Robotics and Control: Theory and Practice
Prof. Felix Orlando
Department of Electrical Engineering
Indian Institute of Technology, Roorkee

Lecture – 38
Flexible Link Kinematics-II

Good morning, today we are going to see about the Flexible Link Kinematics II. The basically what is the solution for the function of the spatial coordinates under three cases of a 2 link planar flexible manipulator.

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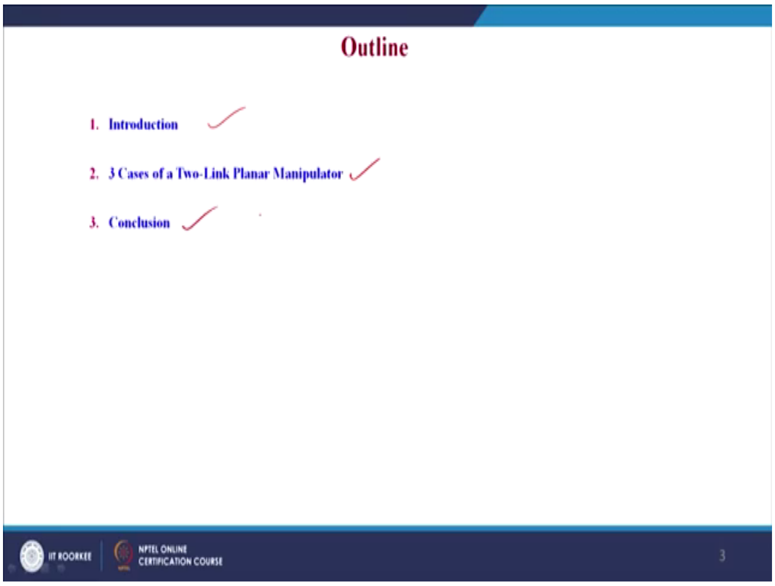

The outline of this lecture will be as follows, first we have the introduction then we have the three cases associated with the boundary conditions for a 2 link planar flexible manipulator. Then finally, we draw the conclusion from this discussion.

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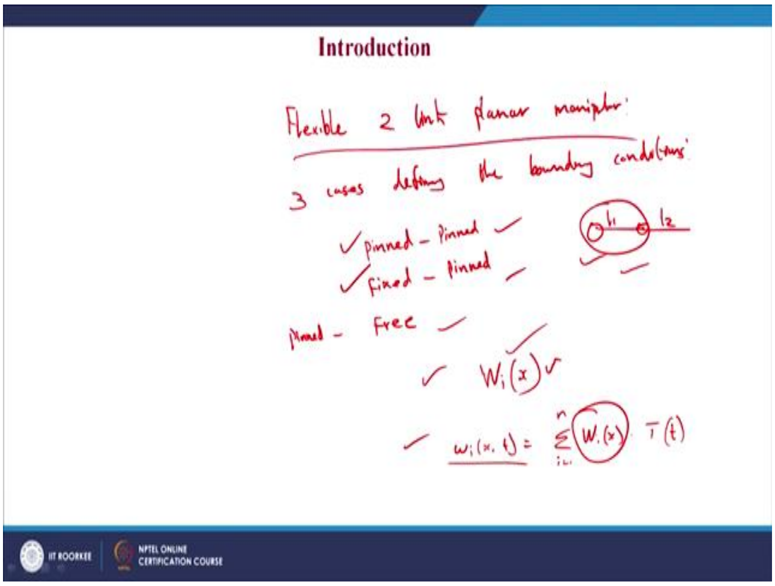

Coming to the introduction, so we are going to see about the flexible 2 link planar manipulator with 3 cases defining the boundary conditions such as pinned, fixed and free. So, the combination of the 2 links, see this is link 1, link 2 will be with this formation; pinned pinned for first link and the situation will be fixed pinned and pinned free.

So, these are the cases we are going to discuss in this lecture, precisely I will mention what are the three different cases. So, that the solution for $W_i(x)$ can be obtained that is the solution for the function of the spatial coordinate can be obtained for these two links with this boundary conditions.

Whether the one end is clamped that is fixed or pinned, or the other end is free pinned or fixed. Based on these three combinations we can obtain the solution for the function of the generalized spatial coordinate that is $W_i(x)$. Because $W_i(x, t)$ is given by that is a deflection part of a flexible manipulator or a flexible link. Generally the deflection part of the flexible link is given by $W_i(x, t) = \sum_{i=1}^{n} W_i(x)$.

Where this defines the number of modes because, for a link there can be n number of or infinite number of modes and each one with the independent natural frequency. So, this definition is given by $i(x)T(t)$. So, this solution which is a function of the spatial coordinate is the one we are going to see for these three cases for the particular flexible 2 link planar manipulator in this lecture.

