Robotics and Control: Theory and Practice
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Lecture – 37
Flexible Link Kinematics - I

Good morning, today we are going to see about Flexible Link Kinematics. So, far we have seen rigid robotic kinematics. So, first time we are going to see in our lecture flexible link kinematics. The organization of today's lecture will be as follows.

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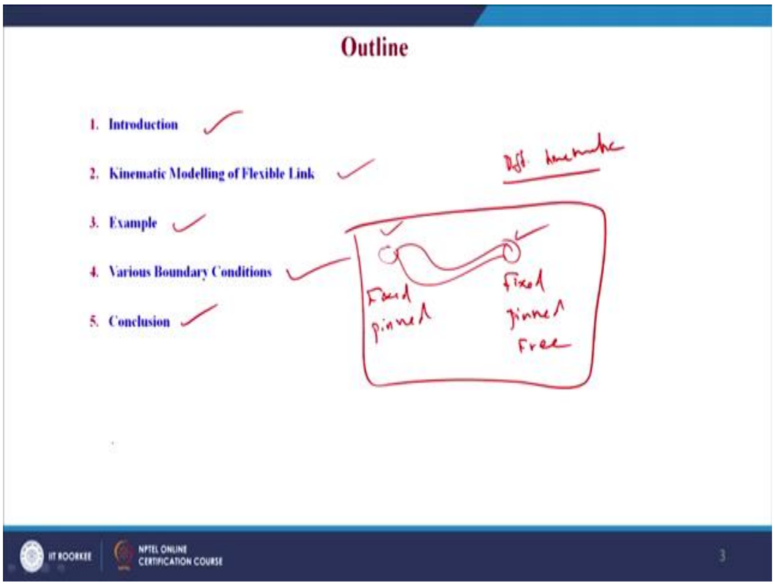

First we have the introduction followed by kinematic modeling of flexible link and then we see an example that is basically two link planar flexible manipulator and we just list out the various boundary conditions, how the links will be connected either in its base also to its end; how the flexible link is connected in its base as well as in its end; either it is fixed pinned here also fixed or pinned or it is free.

So, these are the boundary various conditions we are going to see just the name of those conditions and in that only one condition we are going to see because the rest of the other conditions we will be seeing in the next lecture and finally, conclusion should be done. I repeat the organization again, first we see the introduction then we have the kinematic modeling of flexible link, we go up to the differential kinematics; differential kinematics.

So, in the kinematic modeling we go up to differential kinematics, that is forward kinematics then we go to the Jacobian matrix. And then we have an example that is basically two link planar flexible manipulator and then the various boundary conditions will be listed in this lecture and in that only one condition we are going to see finally, conclusions will be drawn from the lecture today.

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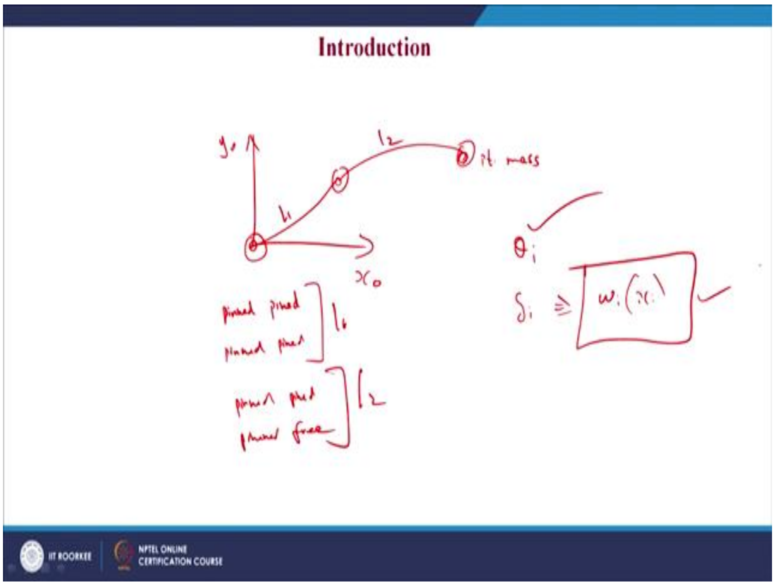

Coming to the introduction part. So, coming to the flexible link. So, flexible link how it gets a shape. So, for example, if this is the base frame; x_0, y_0. So, flexible link will have the shape like this. So, here you have the point mass and here this is link 1 and link 2. This is a base frame and we have the system in such a way that the link 1 is a flexible link, link 2 is a flexible link and they are connected in such a way that pinned that is basically the first link is pinned and the second link also if you see that, the first links end is also pinned.

And this is for link 1 and for link 2 it is pinned and then finally, pinned free because here it is pinned and here it is pinned free because this is a free end here. So, this is for link 2. So, this is how it becomes. And there are variables which are joint angle in the case of rigidity and that is another thing is called deflection. We can say δ_i precisely we can say $w_i(x_i)$.

Where x_i is a point along their flexible link that is called x_i. So, this is the deflection of the flexible link and we know that rigidity based on joint angle θ. So, we know that because of rigid robot kinematics and dynamics we know what is θ_i, which is the generalized coordinate for the robotic system for rigid links and the deflection in the flexible link will be given by the term $w_i(x_i)$; that is a one which determines the deflection of the link l_i having the flexibility.

