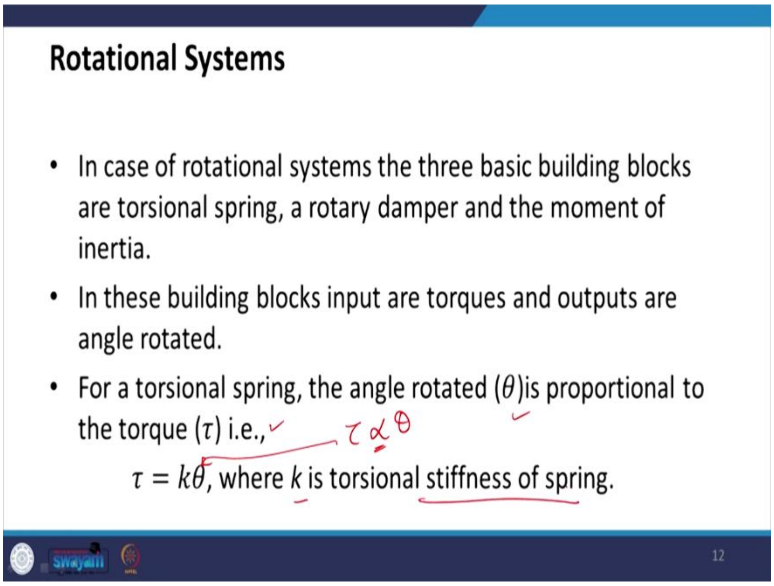

Now, this is this was all about a translational mechanical system, where we have seen a spring, a damper, and a mass as the system building block. We have their counterpart in the rotational systems. So, in the case of the rotational system, the three basic building blocks for the translational spring here we have the torsional spring. For a translatory damper, we have a rotary damper here, and for the mass here, we are using the property of the mass, what we call the moment of inertia. And in these building blocks, the inputs are torques, and output is the angle rotated. So, for a torsional spring, the angle rotated is proportional to the torque. So, what we have is, $\tau \, \alpha \, \theta$, and if I remove this proportionality constant, it is,

$$\tau = k\theta$$

Where this k is the torsional stiffness of the spring. Now, for a rotary damper, that is how it could be your rotary damper. Something like this is being rotated.

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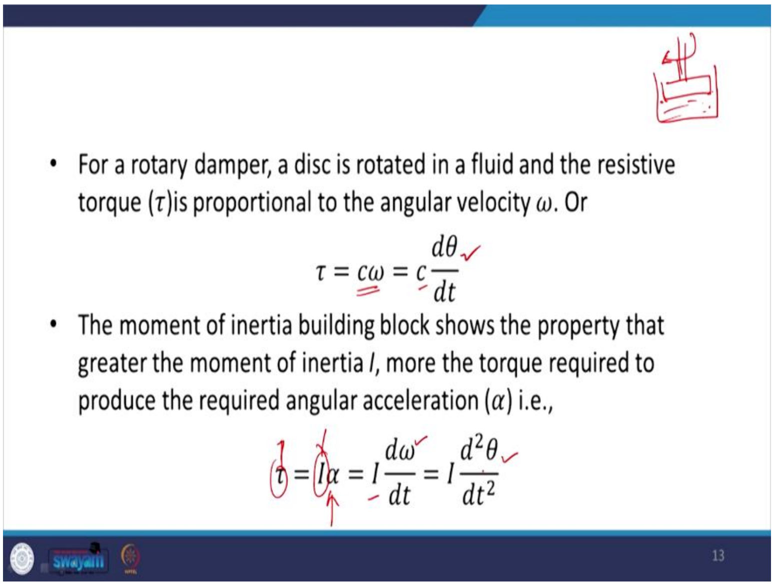

So, for a rotary damper, a disk rotates in a fluid, and the resistive torque here you have got the fluid, and the resistive torque is proportional to the angular velocity ω.

$$\tau = c\omega = c\frac{d\theta}{dt}$$

So, this is about the rotary damper. Now, coming back to the moment of inertia, the moment of inertia building block shows the property that the greater the moment of inertia, the more the torque required to produce the required angular acceleration. So, this is how this property is defined. So, for a given α, if you have more value than I, you require more value of torque.

$$\tau = I\,\alpha = I\frac{d\omega}{dt} = I\frac{d^2\theta}{dt^2}$$

Now, let us look at the energy associated with the inertia as well as rotary inertia as well as the torsional spring, and the power associated with the rotary damper. In the case of the rotary damper, as I said, the torsional spring and the rotating mass store energy, whereas the damper dissipates energy. So, the damper has got the same job.

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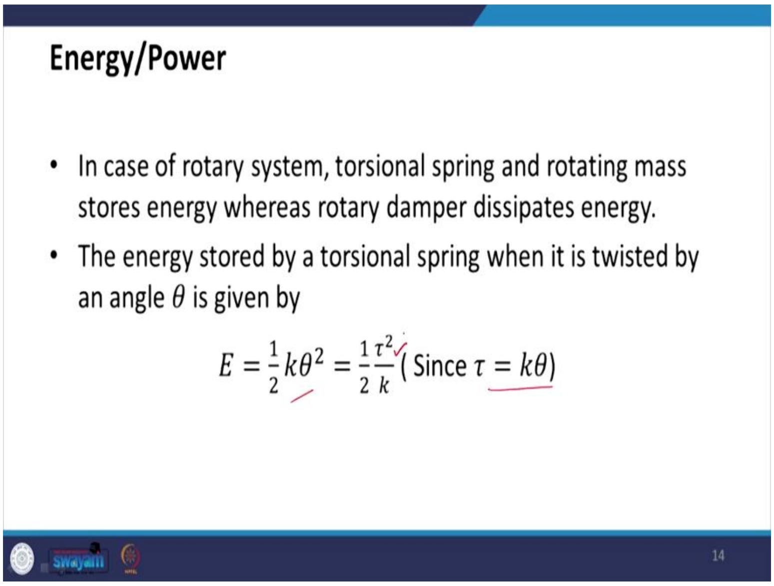