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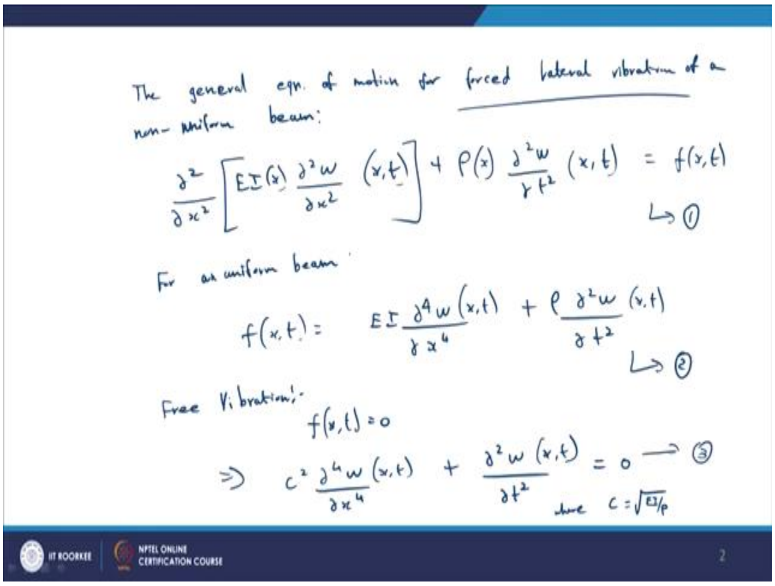

Now, let me start this lecture with the general equation of motion for forced lateral vibration of a non-uniform beam is given by

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 W}{\partial x^2}(x,t)\right] + \rho(x)\frac{\partial^2 W}{\partial t^2}(x,t) = f(x,t) \qquad (1)$$

Where $EI(x)$ is the flexural rigidity of the concerned beam, $\frac{\partial^2 W}{\partial x^2}(x,t)$ $+\rho(x)$ which is the uniform density into $\frac{\partial^2 W}{\partial t^2}(x,t)$ is equal to $f(x,t)$ which is under forced later vibration condition, say this is equation number 1. For an uniform beam this equation can be reduced into

$$f(x,t) = EI(x)\frac{\partial^4 W}{\partial x^4}(x,t) + \rho\frac{\partial^2 W}{\partial t^2}(x,t) \qquad (2)$$

Then under free vibration condition the force part is 0, because a forced vibrate lateral vibration is now not there which is under the condition of free vibration. So, we have $f(x,t) = 0$ which implies

$$C^2\frac{\partial^4 W}{\partial x^4}(x,t) + \frac{\partial^2 W}{\partial t^2}(x,t) = 0 \qquad (3)$$

Where $C = \sqrt{\frac{EI}{\rho}}$.

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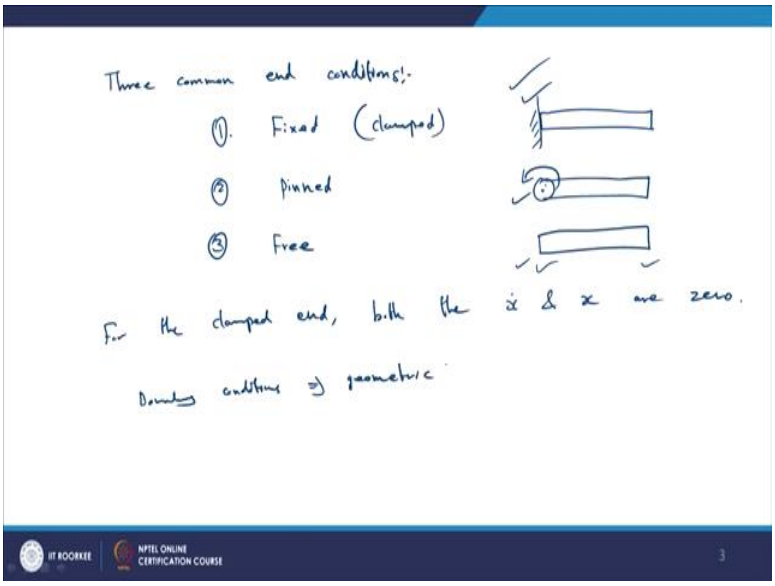

Next we move on to the three common conditions we are going to consider in our study, that is three cases that is basically instead of saying three cases I can tell that it is basically three common end conditions. I can specify this by three common end conditions of a beam, what are the three common end conditions of a beam? Number one it is fixed; it is also called as clamped. Second is pinned, third one is free, so these are the common end conditions.

For example you consider a flexible link, now fixed means it is this it is fixed and pinned means it will have this configuration, That means, it is free to rotate around this hinge point, and the free is this the end is free as you can see here. So, this is fixed or clamped and the pinned one is this configuration and the free one is this configuration.

