Robotics and Control: Theory and Practice

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Lecture - 04

Differential Transformation

Hello viewers, in this lecture we will see the velocity of a point in a coordinate frame and the velocity of a coordinate frame with respect to another coordinate frame using Differential Transformation.

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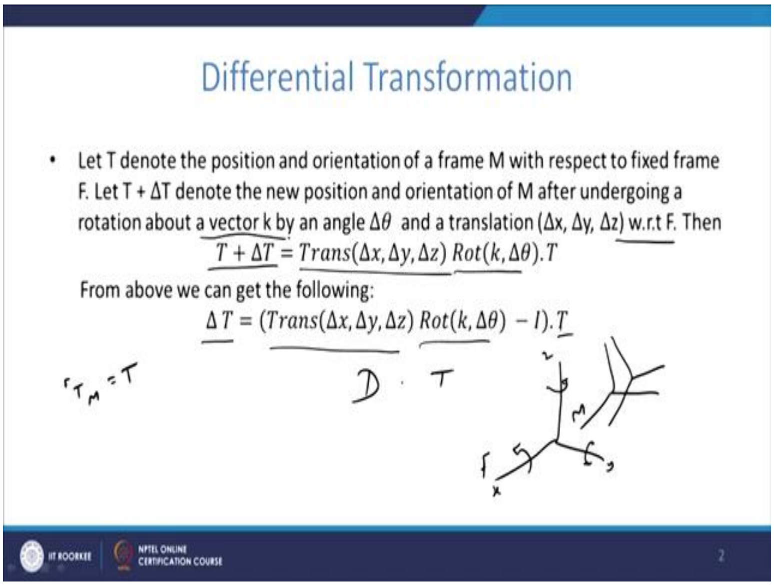

Let T denote the position and orientation of a frame M with respect to fixed frame F. Let T + ΔT denote the new position and orientation of M after undergoing a rotation about a vector k by an angle $\Delta\theta$ and a translation (Δx, Δy, Δz) w.r.t F. Then

$$T + \Delta T = Trans(\Delta x, \Delta y, \Delta z)\ Rot(k, \Delta\theta).T$$

From above we can get the following:

$$\Delta T = (Trans(\Delta x, \Delta y, \Delta z)\ Rot(k, \Delta\theta)\ - I).T$$

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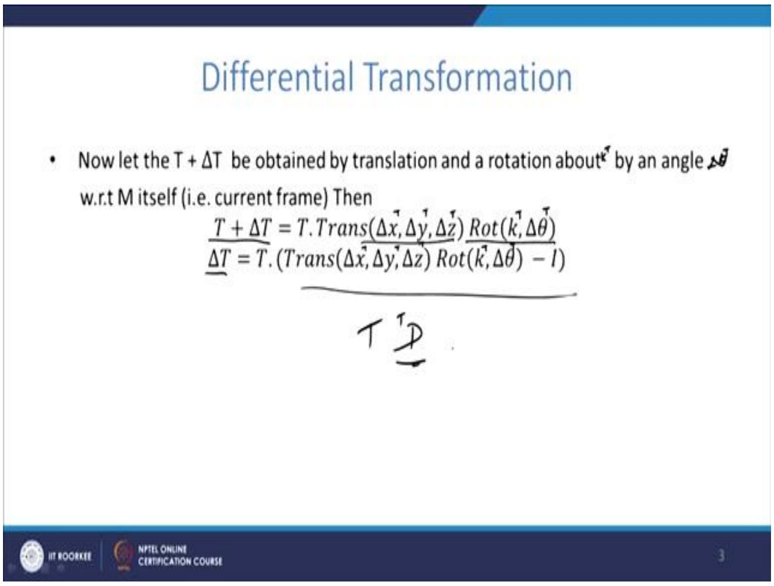

Now, in the similar manner we can obtain the same position T + ΔT by making a translation along the x y z direction of the M frame itself the current frame and rotation by an angle $\Delta\theta^T$ with respect to a unit vector k.

Now let the T + ΔT be obtained by translation and a rotation about by an angle $\Delta\theta^T$ w.r.t M itself (i.e. current frame) Then:

$$T + \Delta T = T.Trans(\Delta x^T, \Delta y^T, \Delta z^T)\ Rot(k^T, \Delta\theta^T)$$
$$\Delta T = T.(Trans(\Delta x^T, \Delta y^T, \Delta z^T)\ Rot(k^T, \Delta\theta^T)\ - I)$$

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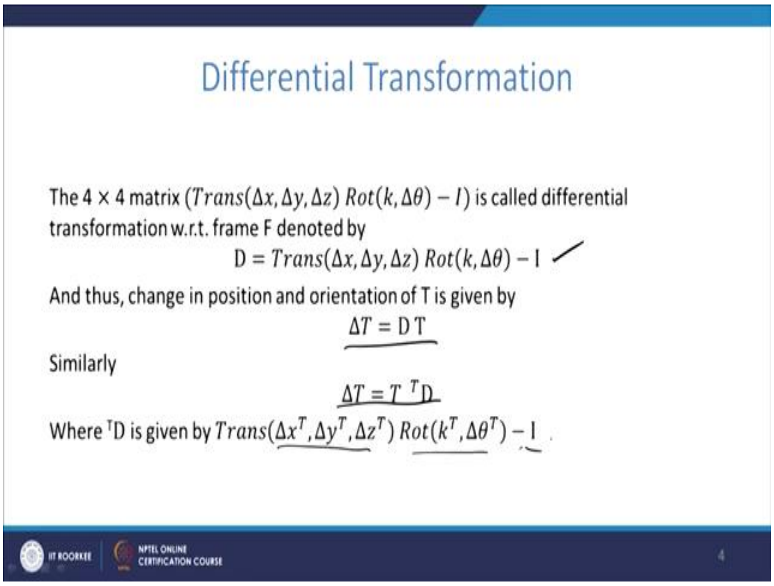

