**Robotics and Control: Theory and Practice**
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**Lecture – 33**
**Sliding Mode Control**

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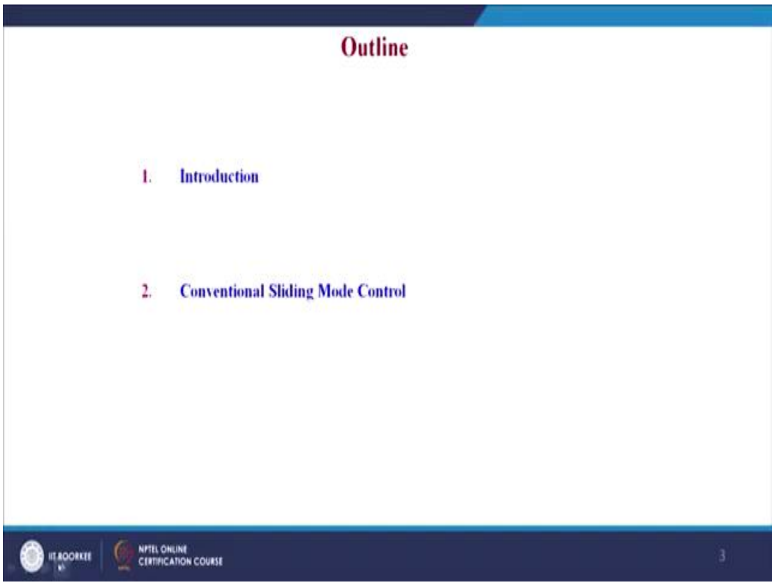


Good morning. Today, I am going to have the lecture in Sliding Mode Control, the outline of this lecture will be as follows. First we have the introduction and then we see the conventional sliding mode control with some example.

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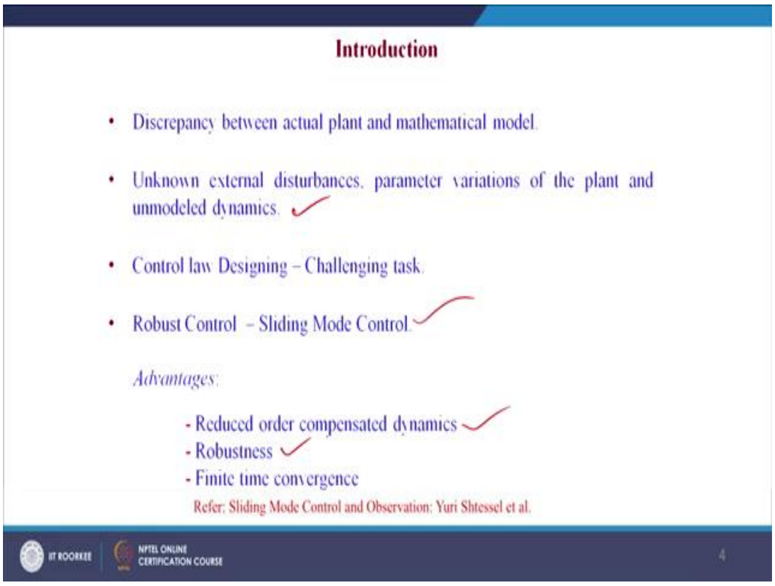


Coming to the introduction, in most of the situations there is a discrepancy between the actual plant and its mathematical model. This discrepancy is due to unknown external disturbances, parameter variations of the plant and unmodeled dynamics. Due to these reasons designing a control law is a challenging task. The control law that can take care of these causes is called robust control law, one among them is a sliding mode control scheme; which has the advantages of reduced order compensated dynamics, robustness to disturbances; I mean the bounded disturbances and finite time convergence.

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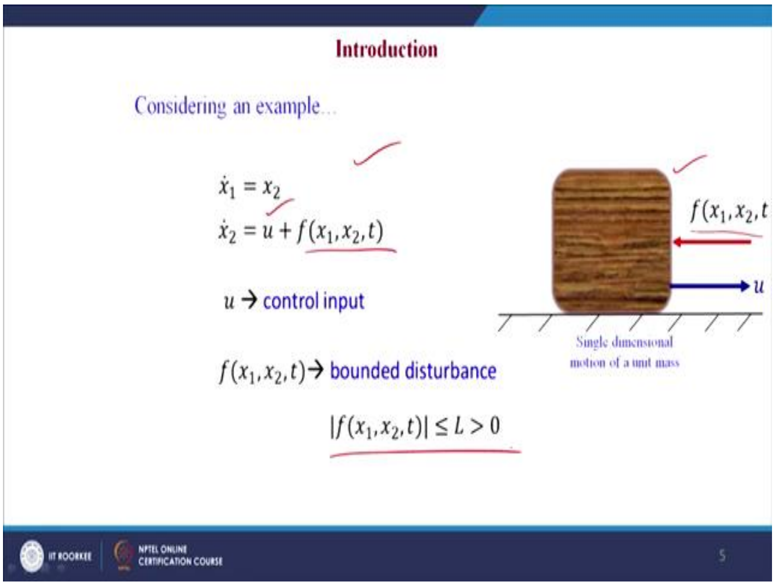