So, the energy is stored in torsional spring when it is twisted by an angle θ is given by,

$$E = \frac{1}{2}k\theta^2 = \frac{1}{2}\frac{\tau^2}{k}$$

And the energy stored by a mass of moment of inertia I when rotating with angular velocity ω is given by,

$$E = \frac{1}{2}I\omega^2$$

This is called the kinetic energy of the rotary motion.

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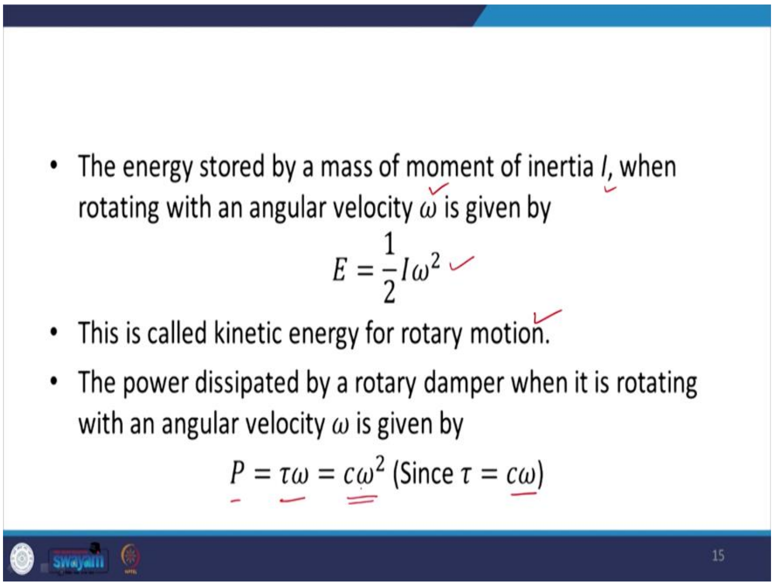

And the damper dissipates energy. So, the power dissipated by the rotary damper when it is rotating with angular velocity is given by,

$$P = \tau\omega = c\omega^2$$

Now, let us see how we build up a mechanical system. So, let us take a spring-mass damper system. After seeing the various building blocks, let us look at the building of a mechanical system. So, I am taking a spring mass damper system. Now, this could be an idealization of some real physical system. For example, the water tank could be modeled as a spring-mass damper system.

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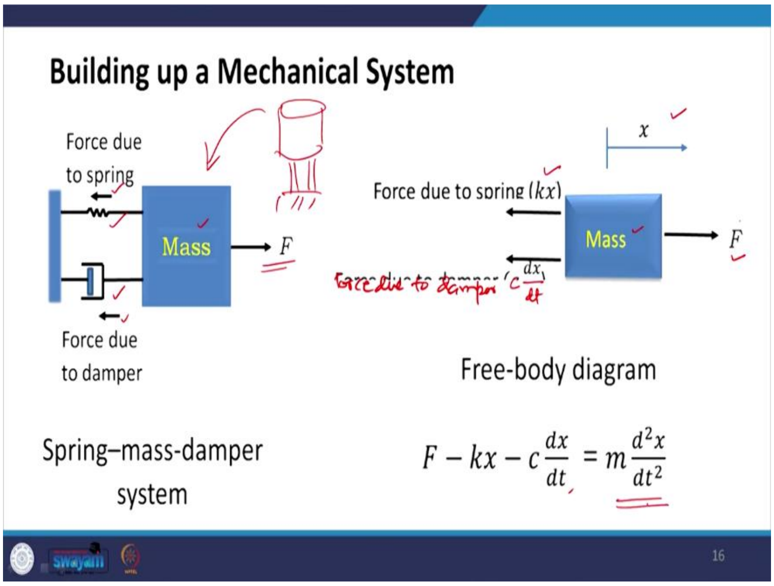

So, to analyze this, you see that when I am applying an input force F in this direction. So, the spring and the damper are going to resist that force. So, they are going to apply the forces in the opposite direction. So, I draw a free body diagram here, so I have a mass, I have a force due to spring kx, and this is the force due to damper that is $\frac{dx}{dt}$. So, that is there. Now, from this free body diagram I can draw, I can write the equation of motion. So, if this is my, this is your force due to damper here. So, I write that is this is my displacement direction x. So, this is my applied force F, so the resistive force by the spring will be k x in this direction, and the resistive force by the damper will be c dx by dt in this direction.

So, what are the unbalanced forces here? That is,

$$F - kx - c\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

So, this is an unbalanced force, and in this direction and this force unbalance force is going to be responsible for the inertial force or accelerate and acceleration. So, this I am equating to $m\frac{d^2x}{dt^2}$.