The 4 × 4 matrix $(Trans(\Delta x, \Delta y, \Delta z)\ Rot(k, \Delta\theta) - I)$ is called differential transformation w.r.t. frame F denoted by

$$\mathrm{D} = Trans(\Delta x, \Delta y, \Delta z)\ Rot(k, \Delta\theta) - \mathrm{I}$$

Thus, change in position and orientation of T is given by

$$\Delta T = \mathrm{D\ T}$$

Similarly

$$\Delta T = T\ {}^{T}\mathrm{D}$$

Where ${}^{T}\mathrm{D}$ is given by $Trans(\Delta x^{T}, \Delta y^{T}, \Delta z^{T})\ Rot(k^{T}, \Delta\theta^{T}) - \mathrm{I}$

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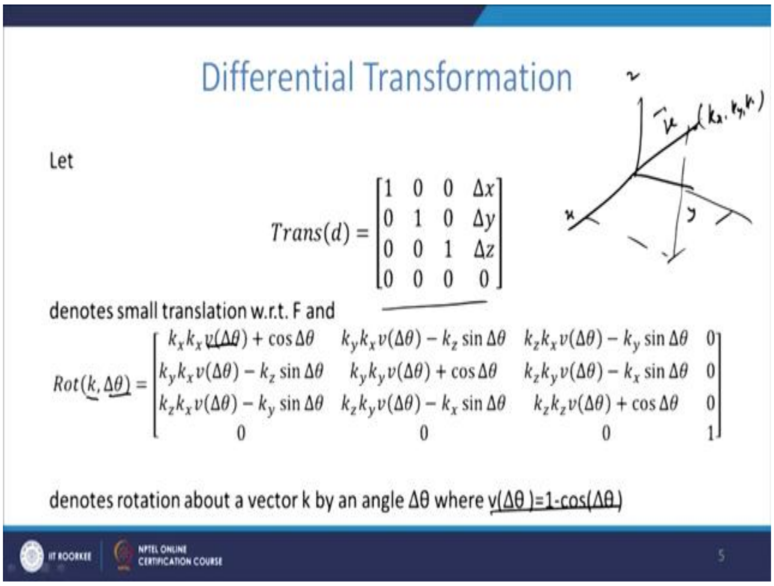

So, the translation matrix is given by:

$$Trans(d) = \begin{bmatrix} 1 & 0 & 0 & \Delta x \\ 0 & 1 & 0 & \Delta y \\ 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and the rotation matrix about a unit vector k by an angle delta theta is given by:

$$Rot(k,\Delta\theta) =$$
$$\begin{bmatrix} k_x k_x v(\Delta\theta) + \cos\Delta\theta & k_y k_x v(\Delta\theta) - k_z \sin\Delta\theta & k_z k_x v(\Delta\theta) - k_y \sin\Delta\theta & 0 \\ k_y k_x v(\Delta\theta) - k_z \sin\Delta\theta & k_y k_y v(\Delta\theta) + \cos\Delta\theta & k_z k_y v(\Delta\theta) - k_x \sin\Delta\theta & 0 \\ k_z k_x v(\Delta\theta) - k_y \sin\Delta\theta & k_z k_y v(\Delta\theta) - k_x \sin\Delta\theta & k_z k_z v(\Delta\theta) + \cos\Delta\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

denotes rotation about a vector k by an angle Δθ where v(Δθ)=1-cos(Δθ), where k_x, k_y, k_z are component of the unit vector. So, the meaning is $k_x^2 + k_y^2 + k_z^2 = 1$.

Here, we are dealing with a small translation and rotation so; that means, the values Δx, Δy, Δz and Δθ are very small values.

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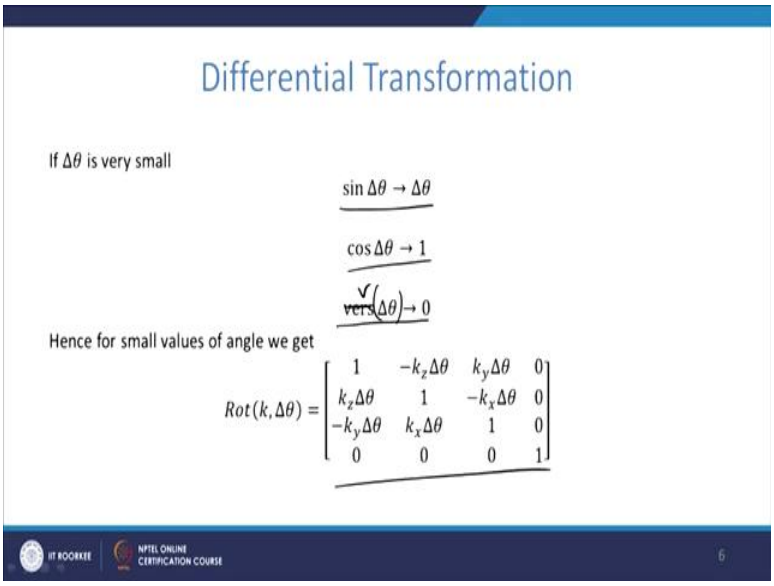