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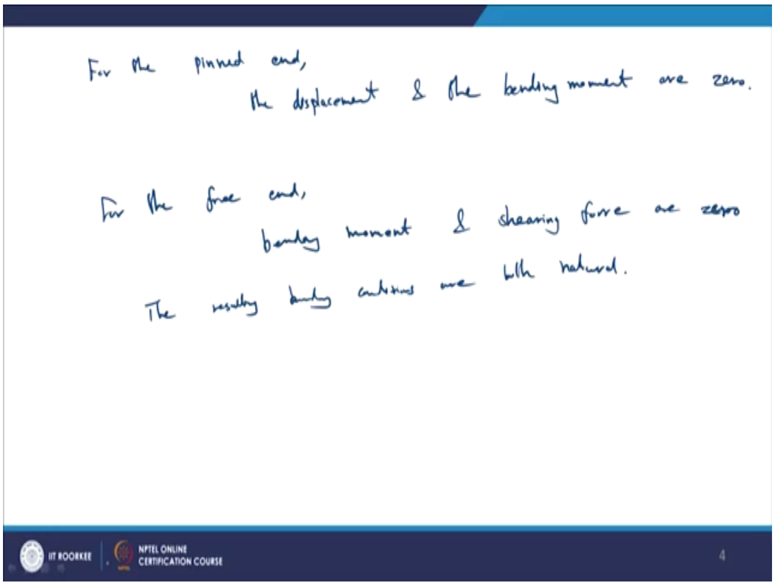

Now, for the clamped end for the clamped end both the displacement and sorry both the velocity and displacement you can see. Both the displacement and the velocity are zero for the clamped condition. So, the resulting boundary conditions are both geometric, so the boundary conditions are geometric in nature.

Now, for the pinned end; for the pinned end, the displacement and the bending moment are zero. Now, for the free end both the bending moment and shearing force are zero. The resulting boundary conditions are conditions are both natural here for the free end.

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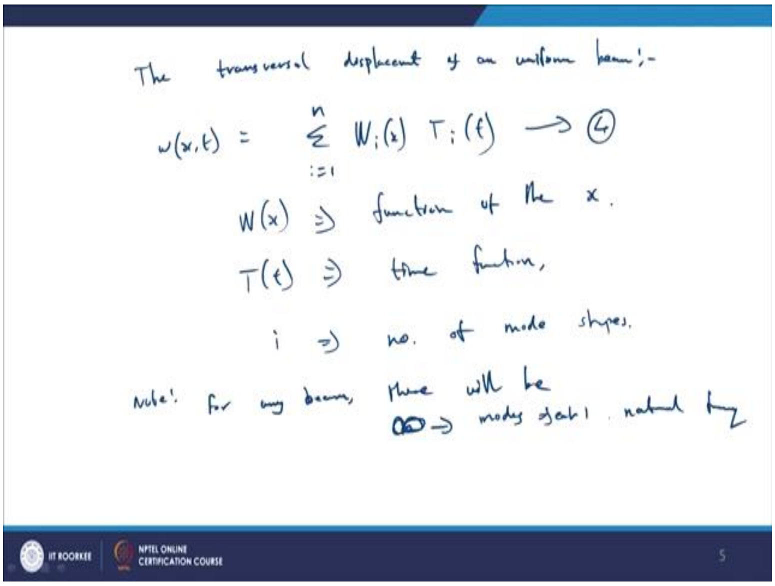

Now, the transversal displacement of an uniform beam is given by

$$W(x,t) = \sum_{i=1}^{n} W_i(x)T_i(t) \qquad (4)$$

This equation is obtained using the method of separation of variables, so that we have this equation number 4 which is the transversal displacement of an uniform beam based on the technique of separation of variables.

We get this equation where capital $W_i(x)$ is function of the spatial coordinate x. And $T_i(t)$ is the time function and i denotes the number of total number of mode shape associated in the concerned uniform beam. Now, it has to be noted that for any beam there will be n number of or you can say instead of n. Being the finite you can say that we can have infinite number of modes with each one with 1 natural frequency, each one will have 1 natural frequency.

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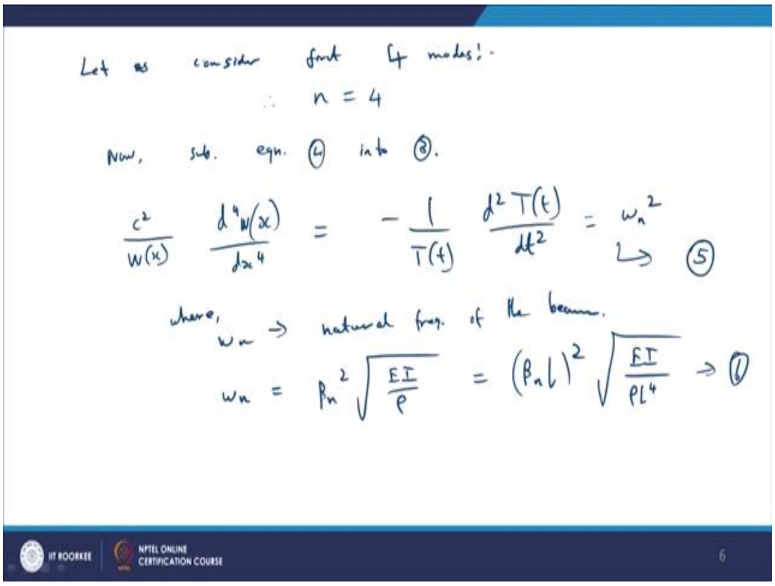

And now, let us consider first 4 modes, so therefore, we are assuming are considering $n = 4$ because we are considering first 4 modes. And now, substitute equation number 4 into equation number 3; we get

$$\frac{C^2}{W(x)}\frac{d^4W(x)}{dx^4} = -\frac{1}{T(t)}\frac{d^2T(t)}{dt^2} = w_n^2 \qquad (5)$$

Say this is equation number 5, where w_n is the natural frequency of the beam of the beam.