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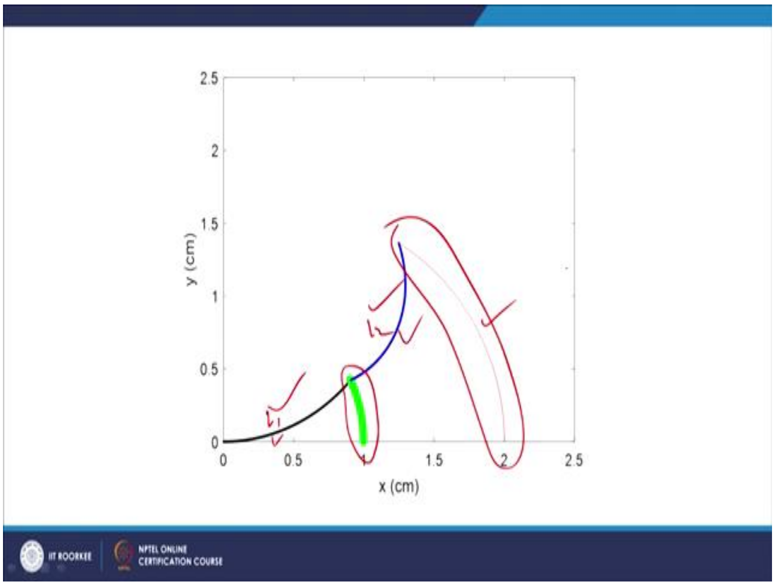

And coming to the modeling kinematic modeling of the flexibility link.

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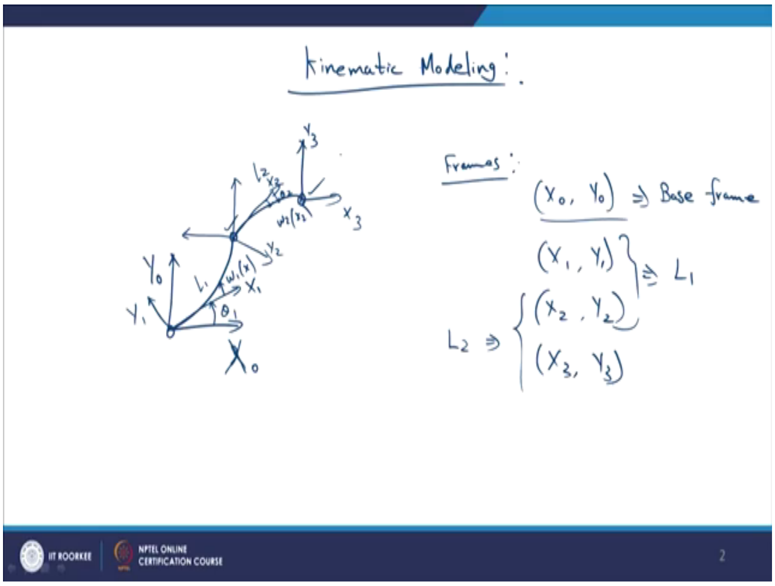

We are going to see first the kinematic modeling kinematic modeling of the flexible link. So, first we consider the schematic which is given precisely by this diagram, which is basically x_0, y_0 and this is the flexible link which is having the revolute joints, which is the point load at the end point and we have the frames attached to the end points.

Now, this is x_1 this is y_1 I am going to define what is frame x_0, y_0 frame x_i, y_i and this is $w_1(x)$. I am going to define each and every point let me make it all the variables associated with this schematic, this is theta 1 the frame attached here and we have the that the frames have finished now.

So, let me define this as the angle θ_2 and you can say this point for example, this is the vector which is given by $w_2(x_2)$. So, now, we have the frames associated with the schematic of a two link plane as flexible manipulator, the frames associated here are first we have the frame x_0, y_0 which is basically base frame for the system.

Both the robotic system with rigid link as well as the flexible link will have a base frame which is the; which is the general rule. We have the base frame attach to the base of the manipulator and we have the frames x_1, $y_1$and x_2, y_2 and frame x_3, y_3. So, if you see x_1, y_1 are connecting the link L_1 and then we see x_2, y_2; x_2, $y_2$and x_3, y_3 are the frames which are connecting link L_2 as you can observe from this schematic this is link L_1 flexibility and this is link L_2 flexible link configuration is shown here.

So, I repeat the base frame is x_0, y_0 and the frames associated to link L_1 of the flexible manipulator is or x_1, y_1 and x_2, y_2 and the frames associated with flexible link L_2 or x_2, y_2 this frame and x_3, y_3 which is the end frame. So, each link has the base frame that is a starting frame and the end fame frame attached.