If $\Delta\theta$ is very small

$$\sin\Delta\theta \to \Delta\theta$$

$$\cos\Delta\theta \to 1$$

$$\mathrm{v}\,\Delta\theta \to 0$$

Hence, for small values of angle we get

$$Rot(k,\Delta\theta) = \begin{bmatrix} 1 & -k_z\Delta\theta & k_y\Delta\theta & 0 \\ k_z\Delta\theta & 1 & -k_x\Delta\theta & 0 \\ -k_y\Delta\theta & k_x\Delta\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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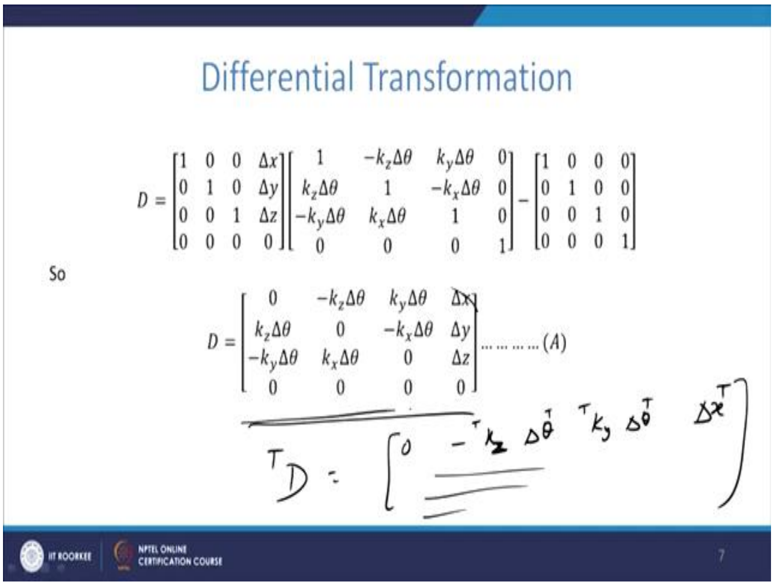

$$D = \begin{bmatrix} 1 & 0 & 0 & \Delta x \\ 0 & 1 & 0 & \Delta y \\ 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -k_z\Delta\theta & k_y\Delta\theta & 0 \\ k_z\Delta\theta & 1 & -k_x\Delta\theta & 0 \\ -k_y\Delta\theta & k_x\Delta\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So

$$D = \begin{bmatrix} 0 & -k_z\Delta\theta & k_y\Delta\theta & \Delta x \\ k_z\Delta\theta & 0 & -k_x\Delta\theta & \Delta y \\ -k_y\Delta\theta & k_x\Delta\theta & 0 & \Delta z \\ 0 & 0 & 0 & 0 \end{bmatrix} \dots\dots\dots (A)$$

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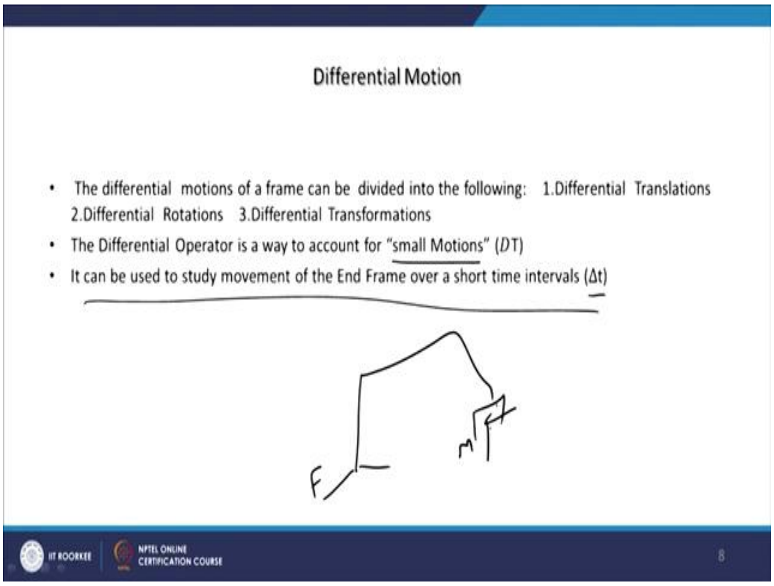

When it comes to the robotics, we can make use of this differential transformation because in a robot manipulator, there will be a end effector coordinate frame and the base coordinate frame it is a fixed frame and this will be like a moving frame. So, for a small time interval when the end effector frame makes a small rotation and translation, we can make use of the relation like this small translation rotation matrix with respect to M itself end effector itself or with respect to the base. So, both the things can be calculated and the relation can be found out using the following.

The differential motions of a frame can be divided into the following:

1. Differential Translations
2. Differential Rotations
3. Differential Transformations

   The Differential Operator is a way to account for "small Motions" (DT).It can be used to study movement of the End Frame over short time intervals (Δt)

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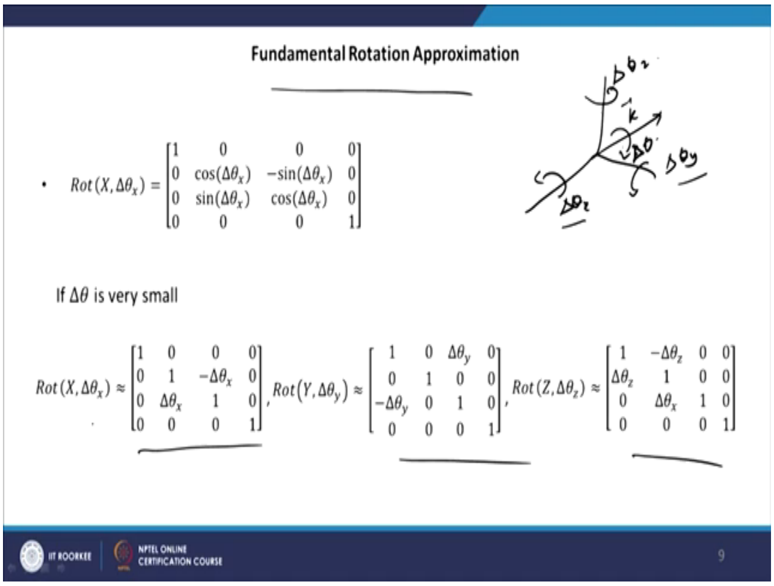