And $w_n = \beta_n \sqrt{\frac{EI}{\rho}}$ which is further given by

$$w_n = (\beta_{n1})^2 \sqrt{\frac{EI}{\rho_1^4}} \qquad (6)$$

which is equation number 6.

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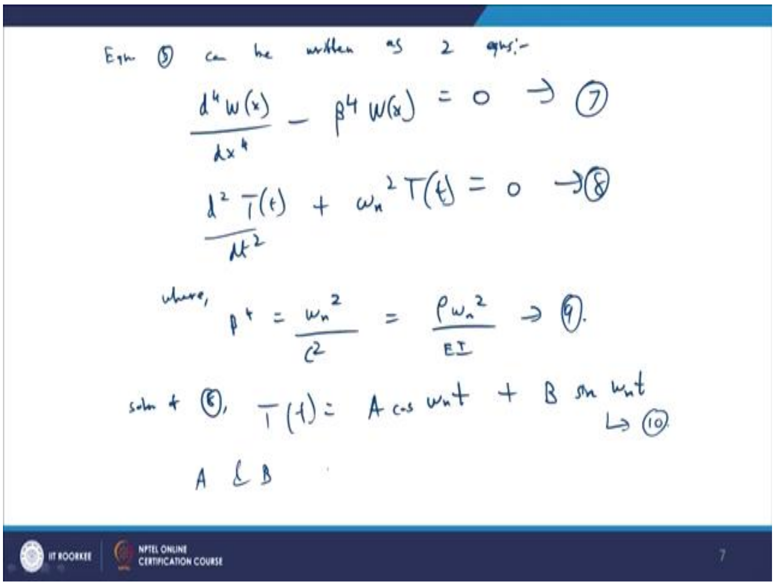

Now, equation 5 can be written as 2 equations, further given by

$$\frac{d^4W(x)}{dx^4} - \beta^4 W(x) = 0 \qquad (7)$$

say this is equation number 7. And the other equation associated from the or derived from equation 5 is

$$\frac{d^2T(x)}{dt^2} - w_n^2 T(x) = 0 \qquad (8)$$

which is number 8 equation number 8. Where,

$$\beta^4 = \frac{w_n^2}{C^2} = \rho \frac{w_n^2}{EI} \qquad (9)$$

say equation number 9.

Now, the solution of equation 8 which is given by that is solution of equation 8 which is given by

$$T(t) = A\,cosw_n t + B\,sinw_n t \qquad (10)$$

Keep this equation that is the solution for equation number 2, where say this is equation number 10. Where A and B are nothing but the initial displacement and the initial velocity of the beam ok and they are determined by the initial conditions of the system.

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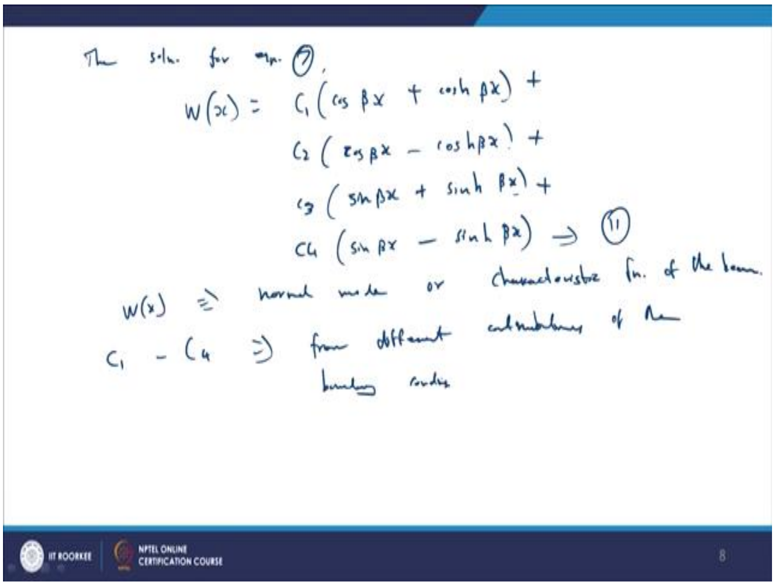

Now, the solution for equation number 7 is given by which is the function of the spatial coordinate which is $W(x)$ is given by

$$W(x) = C_1(Cos\beta x + cosh\beta x) + C_2(Cos\beta x - cosh\beta x) + C_3(sin\beta x + sinh\beta x) + C_4(sin\beta x - sinh\beta x) \qquad (11)$$

say this is equation number 11.

The function $W(x)$ is known as the normal mode or characteristic function of the beam characteristic function of the beam $W(x)$. Now, the unknowns C_1 to C_4 are computed or determined based on the different conditions of the boundary conditions or different contributions of the boundary conditions. They are obtained from different contributions of the boundary conditions; of the boundary conditions of the beam.

