Robotics and Control: Theory and Practice
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Lecture - 05
Transforming Differential Changes Between Coordinate Frames

Hello viewers. In this lecture, we will see the Transforming Differential Changes Between two Coordinate Frames.

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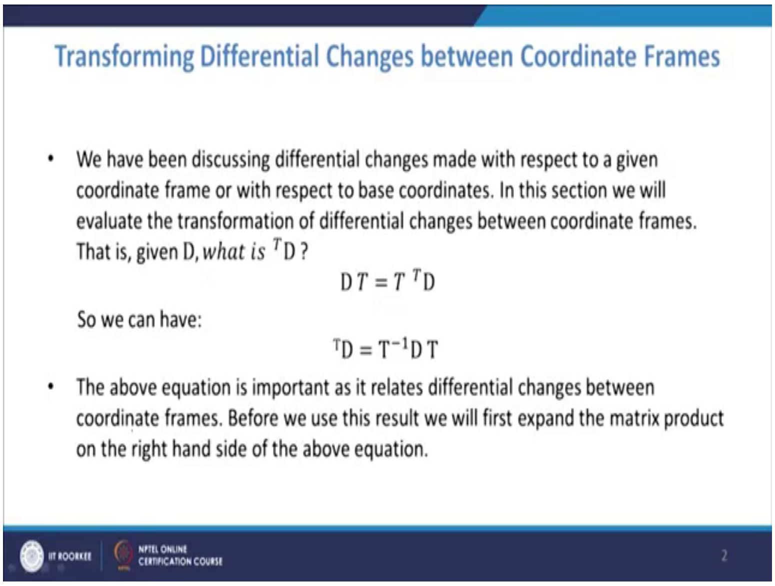

So, in the last lecture we have seen about the differential transformation.

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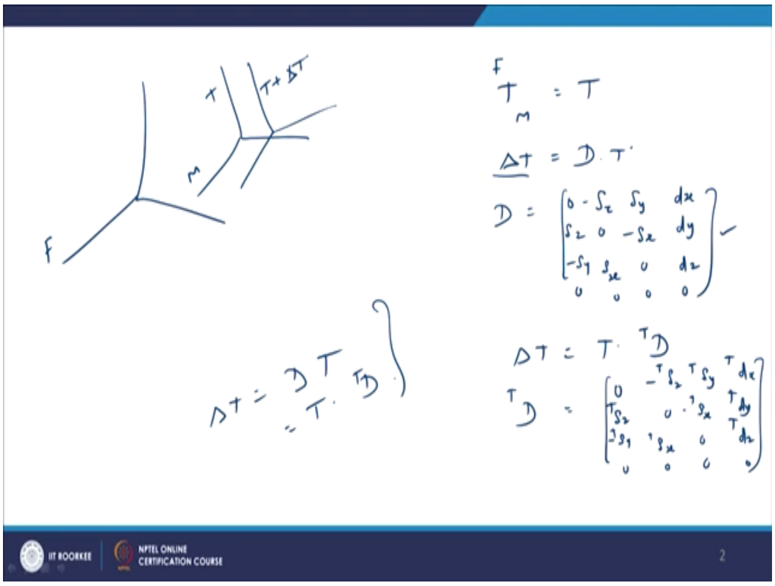

So, to recall that one, if you consider a fixed coordinate frame F and another coordinate frame M, then if you write the frame M with respect to frame F, that is F T M as the homogenous transformation T. The positional orientation of M frame with respect to F frame is given by the 4 by 4 homogeneous transformation matrix T. And then and if we give a small translation and rotation with respect to the fixed frame, then the new position of T will become T plus del T.

And, we have written that in the last lecture we have seen that del T, the increment in the positional orientation of T is given by the differential operator into T, where this D is given by the rotation and a small translations and small rotations. So, the differential operator D is given by the matrix 4 by 4 matrix where d x d y d z represent the small translational components along the x y z direction of the F frame.