Now, another way of calculating the rotation matrix is, using the fundamental rotation matrices given by:

$$Rot(X, \Delta\theta_x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\Delta\theta_x) & -\sin(\Delta\theta_x) & 0 \\ 0 & \sin(\Delta\theta_x) & \cos(\Delta\theta_x) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If $\Delta\theta$ is very small

$$Rot(X, \Delta\theta_x) \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\Delta\theta_x & 0 \\ 0 & \Delta\theta_x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rot(Y, \Delta\theta_y) \approx \begin{bmatrix} 1 & 0 & \Delta\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\Delta\theta_y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rot(Z, \Delta\theta_z) \approx \begin{bmatrix} 1 & -\Delta\theta_z & 0 & 0 \\ \Delta\theta_z & 1 & 0 & 0 \\ 0 & \Delta\theta_x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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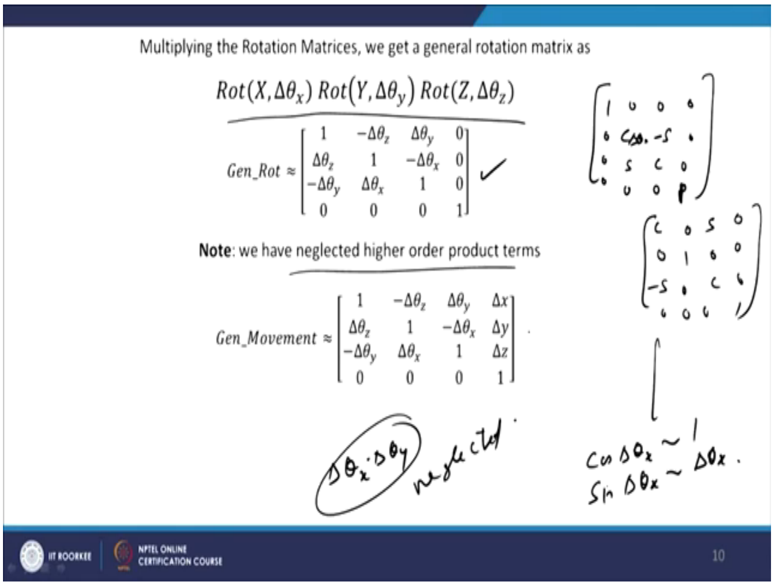

Multiplying the Rotation Matrices, we get a general rotation matrix as:

$$Rot(X, \Delta\theta_x)\ Rot\left(Y, \Delta\theta_y\right)\ Rot(Z, \Delta\theta_z)$$

$$Gen\_Rot \approx \begin{bmatrix} 1 & -\Delta\theta_z & \Delta\theta_y & 0 \\ \Delta\theta_z & 1 & -\Delta\theta_x & 0 \\ -\Delta\theta_y & \Delta\theta_x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Gen\_Movement \approx \begin{bmatrix} 1 & -\Delta\theta_z & \Delta\theta_y & \Delta x \\ \Delta\theta_z & 1 & -\Delta\theta_x & \Delta y \\ -\Delta\theta_y & \Delta\theta_x & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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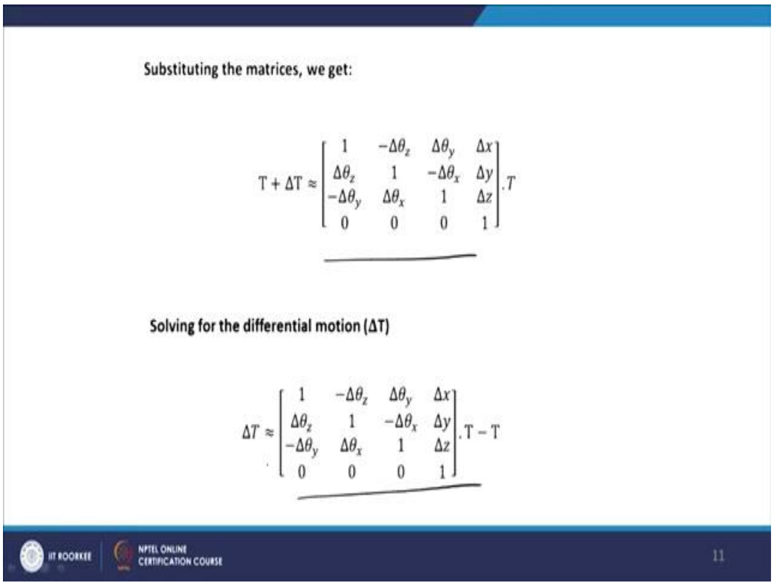

Now, if we substitute for the T + ΔT this is the new position of the M frame. As we have calculated in the beginning, T is a initial position and T + ΔT is the new position after a small change. So, here also we do the same thing the small rotation is given by:

$$\mathrm{T} + \Delta\mathrm{T} \approx \begin{bmatrix} 1 & -\Delta\theta_z & \Delta\theta_y & \Delta x \\ \Delta\theta_z & 1 & -\Delta\theta_x & \Delta y \\ -\Delta\theta_y & \Delta\theta_x & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix} . T$$

$$\Delta T \approx \begin{bmatrix} 1 & -\Delta\theta_z & \Delta\theta_y & \Delta x \\ \Delta\theta_z & 1 & -\Delta\theta_x & \Delta y \\ -\Delta\theta_y & \Delta\theta_x & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix} . \mathrm{T} - \mathrm{T}$$