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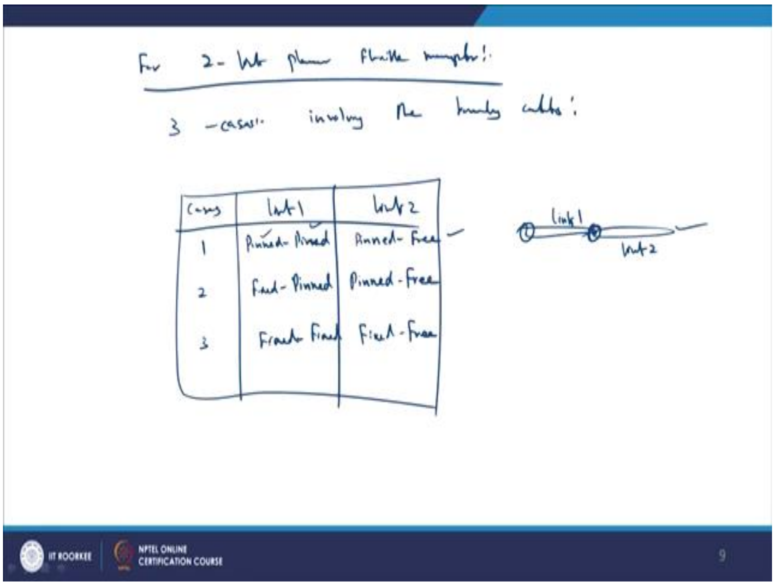

So, now let us consider for 2 link planer flexible manipulator we consider 3 cases involving the boundary conditions, so 3 cases for link 1 and link 1. Let me have a table to show that these are the cases and this is link 1, link 2. First is case number 1 which is pinned and pinned and similarly for link 2 it is pinned and free. And case 2 is fixed and pinned and here it is pinned and free; case 3 is fixed and fixed and link two is fixed and free.

So, these are the combinations of the links; this is link 1, this is link 2. How they are? They are first a pinned and pin and here pinned and free. So, you can see that link 1 link 1 is pinned and pinned you can see that pinned and pinned. And link 2 which is this is link 2 which is pinned here in one end and other end is free that is why.

This is similarly these other cases for the same tooling planar manipulator is fixed pinned, pinned, free; and third case is fixed, fixed, and fixed and free. Now, let us get the solution of the function of the spatial coordinate for these 3 cases of this flexible manipulator.

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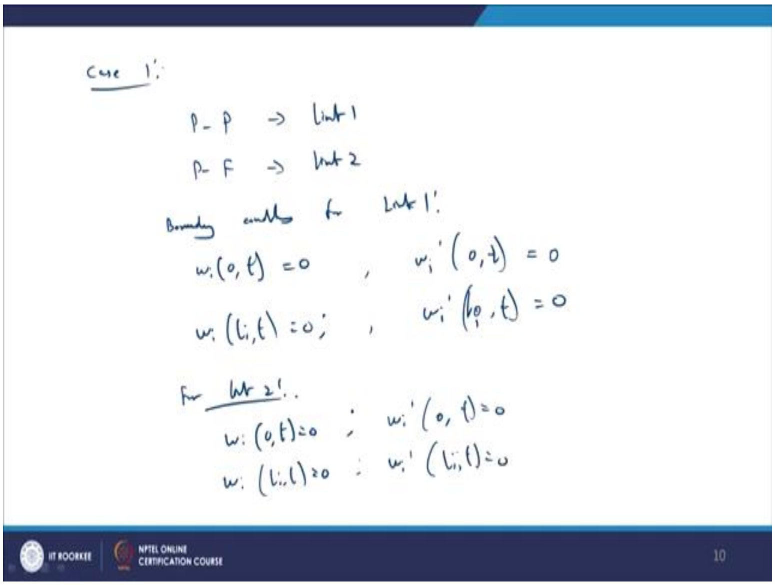

So, now let us consider the case 1 which is pinned, pinned for the link 1, and pinned free for the link 2. So, here it is pinned, pinned for link 1; and pinned, and free for link 2. So, the boundary conditions for the first link are boundary conditions for link 1 are given by $w_i(0,t) = 0, w_i'(0,t) = 0$.

And $w_i(l_i,t) = 0, w_i'(l_i,t) = 0$, these are the boundary conditions for link 1. Similarly, for link 2 the boundary conditions are $w_i(0,t) = 0, w_i'(0,t) = 0$. And $w_i(l_i,t) = 0, w_i'(l_i,t) = 0$; this is the boundary condition or link 1 and 2.