And, del x del y del z are the small rotational component of about the x y z axis of the F frame. So, the same thing can also be written as T times D super fix T, where this D super fix T is given by 0 minus del z super fix T. So, everywhere we will put a T along with the translation and rotation. So, this is these are the linear and angular small translational and rotational movement with respect to the M frame itself.

The, now for example, d x T means the small amount d x travelled by the M frame along the x axis of the M frame itself so, similarly all the translation and rotation. So, these two are the differential d means the differential changes with respect to the fixed frame, D T means the differential changes with respect to the current frame.

Now, we want to relate this two what are the relation between the, because this del T operator is equal to D T on one hand and T into D T on the other hand. So, the relation between D and D T can be easily obtained from this relation. So, the practical meaning of this will be very useful in the practical many robotics problems etcetera.

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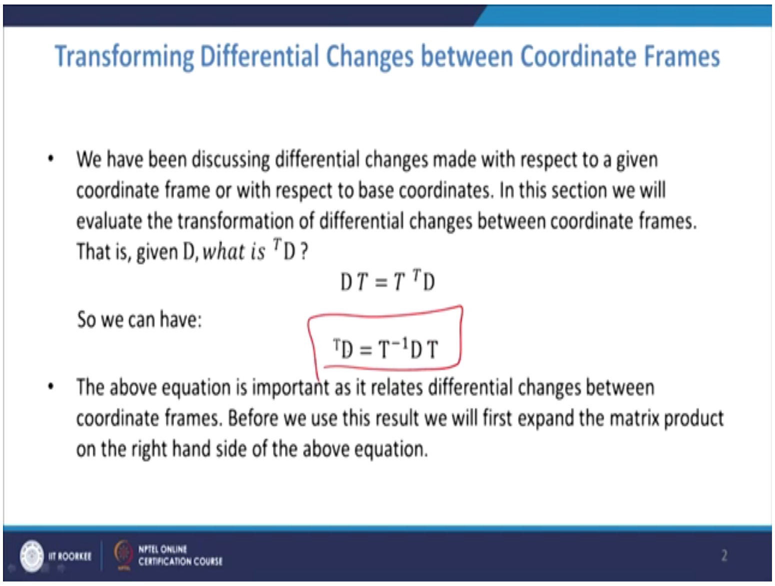

So, here D into T is same as T into D super fix T as seen in the earlier slide. So, D super fix T is given by this formula that T inverse D T. So, if you are able to evaluate this expression in the right hand side expression we get the relation between D and D T. And then we can make use of the many certain problems.

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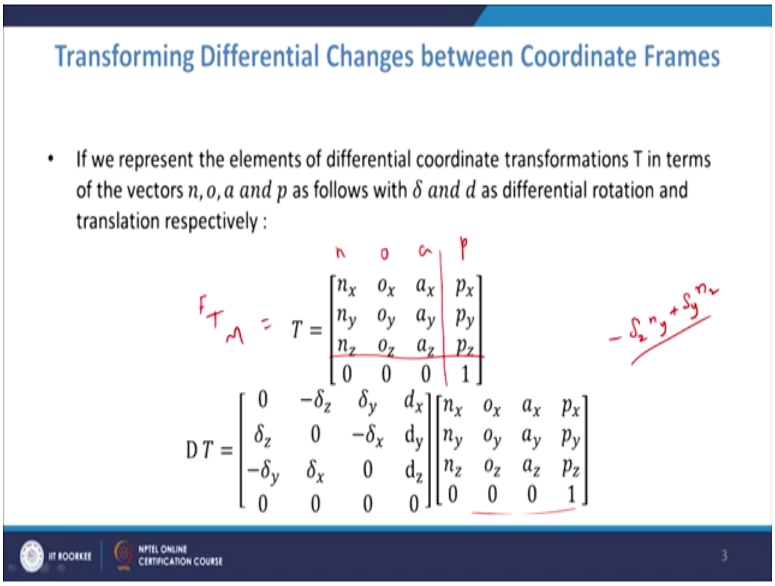

So, we denote this F T M the M frame with respect to F frame by the 4 by 4 matrix like this, where the first column represent the n vector o a p the some standard notation for the 3 4 columns of the homogeneous transformation.