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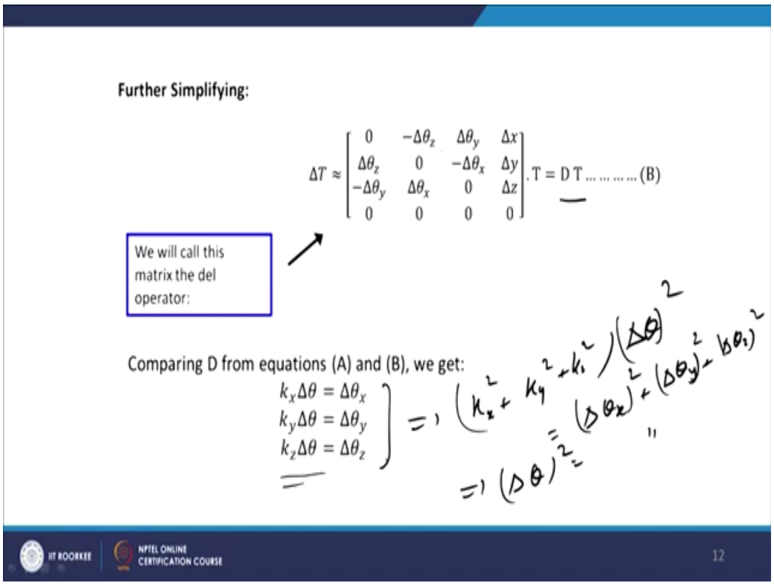

$$\Delta T \approx \begin{bmatrix} 0 & -\Delta\theta_z & \Delta\theta_y & \Delta x \\ \Delta\theta_z & 0 & -\Delta\theta_x & \Delta y \\ -\Delta\theta_y & \Delta\theta_x & 0 & \Delta z \\ 0 & 0 & 0 & 0 \end{bmatrix} . \mathrm{T} = \mathrm{D\ T} \ldots\ldots\ldots\ldots (\mathrm{B})$$

. So, if you compare equations A and B we get :

$$k_x\Delta\theta = \Delta\theta_x$$

$$k_y\Delta\theta = \Delta\theta_y$$

$$k_z\Delta\theta = \Delta\theta_z$$

So, the meaning is when we make a here when we make a rotation by an angle $\Delta\theta$ about the vector k unit vector k, the same thing same effect is obtained by rotating about the x y z axis by this angles. So, the relation between $\Delta\theta$ and $\Delta\theta_x$ $\Delta\theta_y$ $\Delta\theta_z$ and the components of k: k_x, k_y, k_z. So:

$$k_x^2 + k_y^2 + k_z^2 = 1$$

So, knowing these values, the rotation about x y z we can calculate the component of the k vector as well as how much angle we have rotated about the k vector can be calculated.

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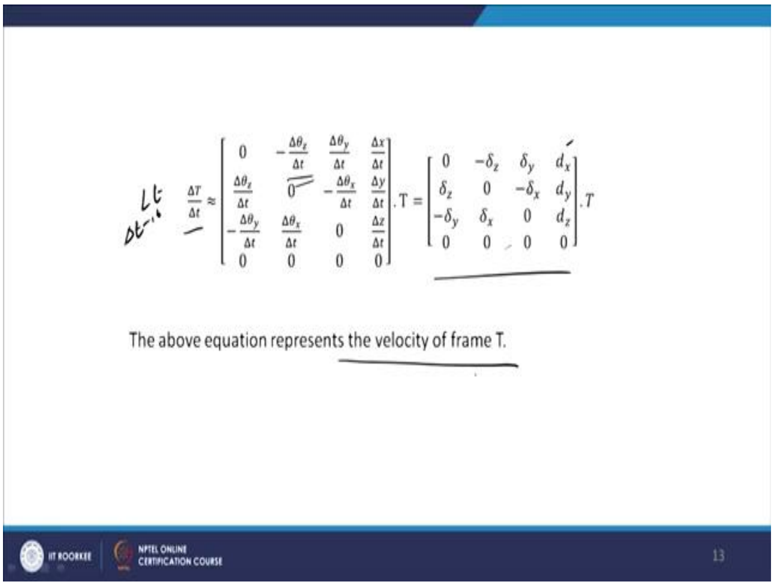

So, now we come to the main formula which represent the velocity of a frame.

$$(lim\_(\Delta t \to 0)\ )\ \frac{\Delta T}{\Delta t} \approx \begin{bmatrix} 0 & -\frac{\Delta\theta_z}{\Delta t} & \frac{\Delta\theta_y}{\Delta t} & \frac{\Delta x}{\Delta t} \\ \frac{\Delta\theta_z}{\Delta t} & 0 & -\frac{\Delta\theta_x}{\Delta t} & \frac{\Delta y}{\Delta t} \\ -\frac{\Delta\theta_y}{\Delta t} & \frac{\Delta\theta_x}{\Delta t} & 0 & \frac{\Delta z}{\Delta t} \\ 0 & 0 & 0 & 0 \end{bmatrix} .\mathrm{T} = \begin{bmatrix} 0 & -\delta_z & \delta_y & d_x \\ \delta_z & 0 & -\delta_x & d_y \\ -\delta_y & \delta_x & 0 & d_z \\ 0 & 0 & 0 & 0 \end{bmatrix} .T$$

Here, linear velocity is we denote by d_x, d_y, d_z.

Similarly, rotational velocity angular velocity about x y z are denoted by $\delta_x, \delta_y, \delta_z$. So, the velocity of a frame is given by the coordinate homogeneous transformation T and in the left side we have to multiply by the velocity along the x y z direction and the rotation about the x y z direction angular velocity and the linear velocity matrix. So, this is a very important formula which represent the velocity of a coordinate frame T.

