**Robotics and Control: Theory and Practice**
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**Lecture – 08**
**Inverse Kinematics**

Hello viewers, in this lecture; we will see two examples for Inverse Kinematics solutions of robot manipulators.

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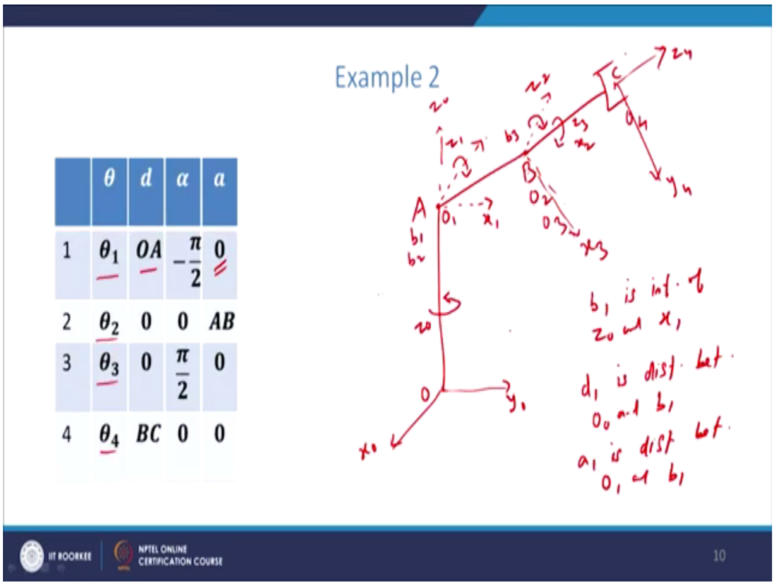


So, in the first example we will see a 3 degree of freedom manipulator and then in the second example we will see a 4 degree of freedom robot manipulator for which we will construct the kinematics equation and then the inverse kinematic solutions.

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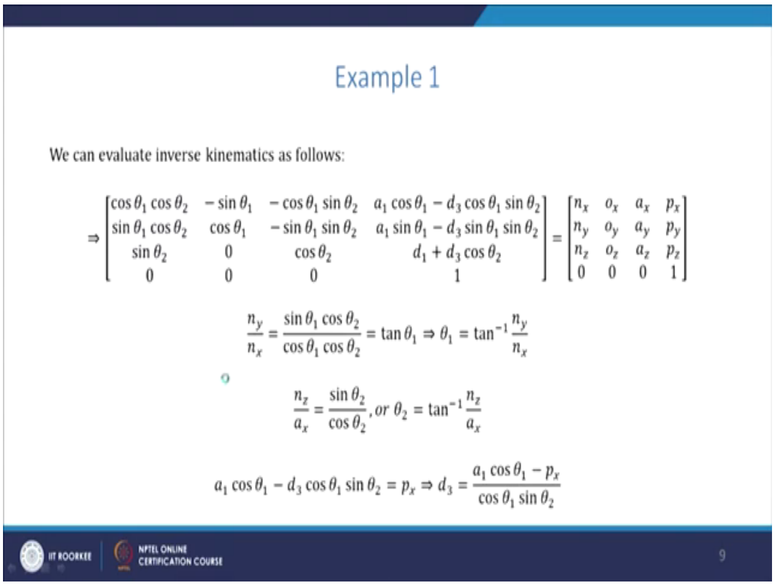

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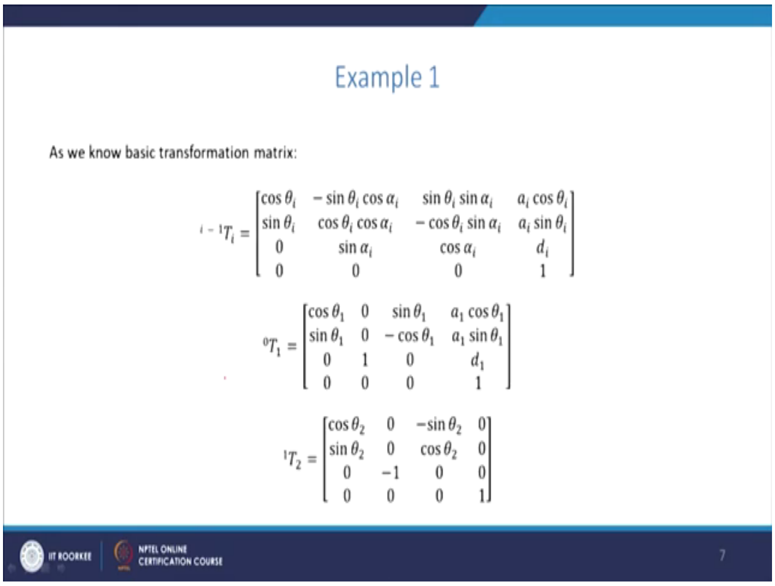