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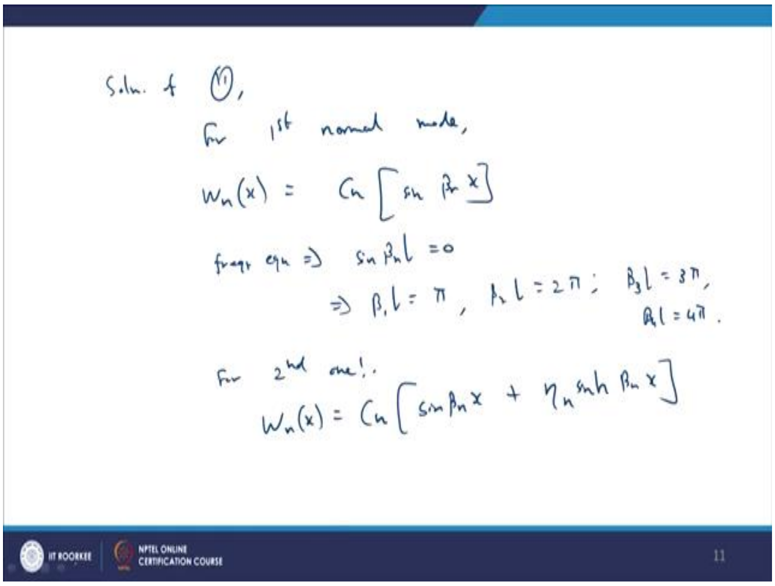

Now, let us go for the solution of equation, so which equation is that equation 11 that is the solution of that equation is given by equation 11 is given by for first normal mode. For first normal mode the solution is given by $w_n(x) = C_n[sin\beta_n x]$. With the frequency equation is given by $sin\beta_n 1 = 0$ which implies $\beta_n 1 = pi$, $\beta_2 1 = 2pi$, $\beta_3 1 = 3pi, \beta_4 1 = 4pi$.

While the solution of the second one for the second one it is given by

$$w_n(x) = C_n[sin\beta_n x + \eta_n sinh\beta_n x]$$

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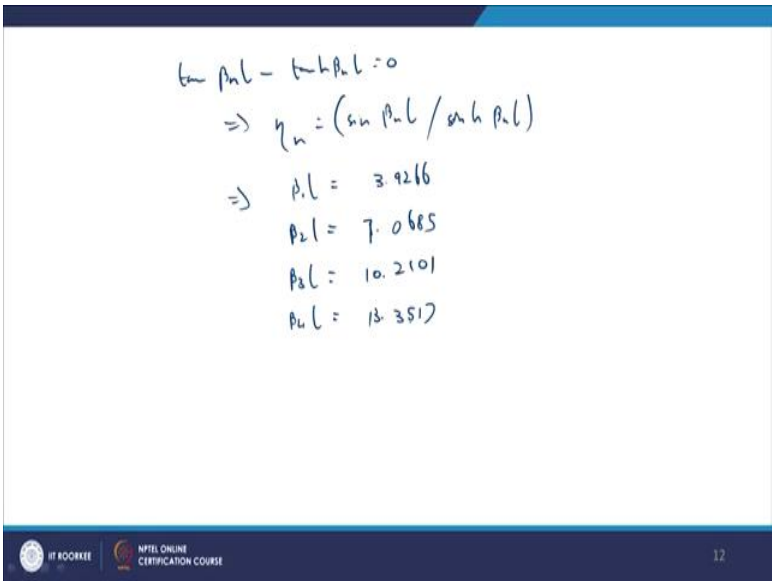

The associated frequency equations are $tan\beta_n 1 - tanh\beta_n 1 = 0$ which implies η_n is given by $\eta_n = {}^{sin\beta_n 1}/_{sinh\beta_n 1}$ from which we get $\beta_n 1 = 3.9266$, $\beta_n 2 = 7.0685, \beta_n 3 = 10.2101, \beta_n 4 = 13.3517$.

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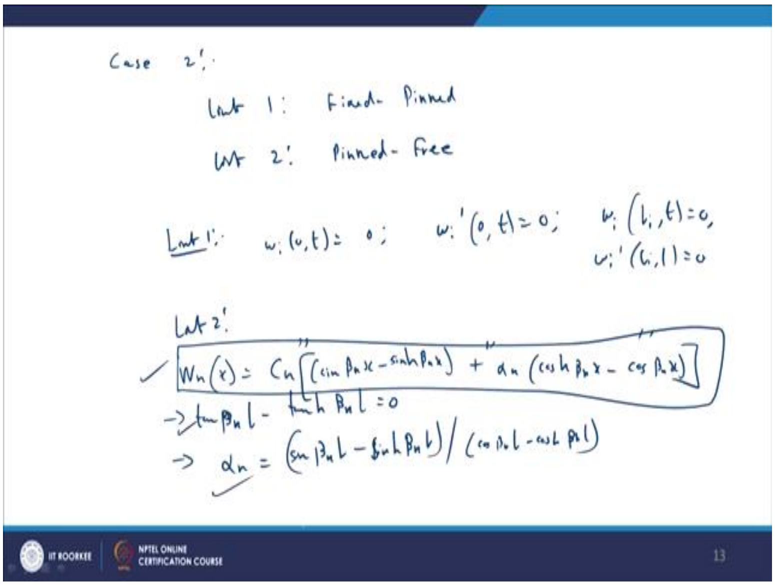