Considering an example of single dimensional motion of a unit mass as shown in the figure here, which is pulled by the control input force and which is resisted by the disturbance force which is given by the term $f(x_1, x_2, t)$. Where the disturbance is a bounded disturbance and this disturbance has is including the viscous friction force as well as the unknown rigid forces associated with this mass and the system states are $x_1$ and $x_2$.

Where, $x_1$ is a position and $x_2$ is the velocity of this mass while pulling, thus the state space model of the system is given by $\dot{x}_1 = x_2$ and $\dot{x}_2$ is having the control input u and the disturbance force which is the bounded. And, is given by $|f(x_1, x_2, t)| \leq L > 0$.

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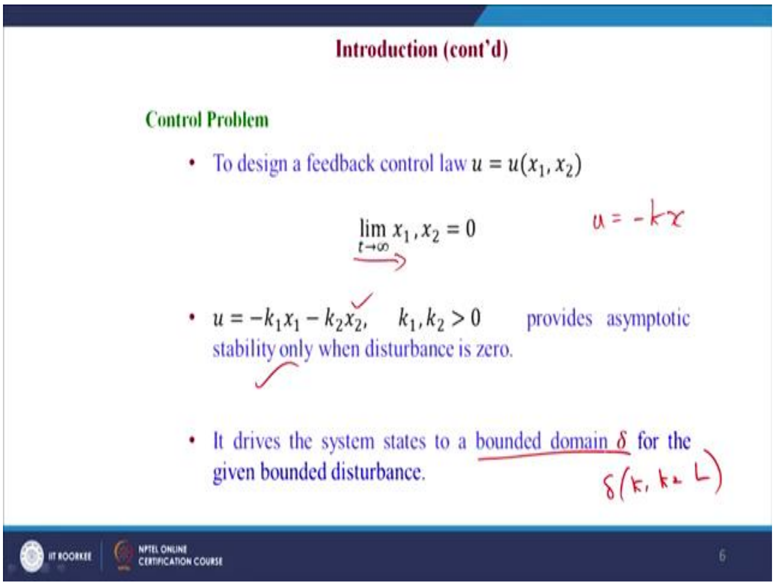


Thus, the control problem is to design a state feedback control law; which is $u$ equal to $u = -k_1x_1 - k_2x_2$ such that it can drive the system states to equilibrium point asymptotically, that is has the time varies from 0 to infinity the system states reaches the equilibrium point.

Considering the state feedback control law which is for these two states system it is $u = -k_1x_1 - k_2x_2$ Where, $k_1 > 0$ and $k_2 > 0$, provides asymptotic stability only when the disturbance part $f(x_1, x_2, t) = 0$. But, if you consider the disturbance existing in the system then this control state feedback control law will drag the system states to a bounded domain delta, we are going to see that what is that bounded domain delta which is a function of $\delta(k_1, k_2, L)$.

So, it is a function of this domain is a function of $k_1, k_2$ and the disturbance bound value for a bounded disturbance. Thus, it will not bring it to the convergence, it will bring it close to the domain of, it brings it to the domain that is all, it will not converge to 0 or equilibrium point provided the system disturbances not equal to 0.

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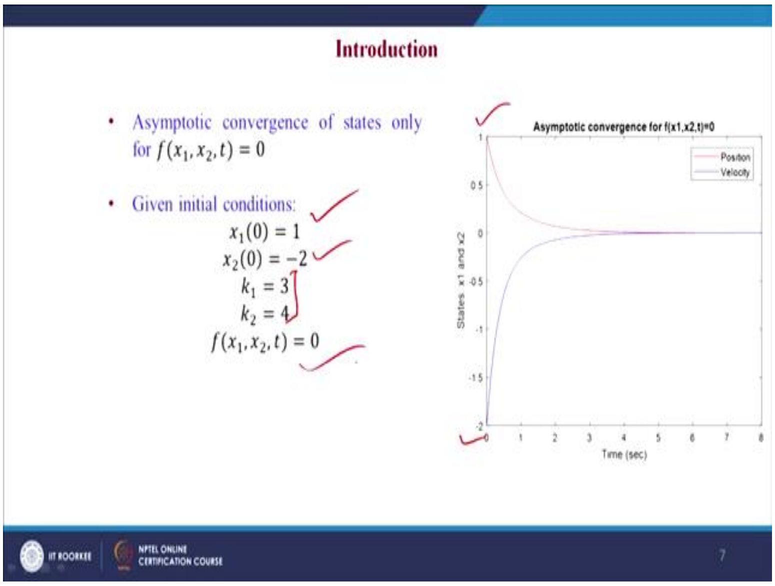