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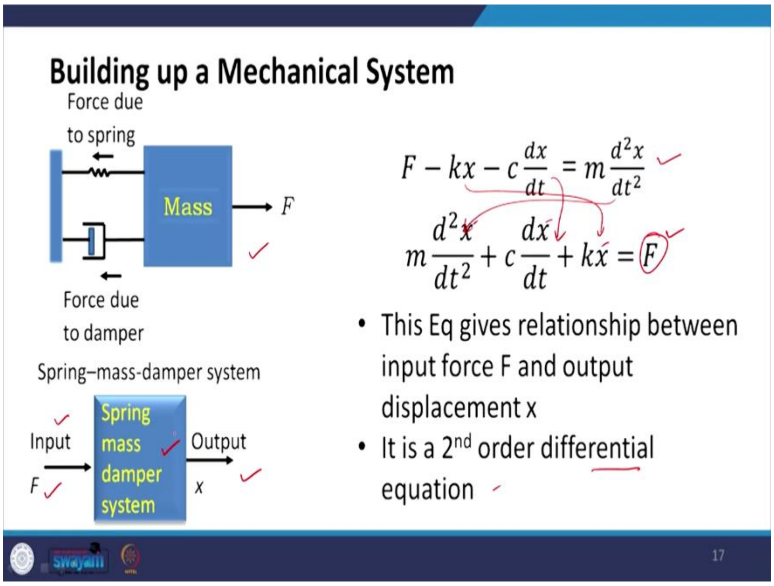

So, for this system, this is my equation of motion,

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F$$

This equation gives you the relationship between the input force F and the output displacement x, as we are seeing. And it is a second-order differential equation. So, if I want to represent it in a block diagram, I have input F and the output x, and this is my spring mass damper system.

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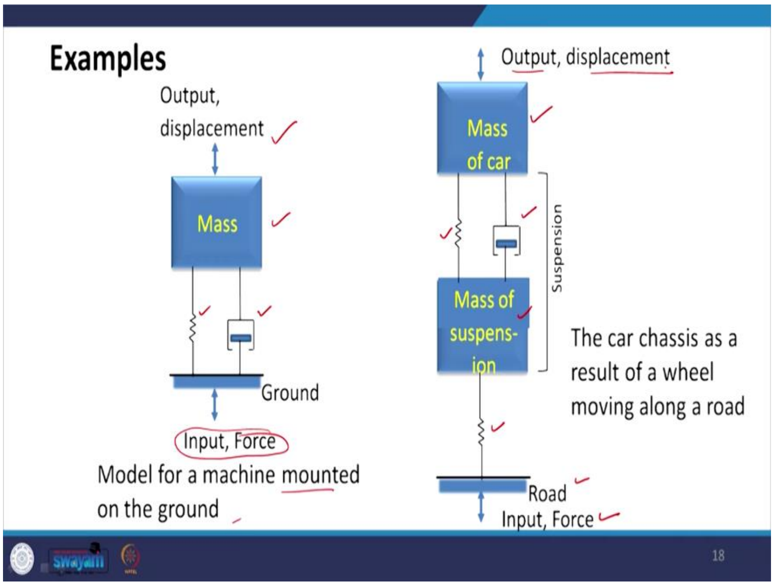

I can take many other examples to model this system. For example, model of a machine mounted on the ground, I have a mass, I have a spring, I have a damper here, and there could be input forces from the ground excitation from the ground because of the vibration of the machine, and you could have the output displacement. Or, if you want to model a car using this basic building block, so the mass of the car, I can take it as a lumped mass. For the suspension system, I can model using a spring damper and the mass of the suspension, and I can model a tyre by a spring. And here we have the road, and we get the input forces excitation from the road, and output is the displacement over here.

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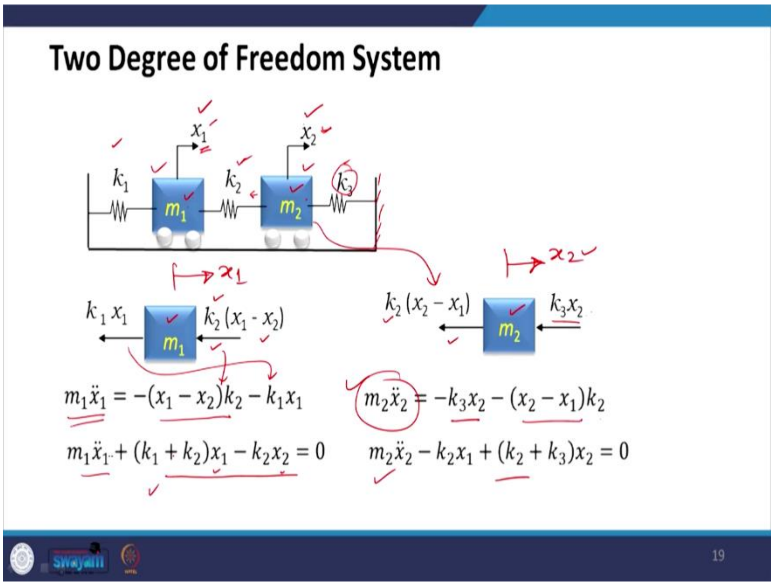

Let us take another example. I have the two masses which are connected by springs over here, here, and here, and I am asked to write the equation of motion for this system. So, let me take a reference direction x_1 for mass 1 and x_2 for mass 2. So, I draw the free body diagram for mass m_1 here. Now, when it is moving in this direction, this spring will resist its motion, and it will apply the force in this direction my k_1, x_1And the force applied by this spring will depend on the relative motion between these two blocks. And that will be again resisting its motion. So, it will be in this direction, and it will be given by k_2 and the displacement between these two masses, so, $x_1 - x_2$. So, what are the unbalanced forces here in this direction?