Likewise, now let us go to case 2, so the case two is now given by which is link 1 having fixed pinned and link 2 boundary conditions are pinned free that is the formation of the pinned link 1 and link 2. So, in this situation the boundary conditions for the first link is given by that is link 1 is given by

$$w_i(0,t) = 0, w_i'(0,t) = 0.\ w_i(l_i,t) = 0, w_i'(l_i,t) = 0$$

And similarly for link 2 this remains the same for link 2 and the with the frequency equation written as

$$tan\beta_n 1 - tanh\beta_n 1 = 0$$

And $\alpha_n = {}^{(sin\beta_n 1 - sinh\beta_n 1)}\!\big/_{(cos\beta_n 1 - cosh\beta_n 1)}$

So, this 2 equations are from the generalized solution for this first link is given by this omega that is

$$w_n(x) = C_n[sin\beta_n x + \eta_n sinh\beta_n x] + \alpha_n[cosh\beta_n x - cosh\beta_n x]$$

So, for this solution for the link 1 this is the solution for the first normal mode that is for the base that is for the link 1 we have this solution coming out for which the frequency equation are written by $tan\beta_n 1$.

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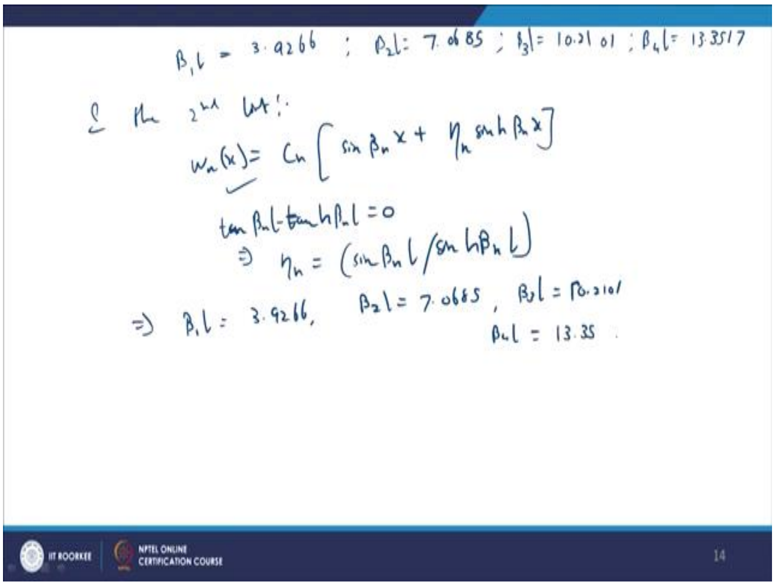

And then α_n is given by this expression which implies

$$\beta_n 1 = 3.9266, \beta_n 2 = 7.0685, \beta_n 3 = 10.2101, \beta_n 4 = 13.3517$$

And the solution for the function $w(x)$ for the second link under the conditions is given by

$$w_n(x) = C_n[sin\beta_n x + \eta_n sinh\beta_n x]$$

So, the frequency equations associated with this link for the solution of the spatial coordinate $w_n(x)$ or

$$tan\beta_n 1 - tanh\beta_n 1 = 0$$

Which implies

$$\eta_n = {sin\beta_n 1}/{sinh\beta_n 1}$$

Where

$$\beta_n 1 = 3.9266, \beta_n 2 = 7.0685, \beta_n 3 = 10.2101, \beta_n 4 = 13.3517$$

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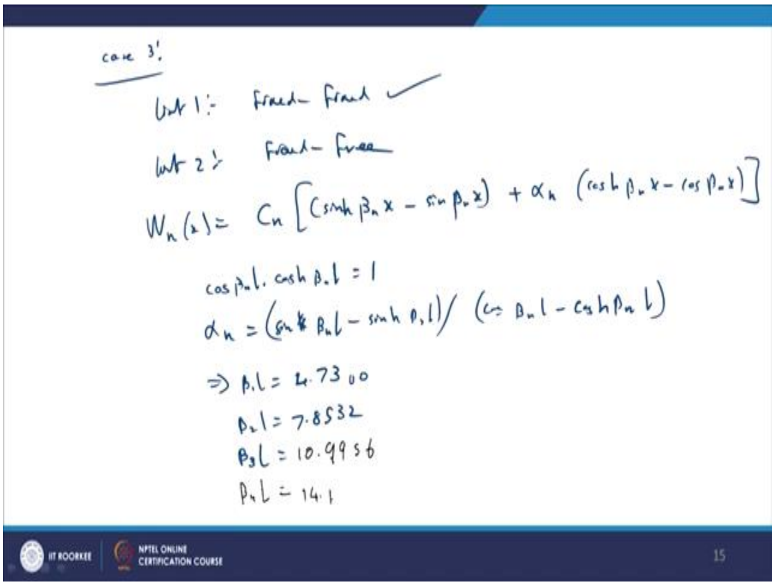

Finally, we are entering into the case 3 which is link 1 fixed, fixed and link 2 fixed, and free. So, in this situation with the same boundary conditions for the other two cases where the position and the velocity at distance $x = 0$ and distance $x = 1$ that is a link length.