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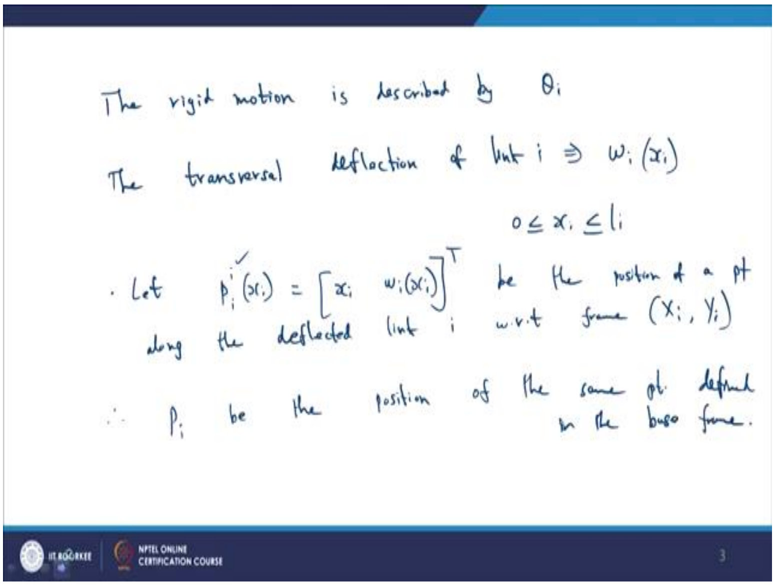

So, now go further that is we say that the rigid motion of this manipulator is described by θ_i with its joint angle θ_i and the transversal deflection of link i is given by $w_i(x_i)$, which is the transversal deflection of link i. So, we have two links which is going to be $w_1(x_1)$ and $w_2(x_2)$, $x_1 x_2$will be varying in such a way that that is each x_i will vary from 0 to the link length l_i.

Now, coming to the point let p that is basically small p_i with respect to i of x_i. See how i define? p_i with respect to i; that means, p_i is a point it is defined in the frame i that is what it means; p_i suffix then super script ok. So, p_i with respect to i of x_i is given by $x_i w_i$ of x_i. So, this is a vector column vector, where x_i is the x_i value which is the function of this variable because we are going to vary this vector by the input argument x_i and $w_i(x_i)$ is the deflection.

So, the point or the position vector p_i with respect to i of x_i is given by x_i $w_i(x_i)$ be the position of a point of a point along the deflected link i with respect to frame x_i comma y_i i with respect to frame x_i comma y_i. Therefore p_i be the position of this point or the vector be the position of this point or the same point I can say defined in the base frame that is it.

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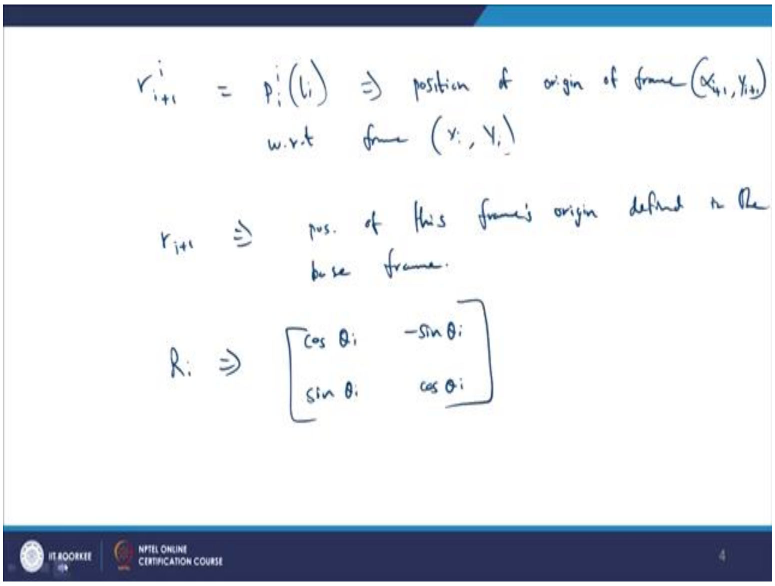

Now, we could have one more notation of a vector which is $r^i_{i+1} = P^i_i(l_i)$, but position of origin of frame x_{i+1}, y_{i+1} with respect to frame x_i, y_i and r_{i+1} is position of this point this frames origin defined in the base frame and so, one.

So, now we see that the joint rotation matrix corresponding to the rigid link is given by r_i; which is basically the joint rotation matrix corresponding to the rigid link is given by r_i is equal to $\cos\theta_i$, $-\sin\theta_i$, $\sin\theta_i \cos\theta_i$.

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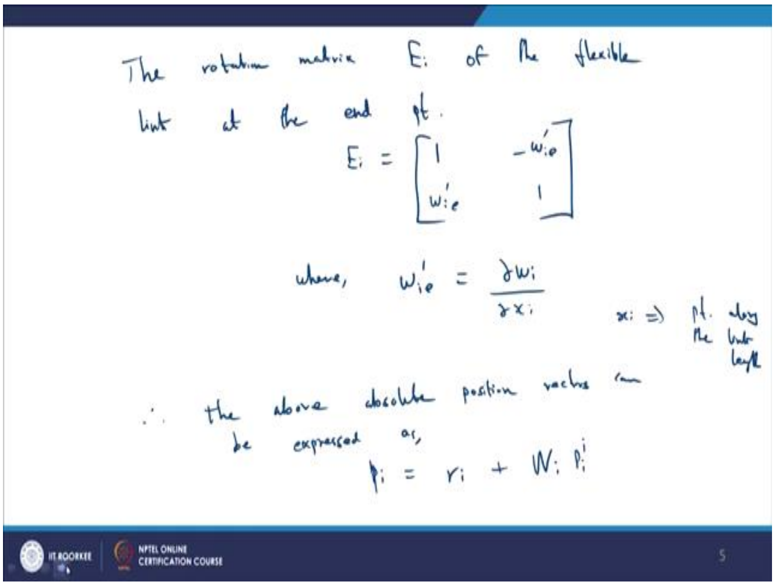