So, now if D is given by the expression here, T is given by this and if you multiply D into T directly we can easily see that the first row into the first column, the product will give minus del z into n y plus del y into n z. Similarly, the first row second column etcetera directly can be multiplied and we can see that this minus del y into n y plus del y into n z.

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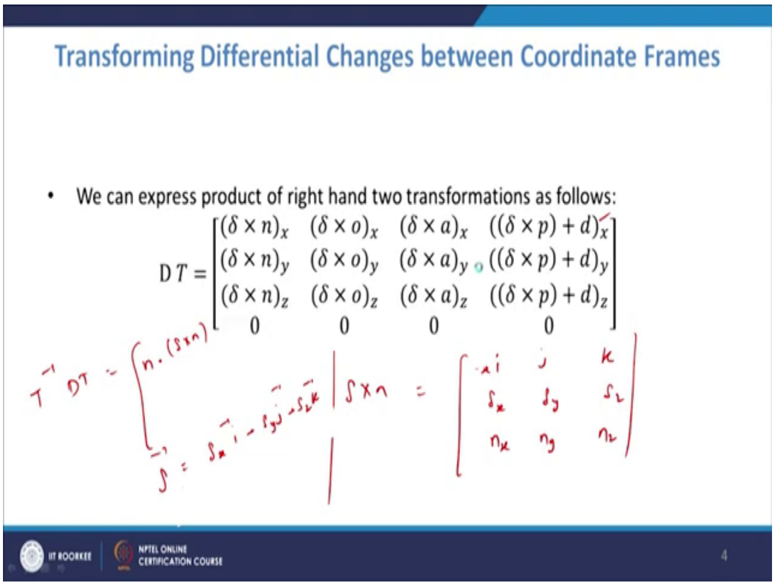

So, that is the first element of this product. And that can be easily seen to be the ex component of the del cross n, this del cross n, it is given by del x i j k, del x, del y, del z n x, n y, n z. So, this determinant if you calculate the ith component is del y n z minus n y del z. So, that is what we have seen here this one. So, this is the first component del cross n x.

Similarly, all the components are given here the product of for example, the fourth first row fourth column. The first row it multiplied by fourth column will give this del cross p vector and the last entry is d x into 1. So, that is given by this one del cross p plus d vector, where this del vector we are calling it as del x I plus del y j plus del z k. And, d vector it is d x I d y j plus d z k vector. So, the z x component of d is given here. So, similarly all the entries can be multiplied D into T can be obtained.

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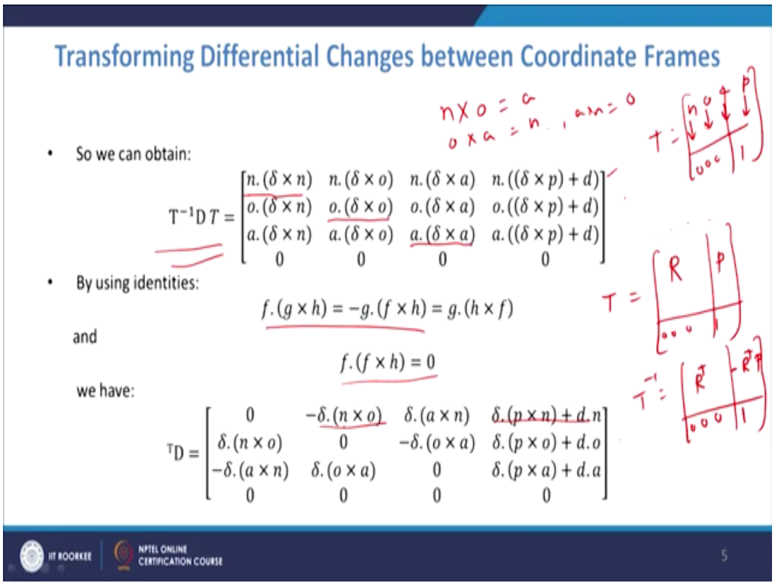

Now, we have to multiply in the left side T inverse. So, T inverse D T can be obtained like this, it is because the T inverse if T is given by the rotation matrix the 3 and then P vector 0 0 0 1, where R contains the first row is first column is n o a and then this is p as shown earlier. So, the T inverse, if you recall it is nothing, but R transpose the 3 by 3 matrix and minus R transpose into p vector then 0 0 0 1.