We have the boundary conditions being the position and the velocity both are 0 with that we have the common solution for link 1 $w_n(x)$ is given by for the link 1 I am saying. It is given by under this condition of the link 1 the solution is obtained as

$$w_n(x) = C_n[sin\beta_n x + \eta_n sinh\beta_n x] + \alpha_n[cosh\beta_n x - cosh\beta_n x]$$

With the frequency equations having the form

$$cos\beta_n 1. cosh\beta_n 1 = 1$$

$$\alpha_n = {}^{(sin\beta_n 1 - sinh\beta_n 1)}\Big/_{(cos\beta_n 1 - cosh\beta_n 1)}$$

which implies

$$\beta_1 1 = 4.7300, \beta_2 1 = 7.8532, \beta_3 1 = 10.9956, \beta_4 1 = 14.1371$$

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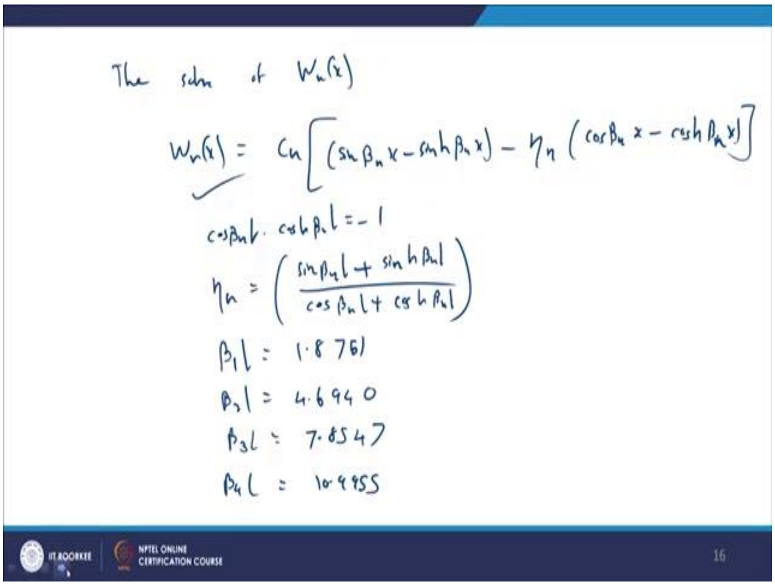

And the solution of $w_n(x)$ for link 2 is given by $w_n(x)$is equal to

$$w_n(x) = C_n\big[[sin\beta_n x - sinh\beta_n x] - \eta_n[cos\beta_n x - cosh\beta_n x]\big]$$

And the frequency equation associated with this solution is given by

$$cos\beta_n 1 - cosh\beta_n 1 = -1$$

$$\eta_n = {}^{(sin\beta_n 1 + sinh\beta_n 1)}\big/_{(cos\beta_n 1 + cosh\beta_n 1)}$$

So, in this situation

$$\beta_1 1 = 1.8751, \beta_2 1 = 4.6940, \beta_3 1 = 7.8547, \beta_4 1 = 10.9955$$

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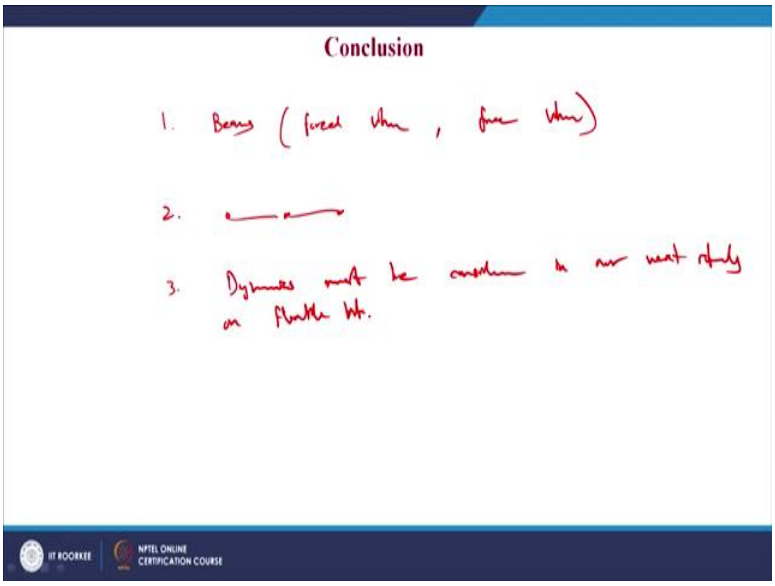

So, coming to the conclusion part with this we are just going to conclude that in such a way that is. In this lecture we have seen the introduction of the beam equations associated with forced vibration and free vibration with uniform and non-uniform beam and then we confine ourselves to the 3 cases of a 2 links flexible manipulator.

So, that we have seen 3 cases of the links being connected with their boundary conditions. And for these 3 cases we have find the generalized solution of the function of the spatial coordinate for this 2 link planar flexible manipulator. And the dynamics of this tooling planar manipulator with the flexibility must be considered for the next study. That is dynamics must be considered in our next study on flexible link that is all, so with this we wind up this lecture on the flexible link kinematics.

Thank you.