Consider the example which is the single dimensional motion of the mass with states position and velocity with the given initial conditions, $x_1(0) = 1, x_2(0) = -2$ with the controller gains $k_1 = 3, k_2 = 4$ respectively. And, with the disturbance being 0 it will converge the system to stability to equilibrium point at time tends to infinite, that is the asymptotic convergence for the system is obtained for the disturbance being 0 by this state feedback controller.

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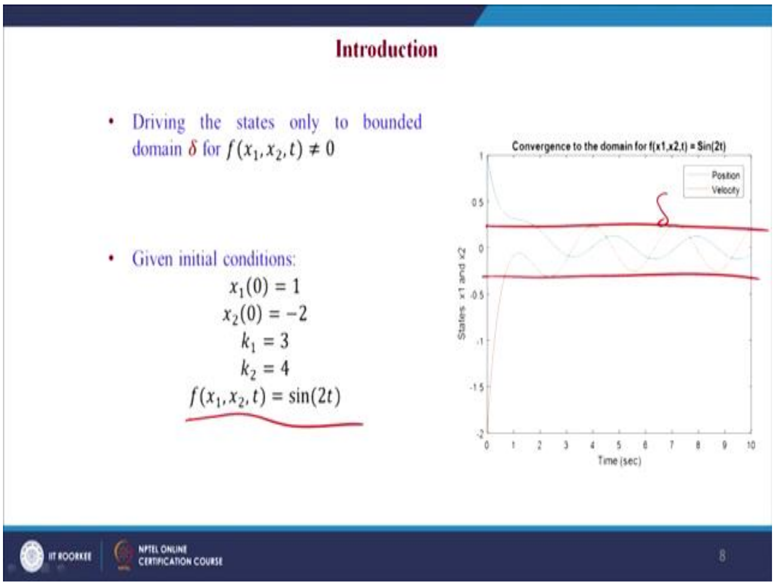


And, what happens when the disturbance is given that is $f(x_1, x_2, t) = \sin(2t)$. When this disturbance is given that straight feedback controller will make the system states common state or stay in this domain, which is this domain is delta which is a function of $k_1, k_2$ and the bound value L. It is not getting converged whereas; it is going to this domain and stays there by this control law.

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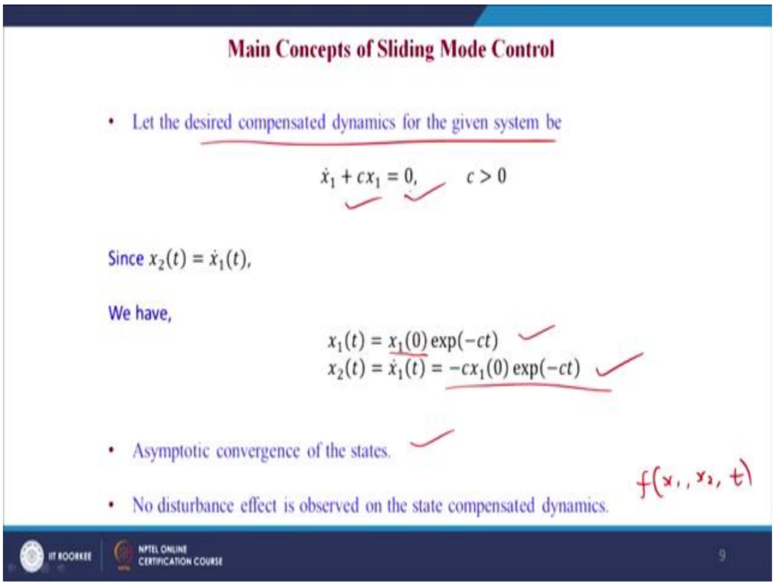


Thus now, we are getting into the main concepts of sliding mode controller. Let the desired compensated dynamics for the given system 1 dimensional motion of this given 2 dimensional 2 degrees of freedom system for the 2 degrees of freedom system, we are considering the decide composited dynamics state dynamics of the system being the first order given by $\dot{x}_1 + cx_1 = 0$. It is a first order homogeneous differential equation, where $c > 0$ and here $x_2(t) = \dot{x}_1(t)$ for this system, we have the solution being $x_1(t) = x_1(0)e^{-ct}$. And, the derivative of the solution is $x_2(t) = -cx_1(0)e^{-ct}$. This shows that the states system states $x_1, x_2$ converges to thus a equilibrium point asymptotically. From these two expressions we can from this solution of this differential equation we observe that these two states converge asymptotically. And, also what we have observed in the system decide compensated state dynamic equation is no disturbance effect that is the $f(x_1, x_2, t)$ that is a disturbance effect is not observed on the state dynamic equation, compensated equation.