So, if we multiply T inverse with this D T D into T. So, we can see that T inverse D T. The first row into the first column will give del cross n is the vector in the first column here the D D T. And, the first row of T inverse, first row of T inverse is nothing, but the first column of R, that is n n x, n y, n z.

So, if I multiply n n x n y n z with this del cross n vector it is nothing, but the first entry will be n dot del cross n only is not it. The first column here is del cross n vector, the first row of

T inverse is n vector. So, the product will give you n dot del cross n. So, similarly all the other components, the first row second element is given by n dot del cross o and the third is n dot del cross a etcetera, just by observing the D T itself you can we can easily find. So, it is the it is very use to verify T inverse D T is given by this expression.

Now, if you observe the entries inside the matrix, it is n dot del cross n. So, we know if there are 3 vectors f g h. Then if f is or anyone of them is repeated 2 times in the this product, the triple product it is called, then it is 0, the value of the it is for example, f dot f cross h is equal to 0.

So, wherever this is repeated n dot del cross n is 0, and then o cross del cross o is 0, and a dot del cross a that this 3 will be 0, here n dot del cross p can be written as del dot p cross n and plus n dot d as it is here. So, we can write from each entry the values can be obtained as given here. So, the D super fix T is given by this expression.

Now, again observing this vectors are unit vectors n o a are unit vectors and their dot products are 0 they are orthogonal to each other, that is the property of the homogeneous transformation. So, we can easily see that n cross o, these are mutually perpendicular vectors. So, we can get that n cross o will be the a vector, and o cross a will be n vector, and a cross n will be o vector.

So, this can be utilized from for this expression and then substituting n cross o equal to a here, it is del minus del dot a vector. And, this is del dot a cross n is; a cross n is o vector. So, this is del dot o.

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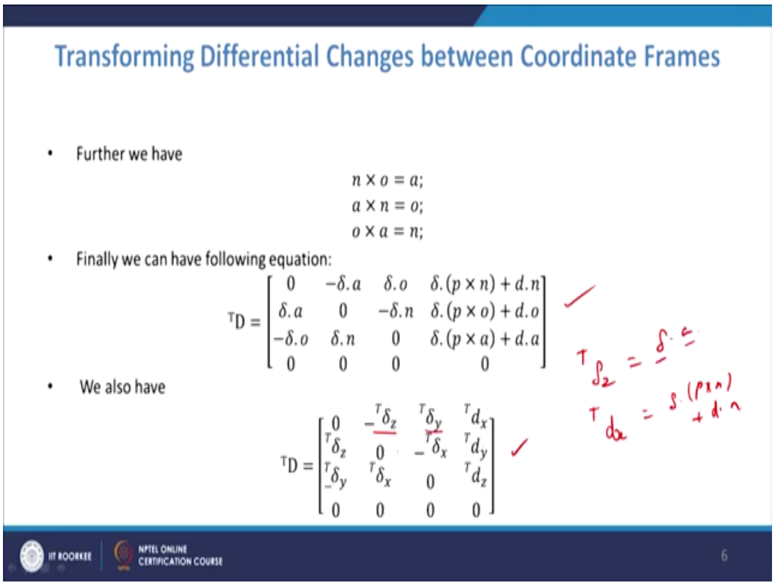

So, what we ultimately obtain is the d super fix T is given by this matrix 0 minus del dot a del dot o del dot p cross n plus d dot n etcetera. So, it is a very straight forward simple formula, which is relating d super fix T and the D matrix.

So, ultimately if you observe here D super fix T the notation for that is given by this one. This position represent the z rotation with respect to the current frame. This position represent the y rotation with respect to the current frame and this is the negative of x rotation with respect to the current frame etcetera, these are the translation with respect to the current frame.