I can write this equation in this form to simplify this one,

$$m_1\,\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$$

Similarly, I can draw the free body diagram for the second mass here and we get equation of motion,

$$m_2\ddot{x}_2 + (k_2 + k_3)x_2 - k_2x_1 = 0$$

So, this is a two-degree of freedom system. And so, for this two degree of freedom system, we get the two-equation of motions.

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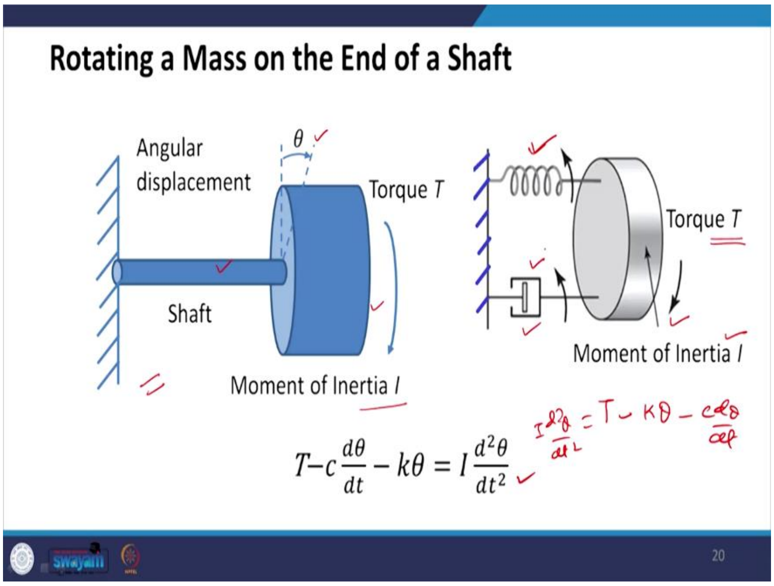

Now, next, after taking the example of the translatory system, let us take an example of a rotary system, so, rotating a mass on the end of the shaft. So, suppose I have got a mass here, which has got the moment of inertia I, and this is a shaft. So, if I want to model it as a spring-mass-damper system, this mass has got a moment of inertia I. So, I model it as the shaft you see they have got a torsional stiffness. So, torsional stiffness can be modeled by a torsional spring, and the damping provided by the shaft can be modeled by a rotary damper. And so, for this displacement, if this is the direction of torque applied, what is the equation of motion,

$$T - k\theta - c\frac{d\theta}{dt} = m\frac{d^2\theta}{dt^2}$$

So this way, we can model a mechanical system, a rotary mechanical system, by identifying the moment of inertia, identifying the torsional stiffness, identifying the torsional damping and rotary damping, and we can write the equation of motion.

Now, let us take up the combinations. So, we have seen the translational system separately, we have seen the rotational system separately. Now, let us take a combination of rotational and translational systems.

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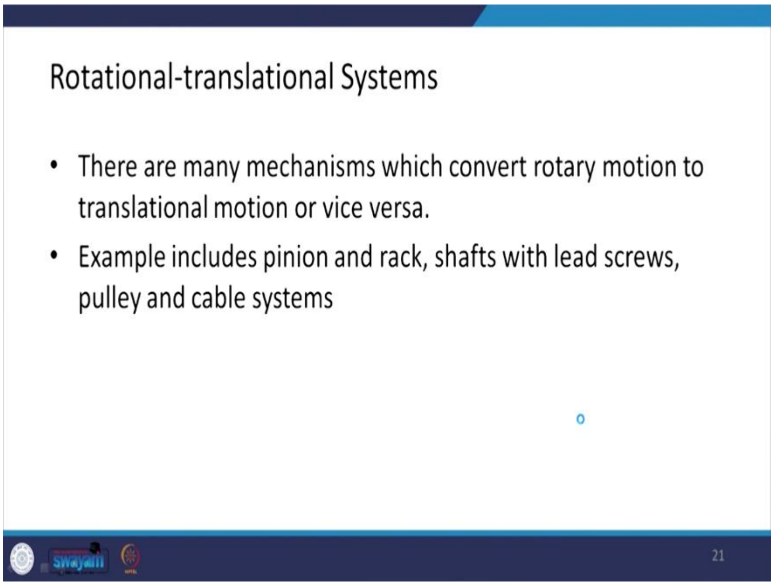

So, there are many mechanisms that convert rotary motion to translatory motion or vice versa. For example, a pinion and rack system converts rotary motion to translatory motion, pinion has got a rotary motion, and it converts into translatory motion with the help of a rack or shaft with lead screws, or we could have a pulley and cable system where the pulley has got the rotary motion, and the cable has got the translatory motion. So, this is there. So, let us take first the rack and pinion system.