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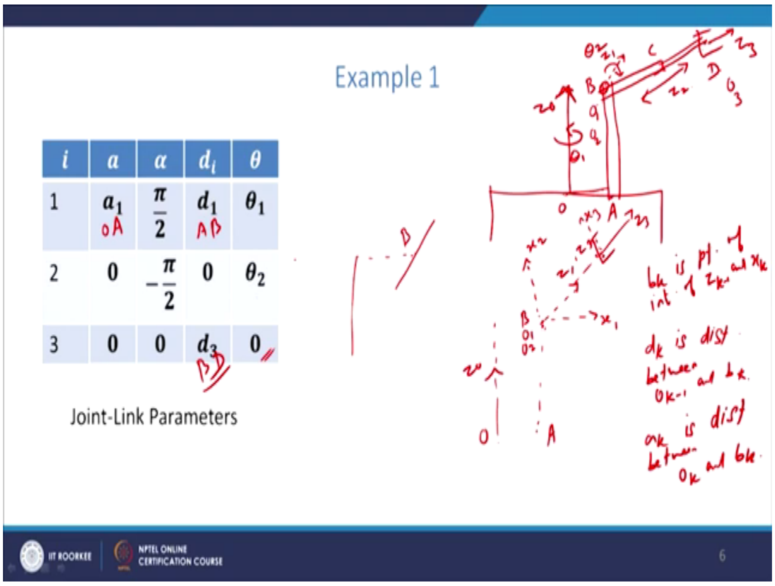


| $i$ | $a$ | $\alpha$ | $d_i$ | $\theta$ |
|---|---|---|---|---|
| 1 | $a_1$ | $\frac{\pi}{2}$ | $d_1$ | $\theta_1$ |
| 2 | 0 | $-\frac{\pi}{2}$ | 0 | $\theta_2$ |
| 3 | 0 | 0 | $d_3$ | 0 |

So, for this first example; we see the following robot manipulator. So, we consider a platform in which the axis of rotation is the vertical direction; we call it as the z 0 direction. So, the entire platform rotates in the horizontal plane with the vertical line as the axis of rotation. Then the robot manipulator is placed here. The stem of the robot manipulator is placed here for which there is a joint which rotates with the axis of rotation perpendicular to the plane of the paper.

So, it is a z 1 is the direction which is perpendicular to the plane of the paper; that means, z 0 and z 1 they are perpendicular to each other. So, at this position; if we call the point this point as O and this point as A and this joint as B and this joint as C and another joint which is the end effector D the final one. So, we assume that there is a rotation about the z 0 axis. So, that angle we call it as theta 1 and there is a rotation about the z 1 axis.

So, that we call it as theta 2 and the joint C will move forward and backward that is a prismatic joint. So, we call that direction as z 2 direction, then at the end effector we have the approach direction that is z 3 direction. So, now, using the D H procedure we can fix the coordinate frames and then find the parameters of the robot manipulator as given in this table. So, the first thing is z 0 and z 1 they are perpendicular to each other, they do not intersect.

And so, the shortest distance if we draw, the intersection of the shortest distance line with z 1 axis can be taken as the origin. So, if you take the shortest distance point the point B is the point of the shortest distance because z 0 line is like this, z 1 is perpendicular line. So, the shortest distance is intersecting at the point B here. So, that point is called the origin O 1 for the second joint. Now, z 1 and z 2 they are also intersecting at the same point B.