So, if you compare the corresponding values here, we can easily see that the angular velocity about the z axis of the current frame is given by del dot a ok. Where del is the rotational velocity with respect to the fixed frame and a is the vector in the T matrix. So, if we take the dot product between the 2 that will give the angular velocity about the z axis of the current frame T that is given by this one.

Similarly, we compare for example, d x T. The linear velocity along the x direction of the current frame is given by this one, it is del dot p cross n plus d dot n. So, you can directly compare this thing and then we obtain the following formula.

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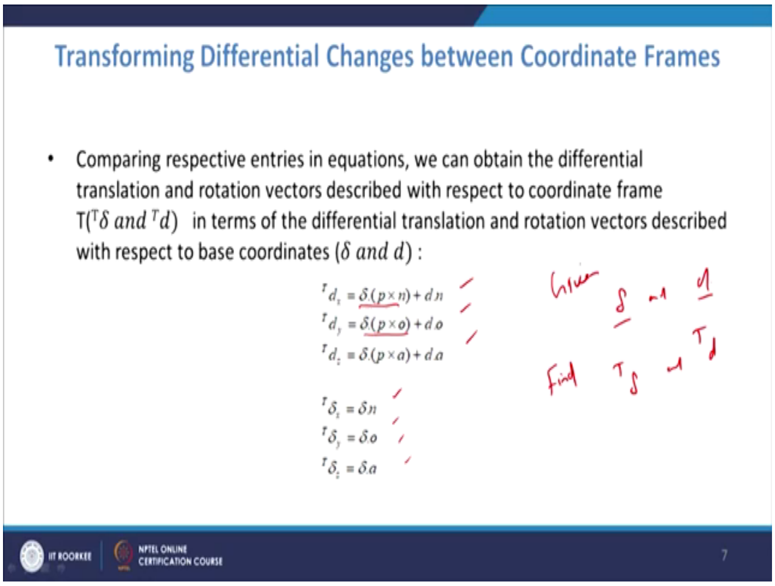

This gives the formula that the these are the linear velocities and then these are the angular velocity. So, what is important in this formula is if we know del and d. If it is given; that means, the translational velocity and rotational velocity with respect to the fixed frame, if it is given, then we can find the translational velocity and rotational velocity with respect to the current frame.

Given the small changes with respect to the fixed frame, we can find the corresponding small changes with respect to the current frame. So, that is the important of this result. And, here again we can make use of the form of this one, we can write del dot p cross n as n dot del cross p this can be changed, this also can be written as o dot del cross p etcetera.

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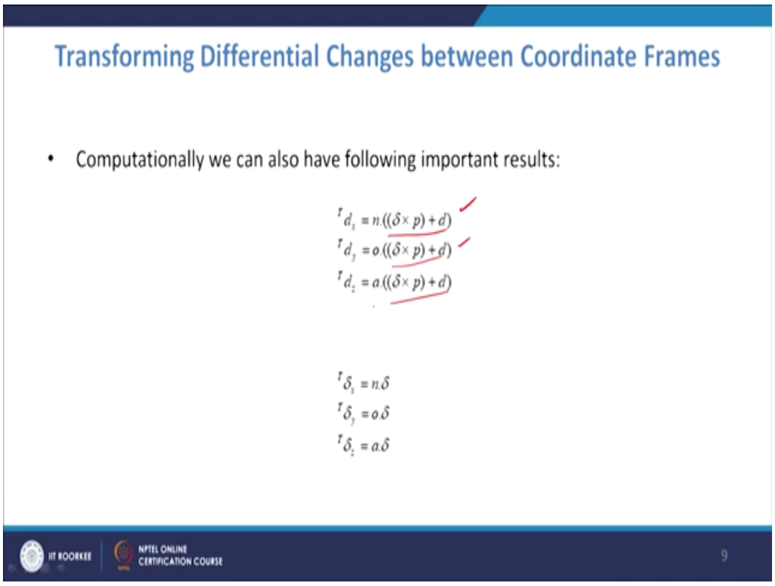

And then we can take this n as a common factor, we can get the res the result like this. Del cross p plus d is a vector which we can calculate and then if you take with the dot product with n that is the linear velocity, along the x direction dot product with o will give the linear velocity along the y direction of the current frame etcetera.