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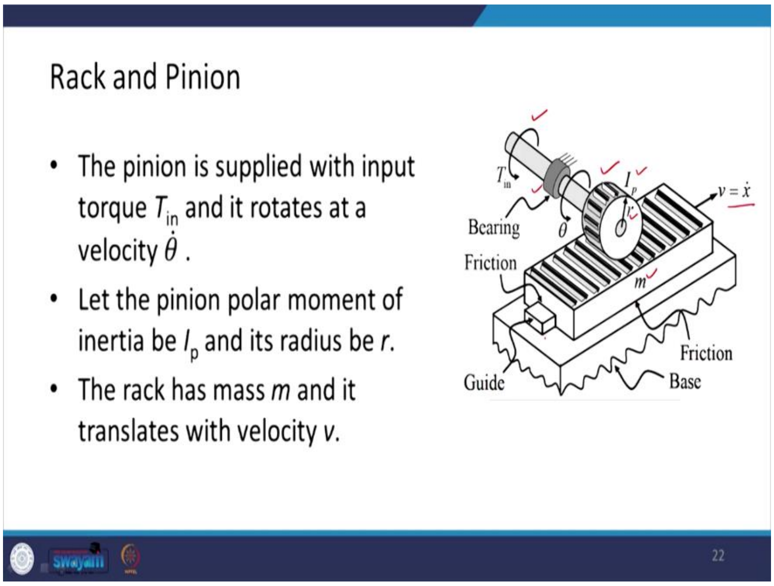

So, here we can see a schematic diagram. I have got a pinion with the moment of inertia, I_p, the radius is r a and torque T_{in} is input torque is applied over here. There is bearing friction. And I have got a rack which has got a mass m, it has got a velocity, and there is friction over here, and there is a guide.

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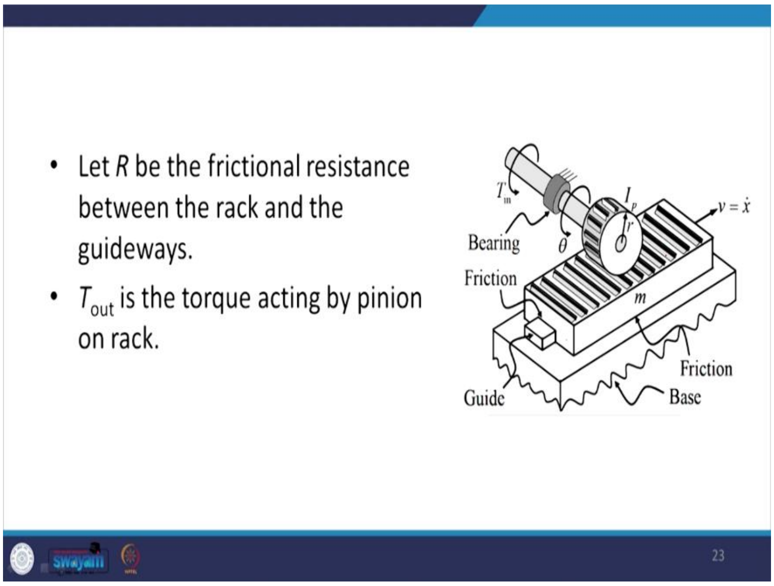

So, let R be the frictional resistance between the rack and the guideway. And T_{out} is the torque acting on the pinion torque acting by the pinion on the rack.

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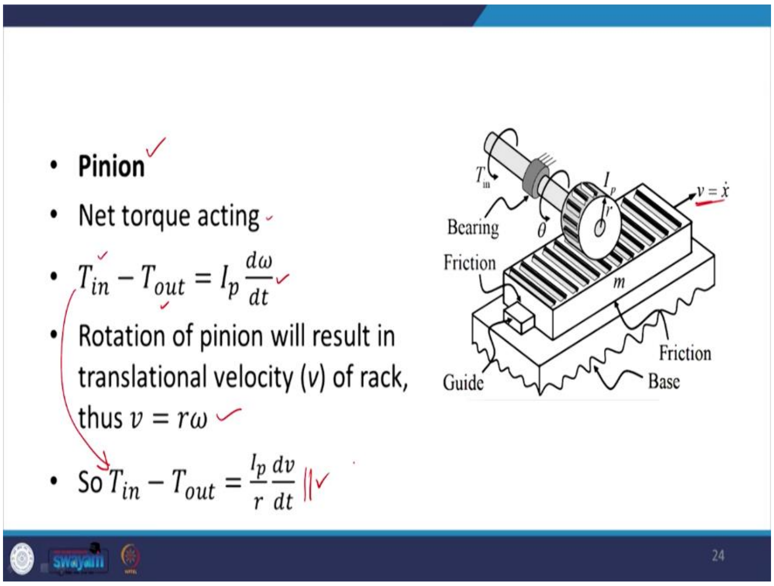

So, let us write the equation of motion first for the pinion,

$$T_{in} - T_{out} = I_p \frac{d\omega}{dt}$$

And you see, the rotation of the pinion will result in the translational velocity of the rack, and that translational velocity of the rack will be, $v = r\,\omega$. So,

$$T_{in} - T_{out} = \frac{I_p}{r}\frac{dv}{dt}$$

Now, let us take the rack. In the force acting on the rack, how much? This is the torque, which is acting on the rack. So, the force acting on the rack,