Likewise the rotation matrix E_i of the flexible link at the end point is given by

$E_i = \begin{bmatrix} 1 & -w'_{ie} \\ w'_{ie} & 1 \end{bmatrix}$; where, $w'_{ie} = \frac{\partial w_i}{\partial x_i}$, I mean partial derivative of the expression w_i with respect to x_i.

And we know that the x_i is a point along the link length that is what x_i. It is in our freedom that we can increment x_i by whatever decimal point we can increment that is in our freedom. And therefore, the above absolute position vectors can be expressed as, in general the position vectors can be expressed as $p_i = r_i + W_i p_i^i$p. Now here the W i is a capital W we can notice that.

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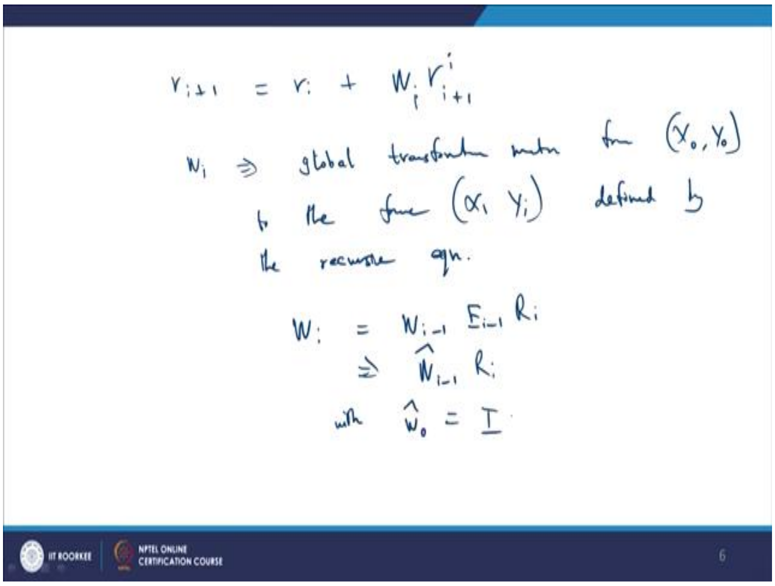

Now we have also $r_{i+1} = r_i + W_i r_{i+1}^i$. You can see that these things 1 and 2 can be cancelled. So, it will be r_{i+1} in the left hand side. So, where W_i is the global transformation matrix, matrix from the base frame $x_0 y_0$ to the frame $x_1 y_1$. Defined by the recursive equation given by $W_i = W_{i-1} E_{i-1} R_i$ which implies $\widehat{W}_{i-1} R_i$ with the condition $\widehat{W}_0$ equal to the identity matrix.

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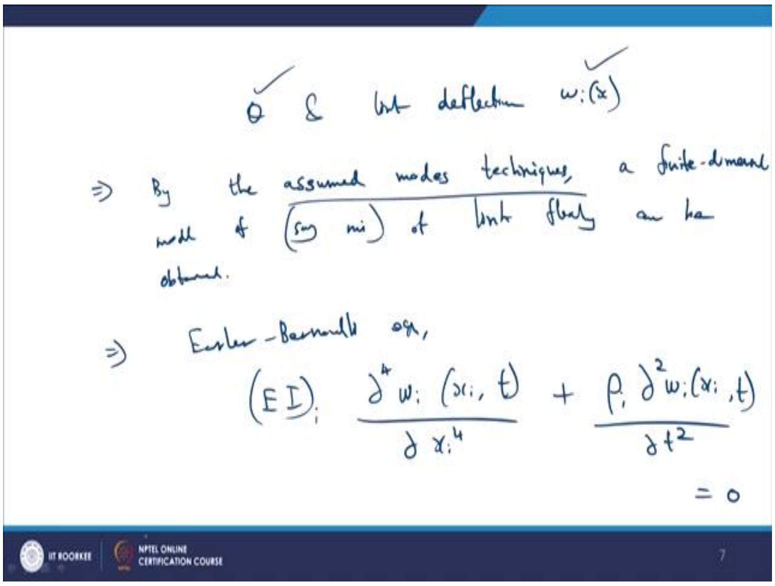

On the basis of the above equations the kinematics of any point along the manipulator is completely specified as a function of a joint angle and link deflection $W_i(x)$. Thus the point on the deflected link can be defined with respect to a joint angle theta and the link deflection $W_i(x)$. Now by the assumed modes technique a finite dimensional model of say an order m_i of link flexibility can be obtained that is by assumed modes method.