The direction; so we can call it as the origin O 2 also. So, the origin O 1 O 2 both of them are at the point B and the base origin is at the point O here and another joint is the end effector that we can call it as O 3 the that is at the point D. So, now, we can fix the x direction that is z 0 and z 1 do not intersect. So, the shortest distance line itself will give the x direction of the coordinate frame.

So, what we can do here is the base coordinate frame it is O z 0 is here and then A the point B is this distance perpendicular direction is z 1. So, here it is O 1 and O 2. So, the shortest distance line, this line itself will give the x 1 axis z, 0 and the z 1 are parallel the shortest distance line, the extension will give the x 1 axis. Similarly z 1 and z 2 are also intersecting here. The line perpendicular to both of them; z 1 and z 2 will give x 2 direction is perpendicular to both is z 2 and z 1 direction.

Now, by fixing the point b k as we have seen b k is the point of intersection of z k minus 1 and x k. And so by fixing the point b 1 b 2 b 3 etcetera we can see that the point d k is distance between o k minus 1 and b k. Similarly, a k is distance between o k and b k. b k is the point of intersection of z k minus 1 and x k. So, this rules can be followed and we can easily obtain the values of all the parameters theta, d, alpha and a from this expression.

So, we can easily see that the distance O A this 1 is a 1, the point O to A this distance is a 1 and A to B that distance is d 1, this is O A and A B is d 1. For the second joint because, both the coordinate origins are coinciding here itself. So, a 2 and d 2 both of them are 0, there is no link length or the joint distance, both of them are 0. But, there is a joint angle, theta 2 there is a revolute angle theta 2 available at this joint B.

Then the third one is the prismatic joint and the point B if we can get d 3 as b 3 and o 2; so, o 2 is the point B and d 3, b 3 is the point d. So, the distance b to the end effector B D this thing; B D will give the joint distance for the third joint. And, a will be 0, the twist angle is 0 because both the z axis are parallel. Then and it is a prismatic joint therefore, the there is no angle of rotation that is 0 here. So, we got the parameters here.

Now, by substituting these parametric values in the i minus 1 T i. The homogeneous transformation for the ith frame with respect to i minus 1th frame we can easily obtain 0 T 1 1 T 2 2 T 3 by just substituting from the table values. Then multiplying all these 3; we get 0 T 3 to be equal to this expression. So, it gives that the end effector frame. The third coordinate frame with respect to the base frame.

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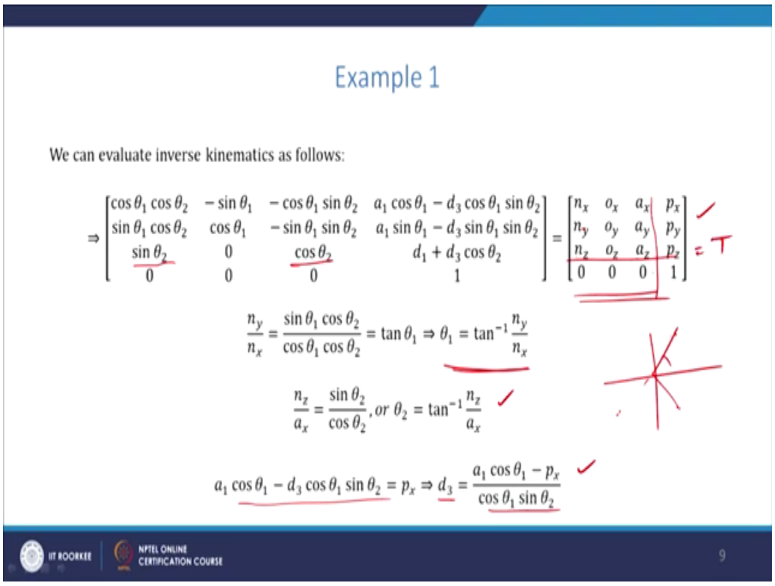


Now, if we equate the end effector frame with respect to the base frame to a homogeneous transformation; so, by giving this n x n y n z all this any the first 3 by 3 matrix is a rotational matrix. And then the fourth column gives their shifting of the origin with respect to the base frame. So, if we give a suitable 4 by 4 homogeneous transformation matrix and equate it with the kinematic expression 0 t 3 here; then we can get the inverse kinematic solution very easily. That is; for example, cos theta 1 into cos theta 2 is given by n x and n y is the sin theta 1 into cos theta 2.