$$F = \frac{T_{out}}{r}$$

and the frictional force here in the guide and in the guide and rack is,

Friction force = Rv

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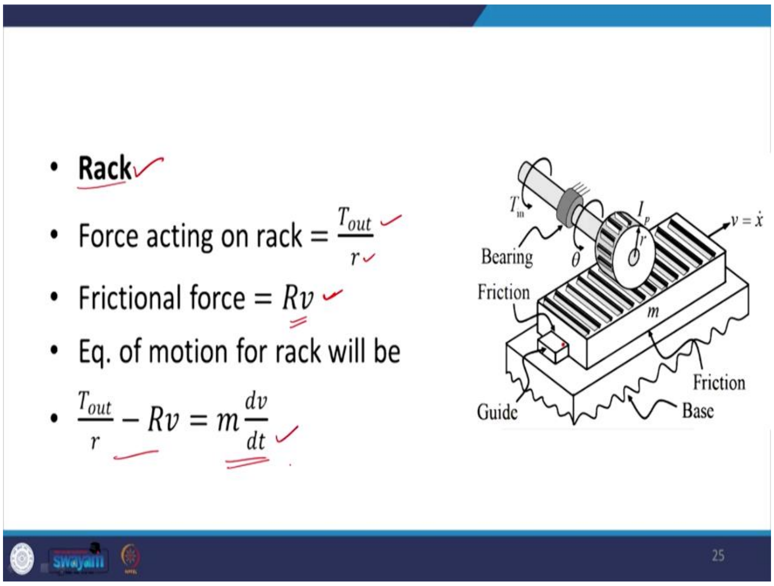

We are modeling it as a damper here. What is my equation of motion?

$$\frac{T_{out}}{r} - Rv = m\frac{dv}{dt}$$

and this is going to be responsible for the acceleration of the rack.

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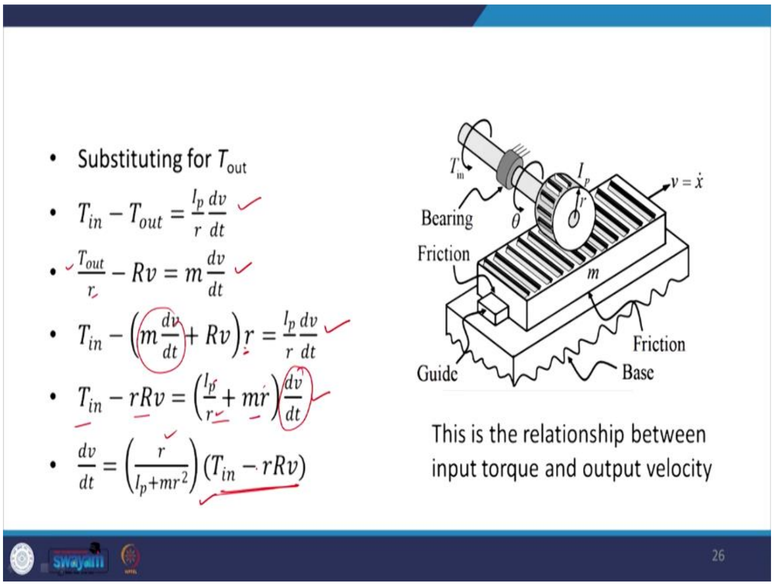

So, we had this equation, and we had this equation. The relation between input torque and output velocity is,

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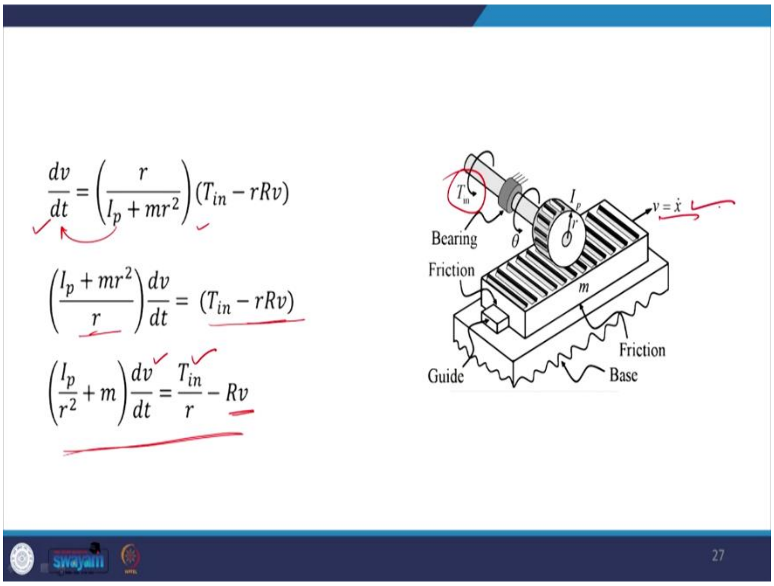

$$(\frac{I_p}{r^2} + m)\frac{dv}{dt} = \frac{T_{in}}{r} - Rv$$

This equation gives me the relationship between the input torque which is supplied to the pinion and what velocity this rack is having.

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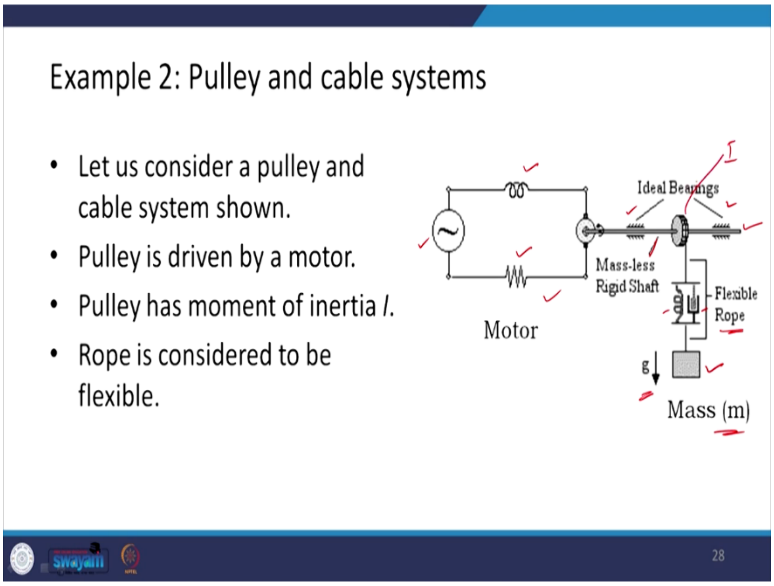