We can define a finite dimensional model or of the link flexibility. Thus by the Euler Bernoulli equation for flexible beams we have $(EI)_i \frac{\partial^4 w_i(x_i,t)}{\partial x_i^4} + \rho_i \frac{\partial^2 w_i(x_i,t)}{\partial t^2} = 0$. Thus by utilizing or exploiting the separable t in time and space of the solutions to the Euler Bernoulli equation Euler Bernoulli equation, we have which is coming out to be the link deflection.

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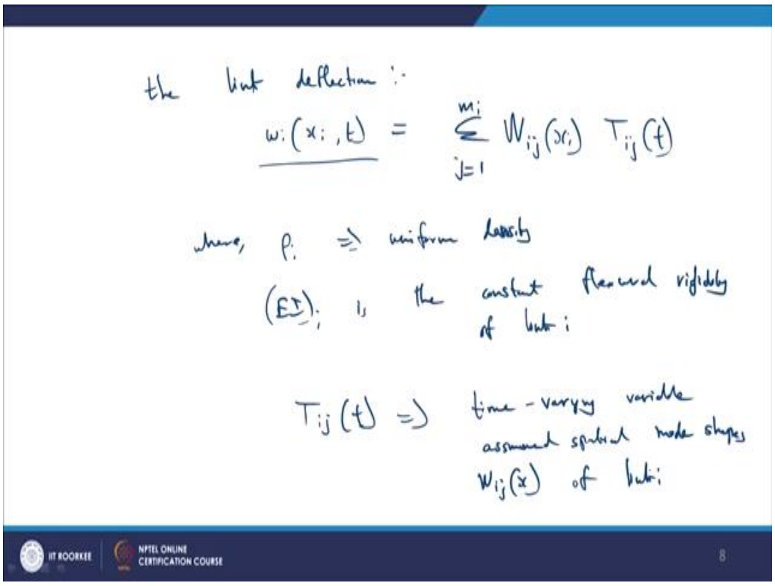

From the solutions of the Euler Bernoulli equation, we have the link deflection given by

$W_i(x_i, t) = \sum_{j=1}^{m_i} W_{ij}\,(x_i)T_{ij}(t)$ where ρ_i is a uniform density and E I with respect to I is the constant flexural rigidity of link i.

And here from this equation of link deflection which is given in terms of $w_i(x_i, t)$ we have $T_{ij}(t)$ is a time varying variable associated with assumed spatial mode shapes $w_{ij}(x)$ of link i.

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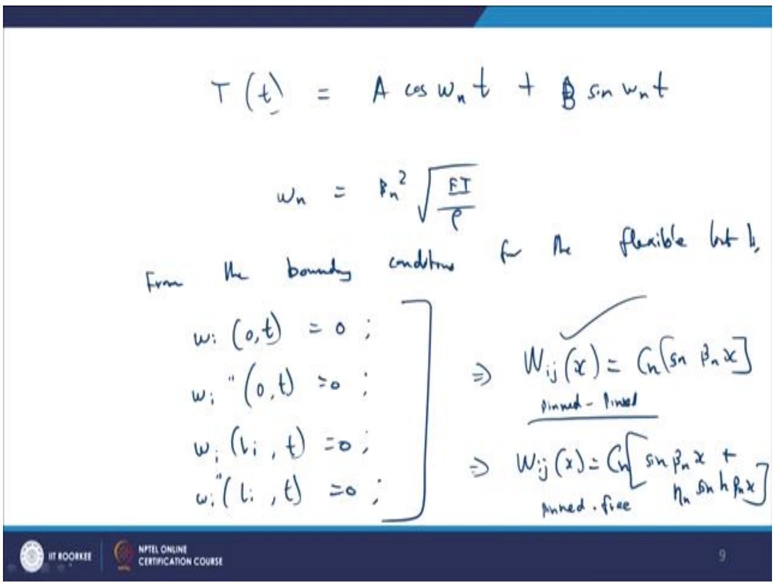

Now, the definition or the expression for these time varying variables are given by $T(t)$ for example, $T(t)$ time varying function is given by $T(t) = A\cos w_n t + B\sin w_n t$. Where w_n is the natural frequency of the beam which is computed by $w_n = \beta_n^2\sqrt{\frac{EI}{\rho}}$ that is the thing and we have from the boundary conditions from the boundary conditions for the flexible link l_i, we have $w_i(0,t) = 0$.

Initially it is 0 and w_i derivative is also 0 and $w_i(l_i,t) = 0$ and $w_i'(l_i,t) = 0$. With this boundary conditions we can have the general solution that is for the first normal mode has the form $w_{ij}(x) = c_n \sin\beta_n x$ that is for the situation which is pinned.

If the first link is pinned then we have the expression for $w_{ij}(x)$ that is the deflection part is given by $c_n \sin\beta_n x$ and the second link is given by $w_{ij}(x) = c_n(\sin\beta_n x + \eta_n h\beta_n x)$. So, this is for the first solution is for the pinned situation of the link and this one the second solution is for pinned free boundary conditions.