Now, dividing both; we will get tan theta 1 equal to n y. So, because we know n y by n x we will get the value of theta 1 by seeing the values of n n x and n y. If both of them are positive; then tan inverse of the positive quantity, we will get between 0 to pi by 2. The first quadrant we if both n x and n y are positive it falls in the first quadrant. So, the tan theta value will be, so theta value will be between 0 to pi by 2 whatever is coming.

If n y is negative and n x is positive in the fourth quadrant. Then, we will get the angle to be the 360 minus theta or we can say that minus theta will be the expression; whatever value we get for tan inverse modulus of n y by n x. Then with a minus sign; we will get the angle because, that number falls in the fourth. If both of them are negative n x is negative n y is also negative it falls in the third quadrant. So, we will get tan inverse of that value plus pi will be the angle to be obtained.

So, depending on n x n y value and its sign; we will get the exact value of theta 1 from here. And similarly we will get n z by a z. If you take n z and the a z quantity this one. So, this will give tan theta 2 from here we can get the value of theta 2 by taking the inverse; similarly, by equating the various expressions. Once we obtain theta 1 and theta 2 from this expression; we can also get the value of d 3 by using the simple manipulation of the various terms here.

So, this is called the inverse kinematic solution of the problem. So, for a given end effector position the inverse kinematic solution theta 1 theta 2 and d 3 can be obtained like this. So, by making suitable rotation of the actuators using these angles, we will reach the end effector position as given in the T matrix like this 0 T 3 expression. So, in this example 2; we will consider a 4 axis robot manipulator which has all 4 revolute joints.

So, first let us consider the first stem o A. It is having revolving axis o A is the axis of revolution and we call it as the z 0 axis. Then at the joint A there is a axis of rotation which will be the perpendicular line to the z 0 axis. So, if o a is on the plane of the paper; then the axis z 1 is perpendicular to the plane of the paper. Then at the joint B; there is a axis of rotation which is parallel to the z 1 axis.

So, z 1 and the z 2 axis are always parallel to each other and it will be perpendicular to the plane of the paper at this position. And at the same joint B there is another axis of rotation, which is in the direction of b c the approach direction itself is the axis of rotation in this particular case. So, we have the joint O A B C 4 joints are there. And then there are 4 revolving axis at these joints. Now, we can easily fix the various joint parameters theta d alpha a as in the case of the d h parameters.

So, if you see that z 0 and z 1 are perpendicular. So, the origin o 1 is at the point A and z 1 and z 2 are parallel. So, the line joining the shortest distance between them is along the line A B. So, the point B is the origin o 2. And the point B itself is the origin o 3, because z 2 and z 3 both of them are intersecting at the point B. And the z 3 and z 4; they are intersecting at all the points. So, a suitable point is selected that point is the end effector position C is called the origin o 4.

Now, we can fix the x direction; arbitrarily, we can fix the x axis x 0 and then corresponding y 0 to form a right hand system at the base. And z 0 and z 1 are perpendicular to each other. We get the x 1 direction as perpendicular to both z 0 and z 1. So, that is the x 1 direction and z 1 and the z 2 are parallel. So, the line a b extension will give the x 2 axis.

Similarly, z 2 and z 3; they are always perpendicular to each other the cross product will give the x 3 axis. A line perpendicular in this direction it is x 3 axis. And at the end effector as the convention; we can fix the y direction as the line joining the fingers and the x direction as the y cross z direction will give the x 4 direction. So, using this, now we can fix the b 1 b 2 b 3 b 4 points as the intersection of.

For example 1 is intersection of z 0 and x 1. So, z 0 is this x 1 is this axis. So, the point A is b 1 and b 2 is z 1 and x 2. z 1 is this direction, x 2 is so, again b 2 is placed at the point A itself. And b 3 is z 2 and x 3 z 2 and x 3, so this point is b 3. The point capital B is b 3. So, using that we can measure d 1 d 1 is distance between o 0 and d 1.

o 0 is the o point and b 1 is A. So, this is o a. d 1 is o a is given and similarly we can easily see a 1 is the distance between o 1 and b 1. So, here we can see that o 1 and b 1 are at the same place. So, the a 1 value is 0. So, like that; we can fix using this rule fix all the parameters a i d i theta i and alpha i here all the 4 joints are revolute joint. Therefore, this theta 1 theta 2 theta 3 theta 4, these are the variables in the system. Now, substituting those values in i minus 1 t i we get 0 T 1 1 T 2 etcetera.