Next, let us take the example that is the pulley and cable system. Now, in this pulley and cable system, I have got a motor over here which is represented by an inductor, a resistor, and there is a voltage source that supplies voltage to the motor. There is a shaft over here that is attached to the motor, and there are ideal bearings supporting the shaft, and we can assume that this shaft to be massless. There is a pulley over here, and there is a flexible rope, and this flexible rope is modeled as a spring-damper system. And there is a mass which is hanging from it, and this is the direction of g. I am assuming that the rope is considered to be flexible, and this pulley has got a moment of inertia I.

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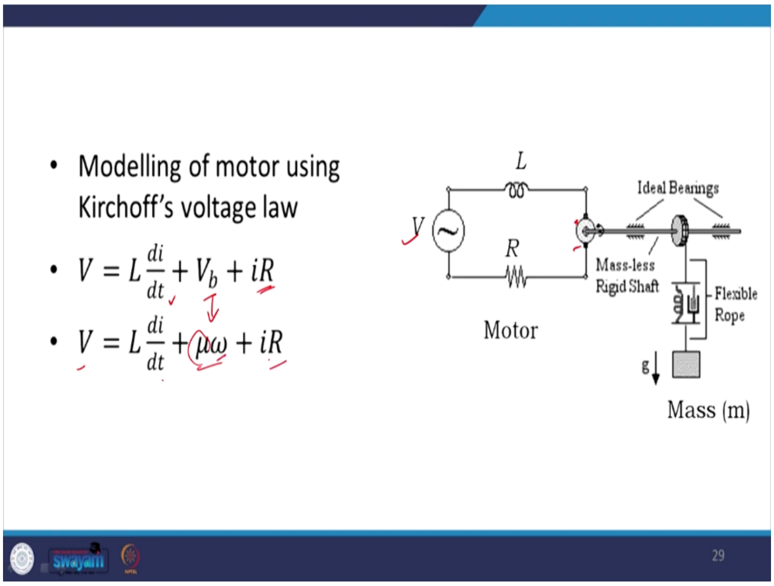

So, modeling the motor is pretty simple. We can use Kirchhoff's voltage law to model it. So, this V will be L$\frac{di}{dt}$. That is the voltage drop across the inductor plus the back emf, which will be generated at the motor plus the voltage drop at the resistor iR. So, this is,

$$V = L\frac{di}{dt} + \mu\omega + iR$$

Here μ is the torque constant or back emf constant.

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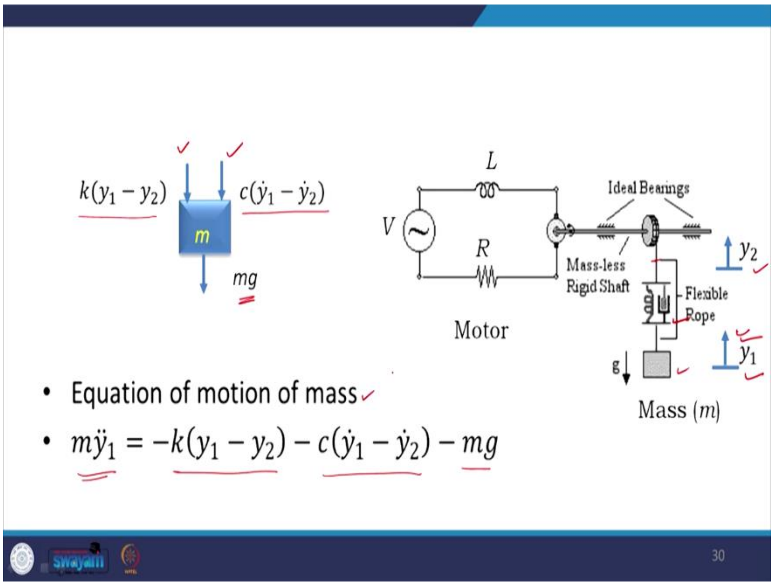

Then we can model this block motion. So, for that, I take this reference $y_1$direction here, at this of motion direction for the block that is mass, and at this end of the rope, this is a direction y_2. And at this end of the rope, this is the in the displacement is y_1, here in this direction. So, I can draw the free body diagram for this one. So, if this block is having motion in this direction, the spring force is going to be, $k(y_1 - y_2)$ that is opposite in this direction, so this direction. Damper force will be c($\dot{y}_1 - \dot{y}_2$) opposite in this direction of motion, and this is the weight. So, I can write the equation of motion.

$$m\ddot{y}_1 =- k(y_1 - y_2) - c(\dot{y}_1 - \dot{y}_2) - mg$$

This is going to be responsible for $m\ddot{y}_1$, that is the acceleration of this mass.