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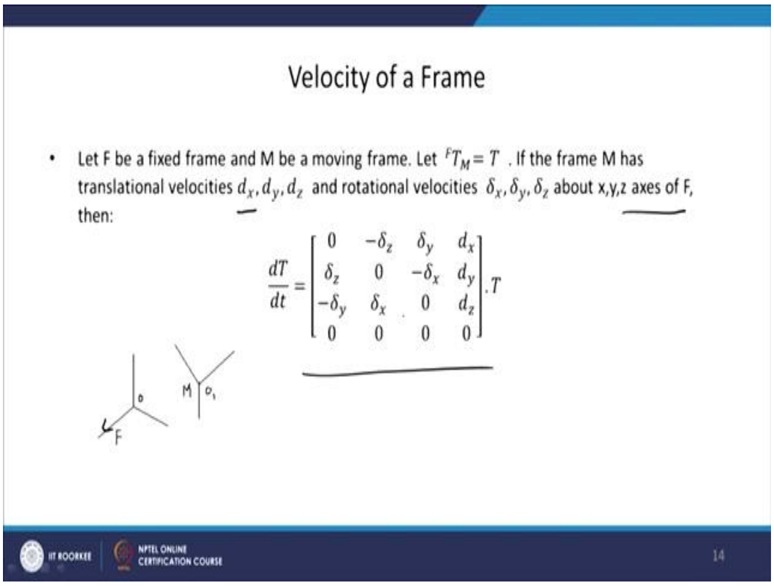

So, now so, this is the velocity of which have seen just now. If you are taking the velocity d_x, d_y, d_z. and $\delta_x, \delta_y, \delta_z$ with respect to a frame F. Therefore, these velocities, which we mentioned here, are the velocity with respect to the x y z axis of the fixed frame. So, that means, d_x for example, means the frame M moves d_x distance in unit time along the x direction of the F frame.

So, like that d_x, d_y, d_z. represent the movement along the x y z direction of the F frame only and similarly the rotation with respect to the F frame. So, we have to multiply in the left side by this matrix to get the velocity of the frame.

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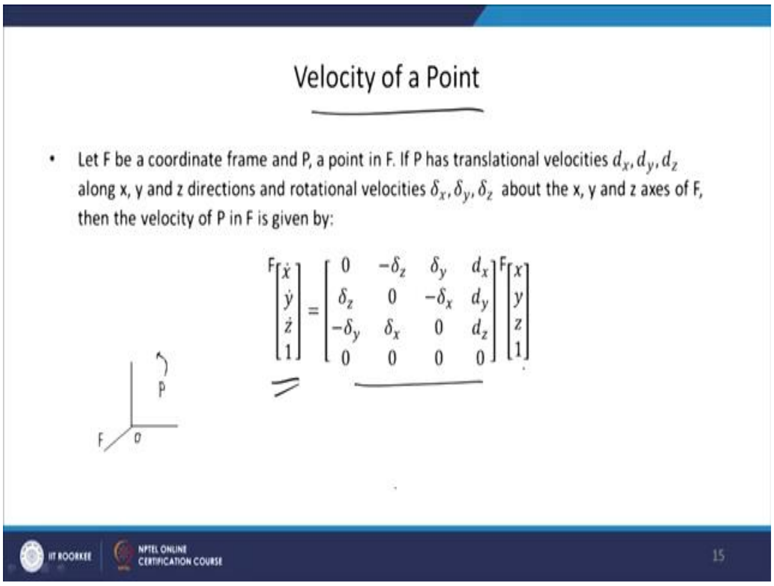

Now, velocity of a point. If you have a point in a fixed frame and it is moving with linear velocity d_x, d_y, d_z. and angular velocity $\delta_x, \delta_y, \delta_z$. we just multiply the point x y z by the matrix here that gives $\dot{x}, \dot{y}, \dot{z}$. denotes velocity of the point instantaneous velocity of the point given by:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -\delta_z & \delta_y & d_x \\ \delta_z & 0 & -\delta_x & d_y \\ -\delta_y & \delta_x & 0 & d_z \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

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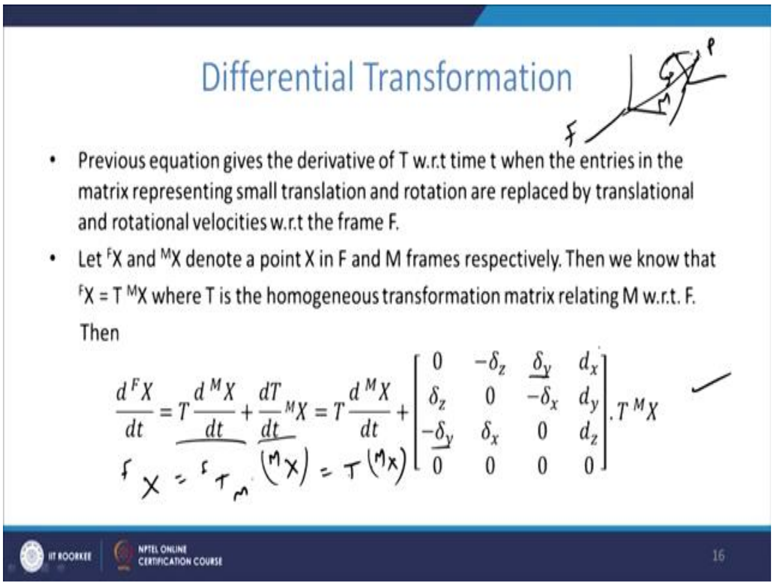

Now, if you combine both if you have a coordinate frame fixed frame and another frame is the moving frame and there is a point p in the moving frame. So, if the point p is moving with respect to the moving th M frame itself let us say it is moving in this frame and the M frame itself is moving with respect to the F frame then how to find the velocity of the point p with respect to the F frame.