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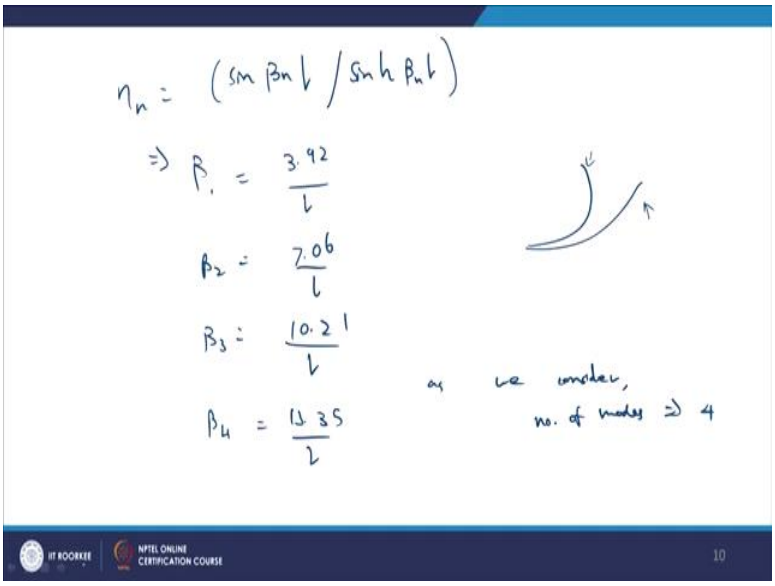

Where $\eta_n = \left({}^{sin\beta_n 1}/_{sinh\beta_n 1}\right)$ which implies $\beta_1 = \frac{3.92}{1}$, $\beta_2 = \frac{7.06}{1}$ and $\beta_3 = \frac{10.21}{1}$ and $\beta_4 = \frac{13.35}{1}$ as we consider number of modes here as 4.

So, what is mode? So, what modes will do? If mode is 1 the flexible link will have the flexibility like this say for example, if the mode is four we have more flexibility of the link this is what the difference between modes. If you have more modes, we have more flexibility of the concerned link if you have single mode we have the flexibility being reduced.

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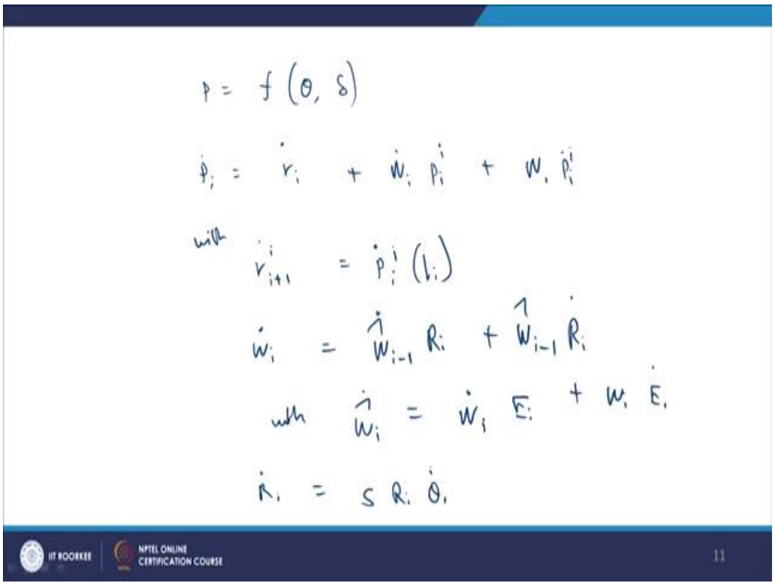

Now we come to the derivation of forward kinematics of the flexible link. Say the forward kinematics is given by p is a function of theta joint variable of the rigid link and the flexible flexibility of the link. So, the absolute linear velocity is given by

$$\dot{p}_i = \dot{r}_i + \dot{w}_i p_i^i + w_i p_i^i$$

because this expression is given already. So, with

$$\dot{r}_{i+1} = \dot{p}_i^i(l_i)$$

So, we can take that

$$\dot{w}_i = \dot{\hat{w}}_{i-1} R_i + \hat{w}_{i-1} R_i$$

with $\dot{\hat{w}}_i = \dot{w}_i E_i + w_i \dot{E}_i$.

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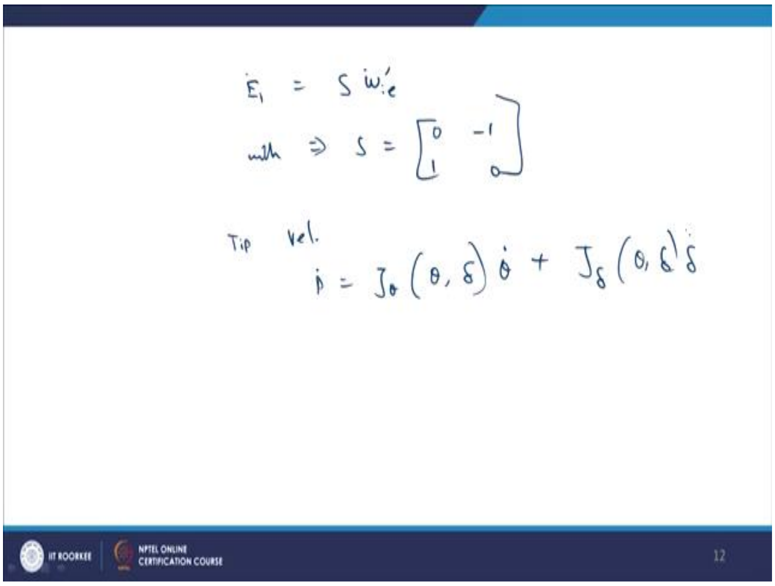