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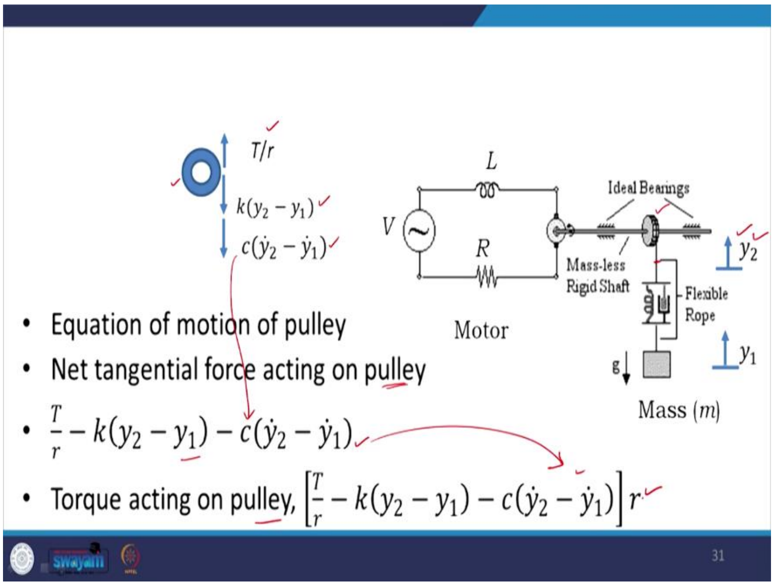

So, this is the equation of motion of the mass. And I can write the equation of motion of the pulley also. So, here we have a pulley, as you can see. So, I have just drawn that pulley over here, and the tangential force acting on the pulley is going to be T/r, where r is the radius of the pulley and T is the torque at the pulley. And this is our y_2 direction half motion of the wire over here, so the resistive forces are going to come in the opposite direction. So, this is the resistive force because the spring will be $k(y_2 - y_1)$ over here, and resistive force because of the damper here at this end is going to be c($\dot{y}_2 - \dot{y}_1$). So, what is the net tangential force in this direction? In this direction, it will be,

$$\frac{T}{r} - k(y_2 - y_1) - c(\dot{y}_2 - \dot{y}_1)$$

So, this is the net tangential force acting on the pulley. So, how much torque is acting on the pulley? From this one value, we multiply by the radius of the pulley we will get the torque acting on the pulley.

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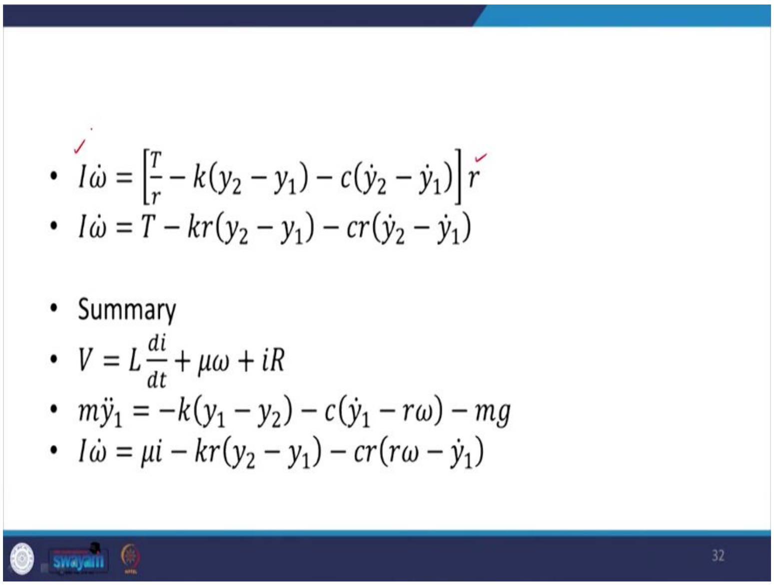

So, and this I can equate to,

$$I\dot{\omega} = T - kr(y_2 - y_1) - \text{cr}(\dot{y}_2 - \dot{y}_1)$$

So, in summary, we have the 3 equations, one is for the motor, one we have the equation of motion for the mass, and last is the equation of motion for the pulley.

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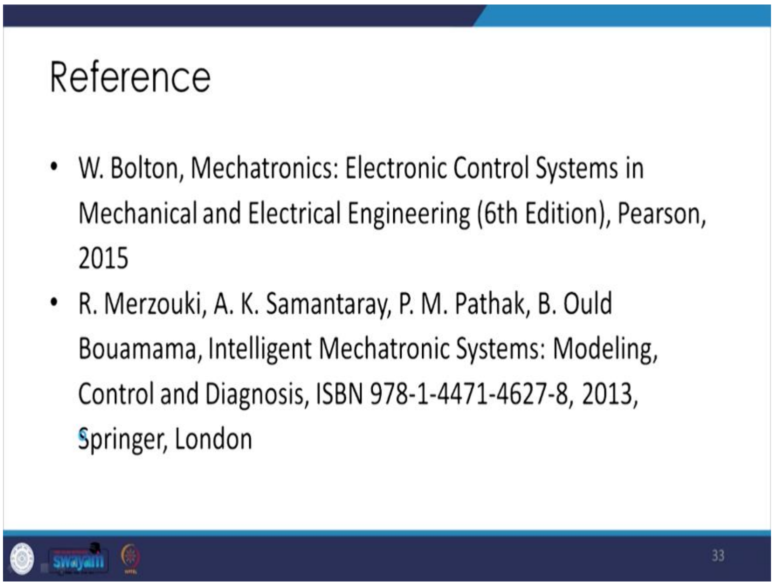

These are further references. You can refer to the Mechatronics book by Bolton, as well as you can refer to our book if you want to read it further.

Thank you.