Let ${}^{F}X$ and ${}^{M}X$ denote a point X in F and M frames respectively. Then we know that

${}^{F}X = T\,{}^{M}X$ where T is the homogeneous transformation matrix relating M w.r.t. F.

Then

$$\frac{d\,{}^{F}X}{dt} = T\frac{d\,{}^{M}X}{dt} + \frac{dT}{dt}MX = T\frac{d\,{}^{M}X}{dt} + \begin{bmatrix} 0 & -\delta_z & \delta_y & d_x \\ \delta_z & 0 & -\delta_x & d_y \\ -\delta_y & \delta_x & 0 & d_z \\ 0 & 0 & 0 & 0 \end{bmatrix} . T\,{}^{M}X$$

So, this a very useful relation to find the velocity of a point with respect to the fixed frame.

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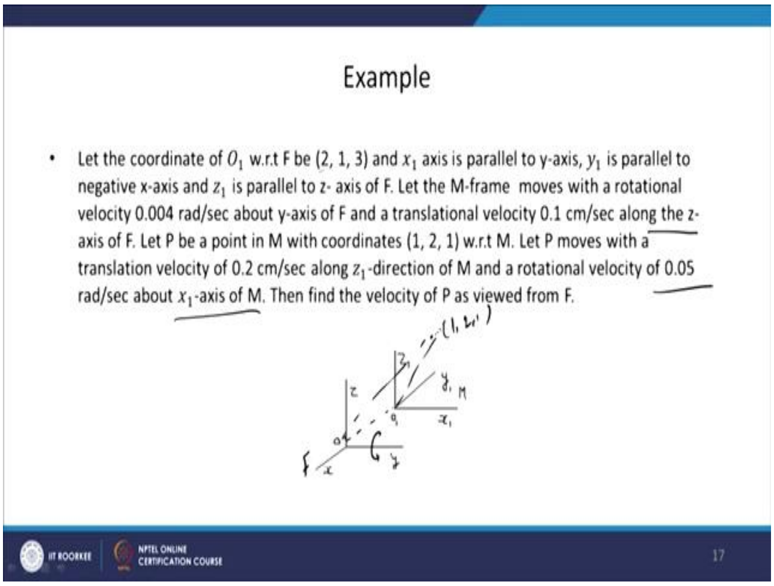

So, in this example it is given that:

Let the coordinate of O_1 w.r.t F be (2, 1, 3) and x_1 axis is parallel to y-axis, y_1 is parallel to negative x-axis and z_1 is parallel to z- axis of F. Let the M-frame moves with a rotational velocity 0.004 rad/sec about y-axis of F and a translational velocity 0.1 cm/sec along the z-axis of F. Let P be a point in M with coordinates (1, 2, 1) w.r.t M. Let P moves with a translation velocity of 0.2 cm/sec along z_1-direction of M and a rotational velocity of 0.05 rad/sec about x_1-axis of M. Then find the velocity of P as viewed from F.

Solution:

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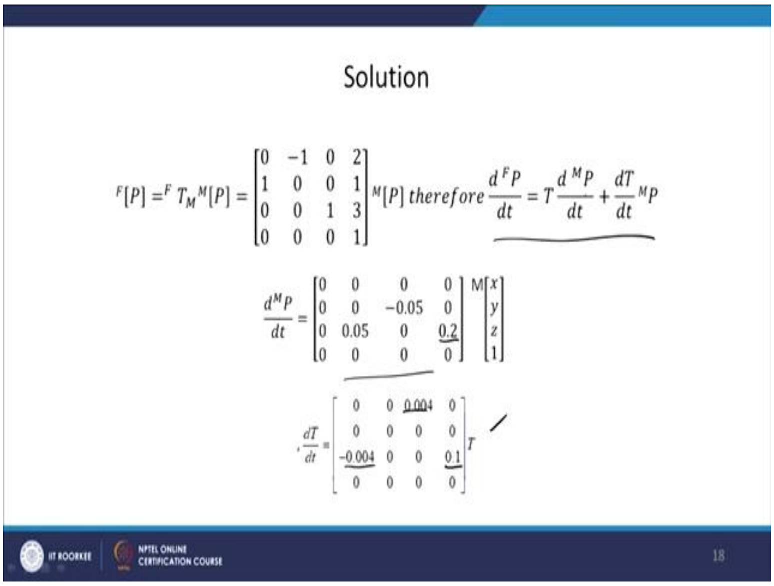

$$^{F}[P] =^{F} T_M M[P] = \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} {}^{M}[P]\ therefore \frac{d\,^{F}P}{dt} = T\frac{d\,^{M}P}{dt} + \frac{dT}{dt}MP$$

$$\frac{d^{M}P}{dt} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -0.05 & 0 \\ 0 & 0.05 & 0 & 0.2 \\ 0 & 0 & 0 & 0 \end{bmatrix}^{M} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$,\frac{dT}{dt} = \begin{bmatrix} 0 & 0 & 0.004 & 0 \\ 0 & 0 & 0 & 0 \\ -0.004 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 \end{bmatrix} T$$