So, it has been noted that $\dot{R}_i = SR_i\dot{\theta}_i$ and $\dot{E}_i = S\dot{w}_{ie}$ with S being the skew symmetric matrix given by $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ that is the tip velocity can be expressed as

$$\dot{p} = J_\theta(\theta, \delta)\dot{\theta} + J_\delta(\theta, \delta)\dot{\delta}$$

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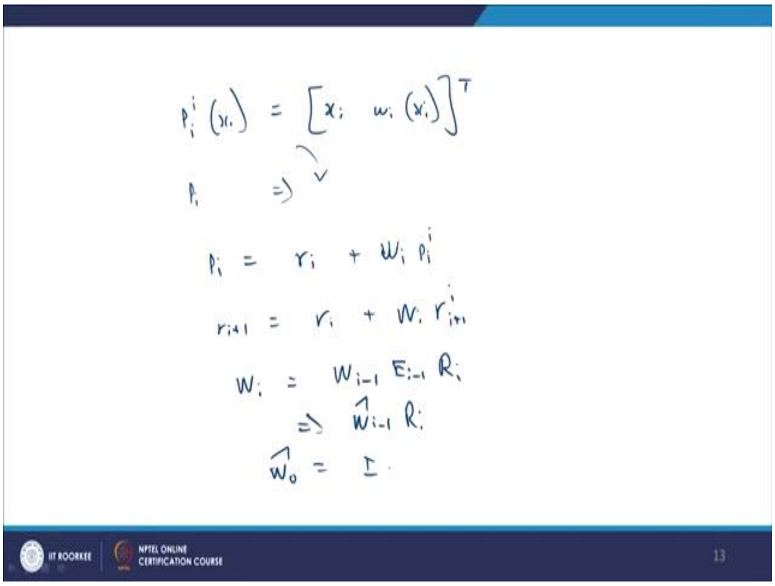

So, this is the expression now we see quickly the two link manipulator flexible ones expression that is the definition. So, this previous velocity expression is given by considering the vector $p_i^i(x_i)$ being the position of a point along the deflected link l_i, which is given by $[x_i \quad w_i(x_i)]^T$. So, with this we have the p_i also which is the definition of that point defined in the base frame.

So, from this we have derived the velocity and the that is the velocity in terms of Jacobean for the flexible link ok. So, now here p_i is this portion of the same point with respect to the base origin. So, now after defining this, we can go for a planar link that is a planar tooling flexible manipulator where the p_i definition for p_i is given by $p_i = r_i + W_i p_i^i$ and all $r_{i+1} = r_i + W_i r_{i+1}^i$ defined in r_i. Here $W_i = W_{i-1}E_{i-1}R_i + W_i r_{i+1}^i$ which implies $\widehat{W}_{i-1}R_i$ where $\widehat{W}_0 = I$.

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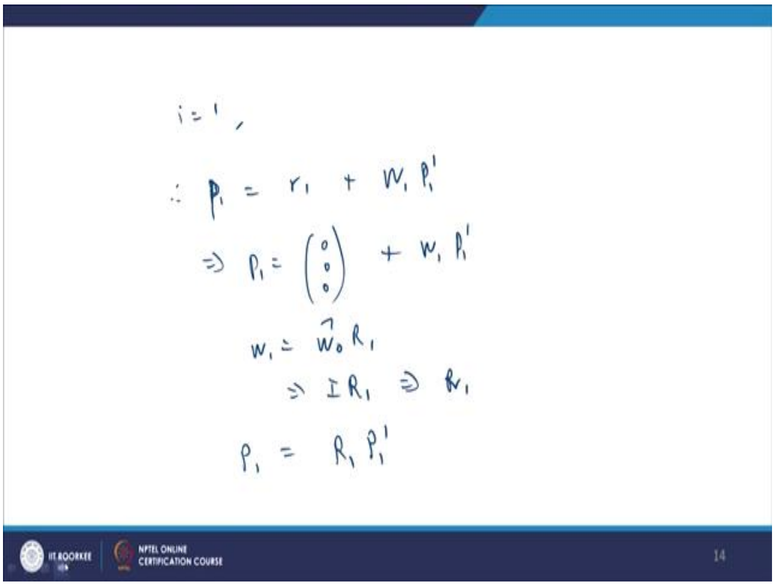

Now we consider the case $i = 1$ therefore, $P_1 = r_1 + W_1 P_1^1$ which implies $P_1 = r_1$ is basically the origin defined in the base frame is the origin which is a zero vector plus $W_1 P_1^1$. So, here $W_1 = \widehat{W}_0 R_1$ which implies IR_1. So, basically it is R_1. So, $W_1 = R_1$ and hence P_1 vector is finally, obtained to be $P_1 = R_1 P_1^1$.