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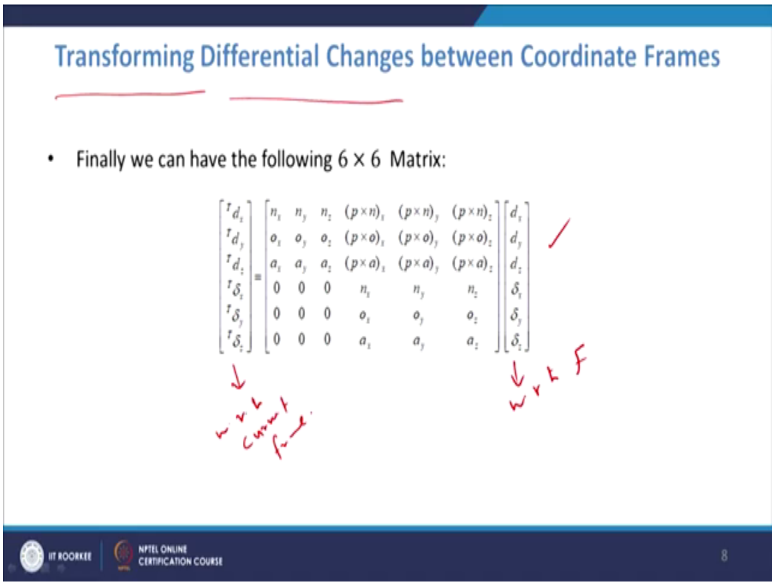

So, now if we summarize this in a matrix form we get here d x, d y, d z r this is with respect to the fixed frame and we can find it with respect to the current frame. So, this formula is a useful one to which is relating. So, it is the transforming the differential changes between coordinate frames F frame and M frame, we are transforming it and this formula is useful in that context.

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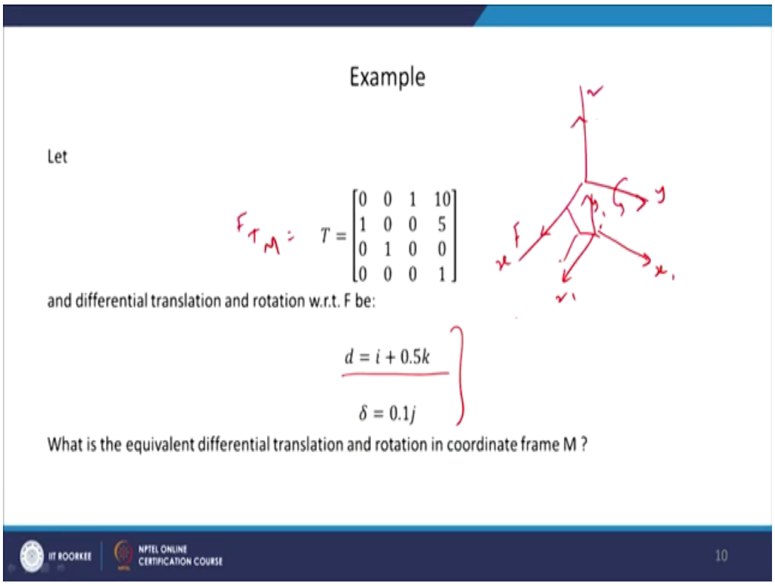

So, for example, if at a particular instant this is a given frame. And, then in that frame you have a the origin is shifted to 10, 5 and 0. Somewhere here and then the coordinate frame new coordinate frame is the x axis of the M frame is 0 1 0, that is this is 1 0 0, 0 1 0, 0 0 1. For example, these are the i j k is the original frame F frame and the new frame, which we are considering it as F T M, it is the x coordinate is 0 1 0. Along this direction and y coordinate is 0 0 1, it is along parallel to the z axis and then 1 0 0 this is d.