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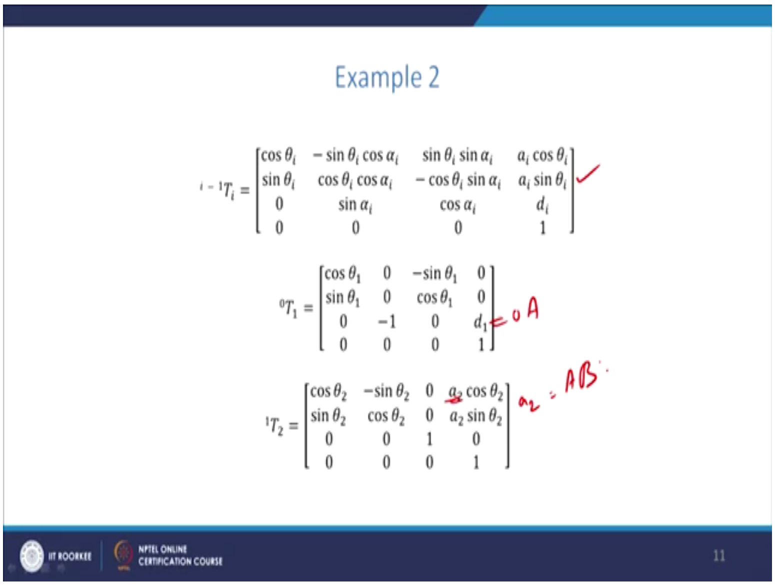


So, this d d 1 value is nothing but o A and a 2 value is A B that distance. So, and we can see that d 4 value is B C.

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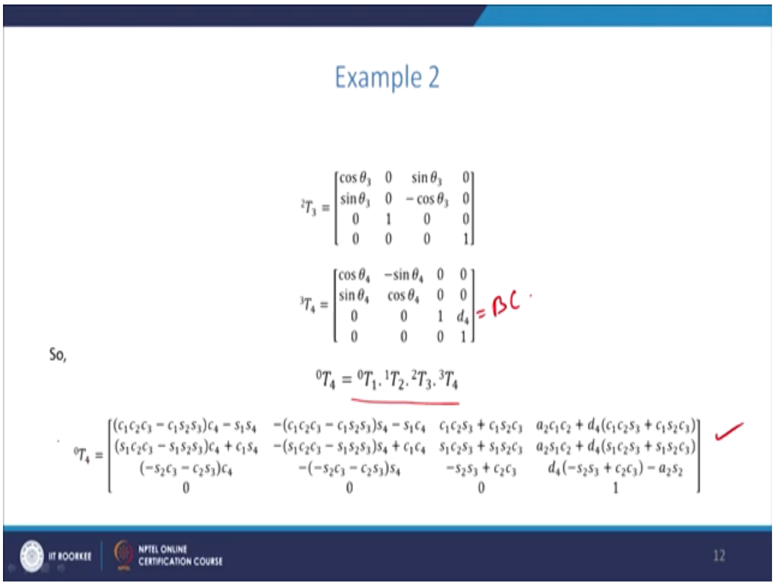


Now, substituting all this values in the product we finally, obtain the direct kinematic problem the equation 0 T 4 is the end effector position with respect to the base of the manipulator.