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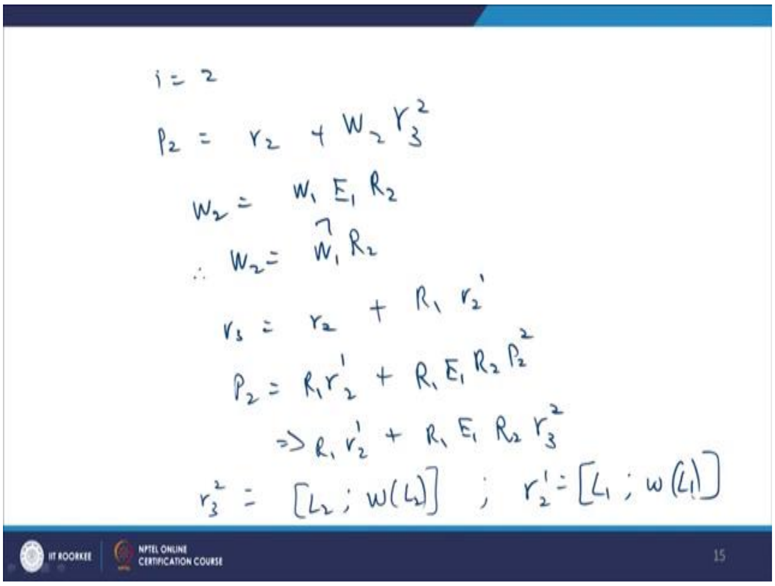

Now, $i = 2$, where the second frame comes we have the vector $P_2 = r_2 + W_2 r_3^2$. So, $W_2 = W_2 E_1 R_2$ having the joint variable θ_2. Therefore, $W_2 = \widehat{W}_1 R_2$ finally, $r_3 = r_2 + R_1 r_2^1$.

So, having defined we have the vector coming out to be

$$P_2 = R_1 r_2^1 + R_1 E_1 R_2 P_2^2$$

which can be written as $R_1 r_2^1 + R_1 E_1 R_2 r_3^2$

where $r_3^2 = [L_2;\ W(L_2)]$ that is at this x equal to the end of the link length L 2, the deflection is $W(L_2)$. Similarly $r_2^2 = [L_1;\ W(L_1)]$.

Thus finally, we have the position vectors thus having defined the two link plane manipulator of the flexible link with the kinematics defined now, we have the simulation obtained with this type of flexibility of the links L_1 and L_2 coming out with the traced trajectory of link 1 and the traced trajectory of the link 2 which is basically the tip of this two link planar flexible manipulator trajectory shown here with its own flexibility link one shown in the black colored trajectory of the link.

And the blue one shows the link 2 flexible link 2 and the tip trajectory is shown by the red trajectory, which is basically the trajectory traced based on the flexibility of the planar manipulator.

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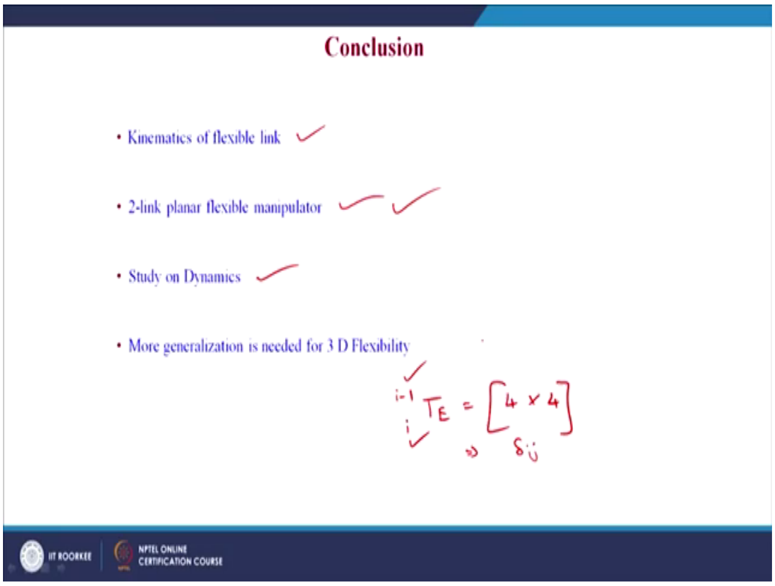

Finally, we come to the conclusion such a way that in this lecture we have seen the kinematics of the flexible link basically the modeling up to the Jacobean matrix and then we have the planar 2 link manipulator which is flexible and that example has been derived to get the tip position and what is needed for this study further is dynamics must be considered for this study because, we have seen in this lecture only the kinematics.

Studying about the dynamics will give you a more broader view about the theory behind flexibility of manipulators and then we have seen in the study only in the tool that is a planar motion of the flexible manipulator, but the more generalized way will be studying the 3 D flexibility. So, that we can have the general homogeneous transformation matrix between he frame $i-1$ and i for a flexible link connecting the frames $i-1$ and i so, that we can have a generalized homogeneous transformation matrix having the flexibility which is δ.

So, in terms of δ_i we can have the generalized homogeneous matrix defining the flexibility between two frames connecting the link.

Thank you so much.