So, we have here x here, y here, z for the new coordinate frame, for this it is x y z. Now, we in this frame, if you are giving a small translational velocity along the x and z direction of the fixed frame, that is 1 unit along the x direction, x axis of the fixed frame 0.5 unit along the z axis of the fixed frame. If you are shifting it and then you are also rotating 0.1 radiance about the j axis of the fixed frame, that is we are rotating it like this.

So, this will obtain a new position somewhere else ok, we you will get a new coordinate frame, after making this much of change with respect to the fixed frame. Now, the same change the new position by giving small changes with respect to current frame itself can also be obtained. So, how much of translation rotation should be given with respect to the current frame, for obtaining the same effect.

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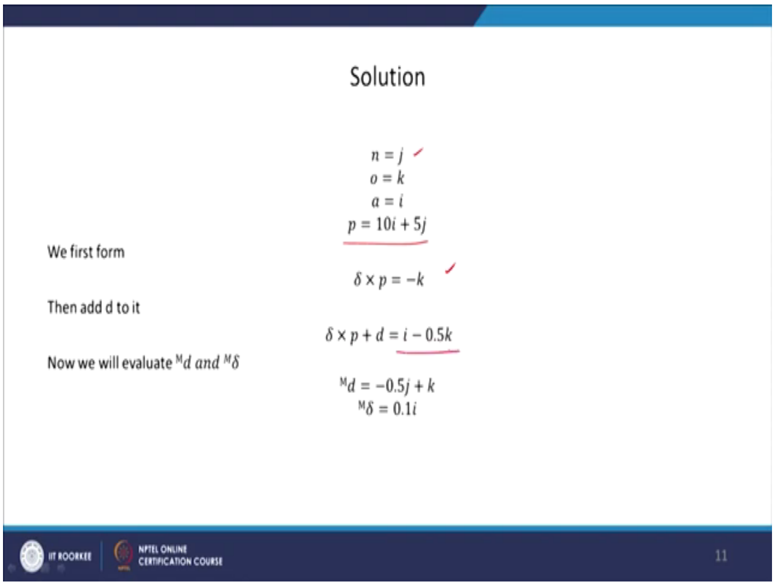

So, that is calculated by this as formula as given here. So, here n matrix is the first column; first column is 010. So, it is j and the o second column is k vector and the third column is i vector and fourth origin is shifted to 10 i plus 5 j position. So, and the del cross p is calculated where del is given by this one, this in vector and d is given by this. So, del cross p can be calculated, del cross p plus d is finally, this one this transformation.

And then by using that formula, this formula we get the rotation with respect to the current frame, this n dot del o dot l etcetera can be calculated again using this expression. This formula is between two coordinate frames. If you have F frame and M frame and the differential changes but if you have a chain of transformations in between then how to do it that we will see now?.

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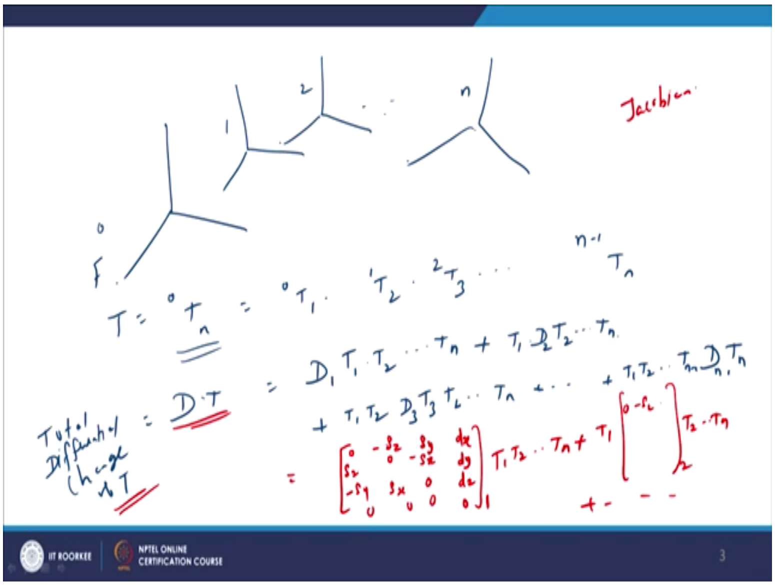