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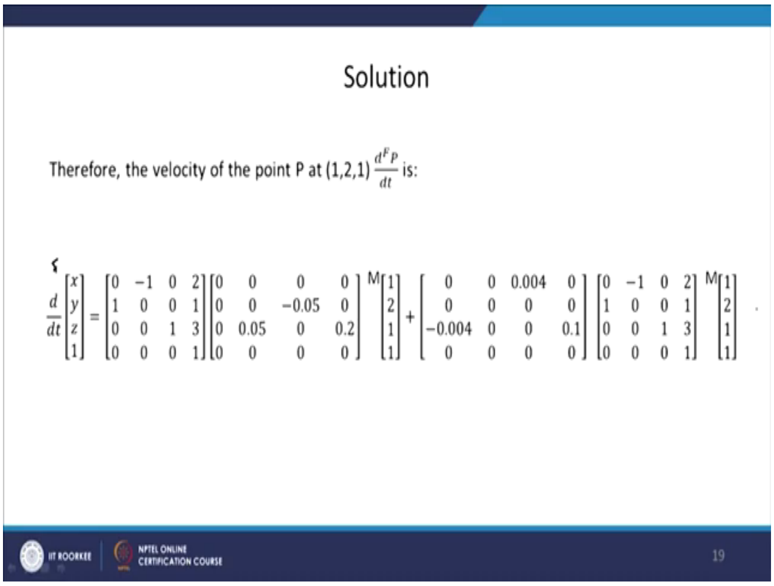

Therefore, the velocity of the point P at (1,2,1) $\frac{d^F P}{dt}$ is:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -0.05 & 0 \\ 0 & 0.05 & 0 & 0.2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0.004 & 0 \\ 0 & 0 & 0 & 0 \\ -0.004 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

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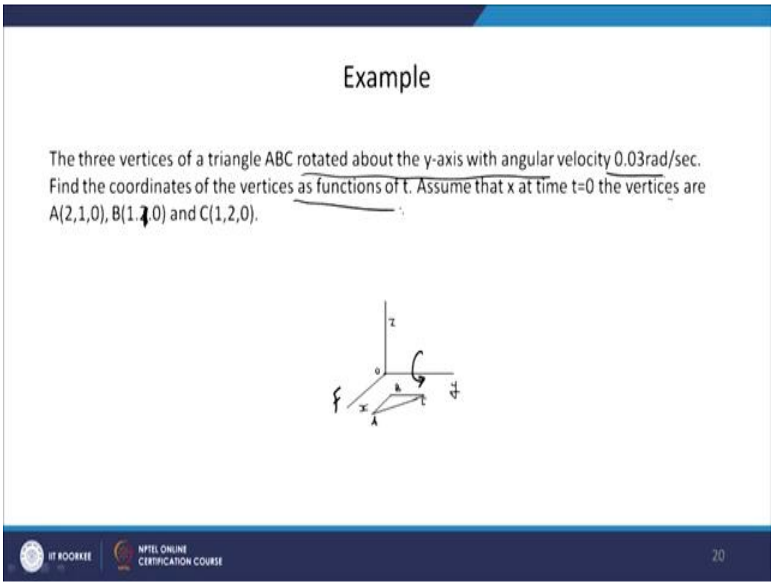

So, similarly we can also see this example:

The three vertices of a triangle ABC rotated about the y-axis with angular velocity 0.03rad/sec. Find the coordinates of the vertices as functions of t. Assume that x at time t=0 the vertices are A(2,1,0), B(1.1.0) and C(1,2,0).

So, we are asked to find the position assuming that this thing, we want to find the position of the vertices at a given time t any another time t.

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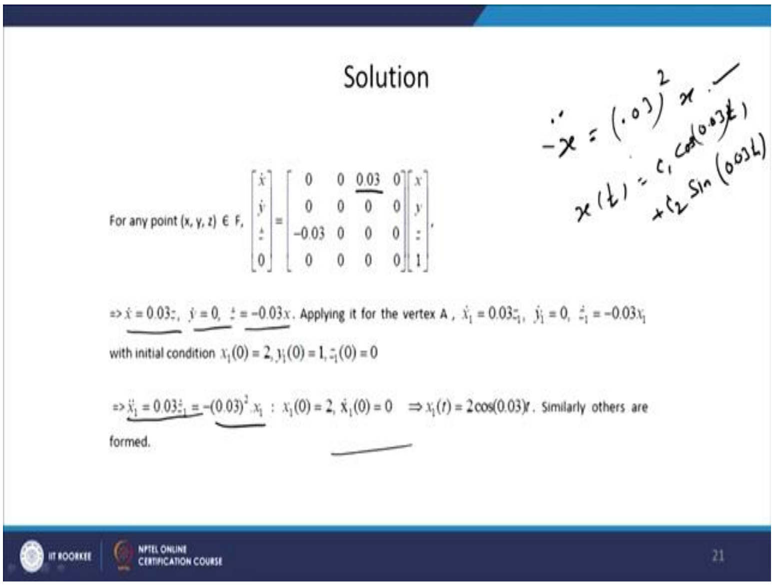

Solution:

For any point (x, y, z) ∈ F, $\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.03 & 0 \\ 0 & 0 & 0 & 0 \\ -0.03 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$,

=> $\dot{x} = 0.03z$, $\dot{y} = 0$, $\dot{z} = -0.03x$. Applying it for the vertex A , $\dot{x}_1 = 0.03z_1$, $\dot{y}_1 = 0$, $\dot{z}_1 = -0.03x_1$

with initial condition $x_1(0) = 2, y_1(0) = 1, z_1(0) = 0$

=> $\ddot{x}_1 = 0.03\dot{z}_1 = -(0.03)^2 .x_1$: $x_1(0) = 2$, $\dot{x}_1(0) = 0$ $\Rightarrow x_1(t) = 2\cos(0.03)t$. Similarly others are formed.

So, in this lecture we have seen the formula for the velocity of a point in a coordinate frame and the velocity of a coordinate frame with respect to another coordinate frame and how to utilize the formulas the standard formula for computing the position and orientation of a point at each instant of time as it is moving using this formulas. So, in the next lecture we will see further related results about the differential transformation.

Thank you.