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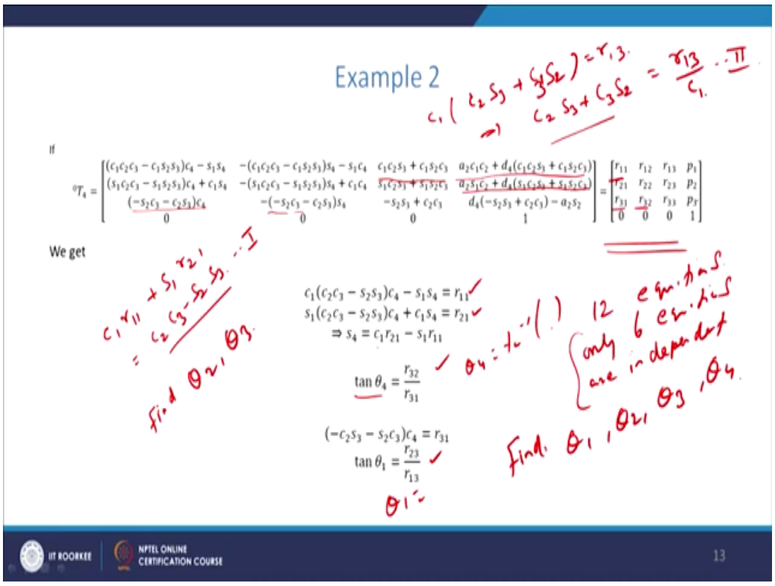


Now, if we are given a specific 4 by 4 matrix representing the end effector position like this n o a p matrix. Then we can directly equate each terms here, we can easily see that even though there are 12 such equations. There are 12 expressions here and in the right hand side corresponding 12 numbers are there. So, if we equate there are 12 such equations, 12 equations are there.

But only 6 of the equations will be independent equation. Because, for any rigid body; the position and orientation will be measured by 6 quantities only. So, only 6 such equations are independent equations. So, using 6 of them we can easily solve the entire equations. We can find the parameters theta 1 theta 2 theta 3 theta 4 can be found out by just equating them.

So, for example, r 1 1; the first number is equated with the first value cos theta 1 into cos theta 2 cos theta 3 minus sin theta 2 sin theta 3 into cos theta 4 minus s 1 s 4 means sin theta 1 sin theta 4 that is r 1 1 similarly, r 2 1. Now, we can easily obtain from the equation; the sin theta 4 is equal to. If you multiply the first the first equation with s 1 and the second equation with c 1 and then subtract we will get sin theta 4 is equal to this expression.

Moreover, we can easily get tan theta 4 from r 3 2 this one. r 3 2 divided by r 3 1. We can see from here r 3 2 by r 3 1 will give tan theta 4 from where we can find theta 4 is tan inverse etcetera. So, depending on this sign of this 2, whether it is in the first or which whichever is the quadrant accordingly we will get the value of theta 4 from here.

Similarly, we can get theta 1 value from this 2 expression. The r 2 3, r 2 3 is this expression and divided by r 1 3 is this one. If you divide this 2; we will get tan theta 1 expression from here; we will get theta 1 is tan inverse etcetera. So, once we get theta 1 value from this expression; we will get cos theta 1 into c 2 is 3 plus c 1 from s 2 c 3 from this equation. This is equal to r 1 3.

So, once we obtain theta 1 and theta 4. Now, from this equation from r 1 3, we can obtain c 1 multiplied by c 2 is 3 plus s 2 c 3 equal to r 1 3. So, that will give you c 2 s 3 plus c 3 s 2 equal to r 1 3 by cos theta 1. So, the right hand side is known. So, we can obtain the left hand side by this number. Now, from the equation here r 1 1 and r 2 1; we get c 1 r 1 one plus s 1 r 2 1 is given by c 2 c 3 minus s 2 s 3.

So, we get for example, we call it as equation 1 and this as equation 2. So, now, we can obtain one of from this 2 equation and combining with any of this or that is p 1 equation from here and p 2 equation p 1 and p 2. Combining this 2; we can find theta 2 and theta 3 also. So, all the values of thetas theta i can be obtained by just by simple manipulation of these algebraic equations and finally, we get the inverse kinematic solution of the problem.

So, the usefulness of the inverse kinematics is as we already know that if the end effector position is given the position and orientation of the end effector is given with respect to the base of the manipulator by finding this theta 1 theta 2 theta 3 theta 4 and adjusting the actuators according to the solution. We obtain the end required end effector position.

This procedure will be very useful in controlling the robot manipulator for tracking objects or in the problem of pick and place type of work while a robot is performing such a task. This will come as a very useful tool ok. So, those controlling of robot manipulators will be seen in the coming lectures.

Thank you.