So, let us consider a fixed frame and there are several other frames let us say etcetera. And, then finally, we get a frame like this. So, let us call is as instead of fixed frame we call it as the 0th frame and this is the first one, this is second and then this is nth frame. Now, the relation between the starting frame that is 0th frame and the nth frame, we can write it as 0 T n.

And the relation between 0 frame and the first frame is 0 T 1. The homogeneous transformation between 0 and 1 frame and then homogeneous transformation between 1 and 2 frame is given by 1 T 2 let us say. And then 2 T 3 etcetera the last, but one is n minus 1th frame and this is the nth frame.

So, let us say there are n such chain of frames available and the first and last frame is related by 0 T n by multiplying all this matrices. So, each 1 is a 4 by 4 homogeneous transformation as we already know. Now, if there is a small change, if you are making changes at each and every frames.

A small translation and small rotation is made in the 0th frame that will affect the last frame also, if you make any change here the last frame will also be affected. Similarly, if you make any small change in the first frame, again the last frame will be affected etcetera. So, the small change at each and every frame will affect the last frame. So, we want to calculate if you call it as the T matrix T is 0 T n.

So, what is the differential change D T? As we have calculated earlier how to calculate the resultant changes that happened between all this frames and then the result on the nth frame that is given by D into T as we have seen earlier. So, how to calculate that is we have to add all such changes, because there are these are all partial changes, if you consider only one change in one of the frames, that is called a partially you are considering a change, that will affect the en the end frame.

Now, if you add all such changes, all such partial changes, you will get the total change the total differential change of T. So, finally, that is equal to D T let us says. So, what is the D matrix, which we have to calculate. So, that is given by the change with respect to the first frame and then plus the change with respect to the second frame etcetera. So, what we should do is the differential change that happened in the first frame, we call it as T 1.

So, instead of let us write 0 T 1 as just T 1, I am mentioning and then 1 T 2 I can mention as T 2 etcetera T n. So, we are just considering the change that has happened in the first coordinate frame and multiply all this plus now we consider the change that has happened in the second coordinate frame, that is 1 D T 2, that is D T 2 up to T n, because D 2 will not affect the first frame anyway, which whatever changes that are happening, in the second frame that is not going to affect the previous frame.

If you change in the second frame, it will affect the third frame fourth frame up to nth frame only not the previous one. The third change is T 1; T 2 will not be affected, but the change in the T 3 that will be up to T n etcetera. So, T 1 T 2 T n minus 1 the last one D n minus 1 D n T n. So, if these are the total changes we have to calculate for this one so, this we can substitute here.

So, whatever changes we are making for the first frame. So, this is given by 0 minus del z 1 we can call and minus that is del y d x d y d z 0 and del x. So, this is with respect to the first frame. So, we can write here 1 along with all these entries. So, I can write 1 here into T 1 T 2 etcetera T n.

So, similarly T 1 multiplied by the similar matrix, but 2 it denotes that all the entries are different for because the translation and rotation changes, which we are making for the first frame may not be the same for the second frame etcetera. So, even though the entries will look similar del z etcetera, but this two indicates that these numbers are corresponding to the changes in the second frame.

So, similarly up to n D 1 D 2 D n they are all different differential changes matrices that can be added here. So, if you add completely all of them that will give the final total change in the end effector the end frame that is 0 T n frame, whatever changes that has happened here will be obtained by adding all this terms. Finally, we will get a consolidated matrix, that is given by the expression.

This is the procedure for finding the total differential change of the frame T by adding all the partial changes with respect to each intermediate frames here. So, using this procedure we will define the manipulator Jacobian in the next lecture.

Thank you.