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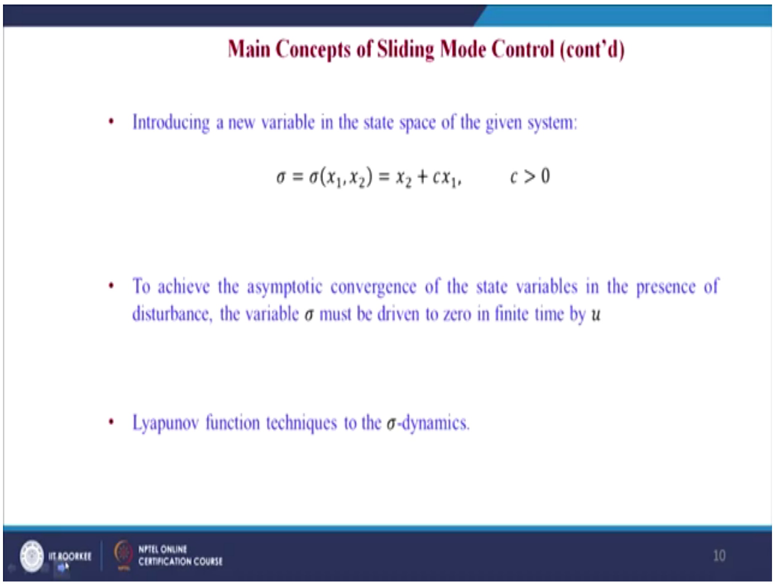


So, here we introduce a new variable where the system dynamics is equal to $\sigma = 0$, as we have seen in the previous slide, it is 0 in this case the dynamic equation; now, we equate the dynamic equation to a new variable called $\sigma$. Thus $\sigma$ is a function of the $\sigma$ $(x_1, x_2)$ here in this example and here it is given by $x_2 + cx_1$, where $c > 0$; to achieve the asymptotic the aim is to achieve the asymptotic convergence of the state variables $x_1$ and $x_2$ in the presence of disturbance, the value of the variable $\sigma$ must be converged in finite time by the control law u.

Now, we have to decide which is the control law that is a sliding mode control law, that drives the state system state to asymptotic stability by driving the stay the $\sigma$ variable to finite time convergence. Now, applying Lyapunov function techniques to the sigma dynamics let the dynamic let the Lyapunov function candidate be $V = \frac{1}{2}\sigma^2$.

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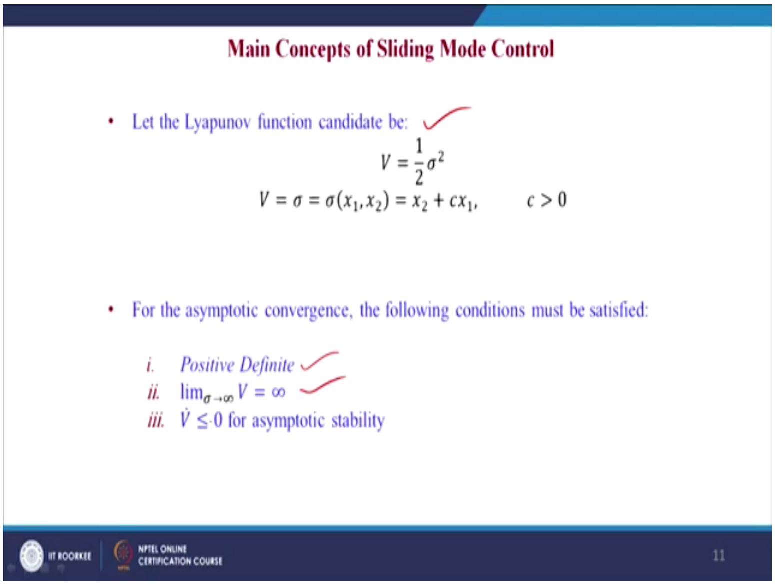


We must note that the selection of the Lyapunov function candidate not only allows us to have the analysis of the stability, but also it helps us in designing the controller. So, here $V = \sigma = \sigma(x_1, x_2) = x_2 + cx_1$ for the asymptotic convergence the following conditions must be satisfied. First is: $V$ must be positive definite and the second condition is as $\sigma$ tends to infinity the value of V must also be infinity. And, the third condition is the $\dot{V} \leq 0$ for asymptotic stability. That is for asymptotic stability $\dot{V} < 0$.

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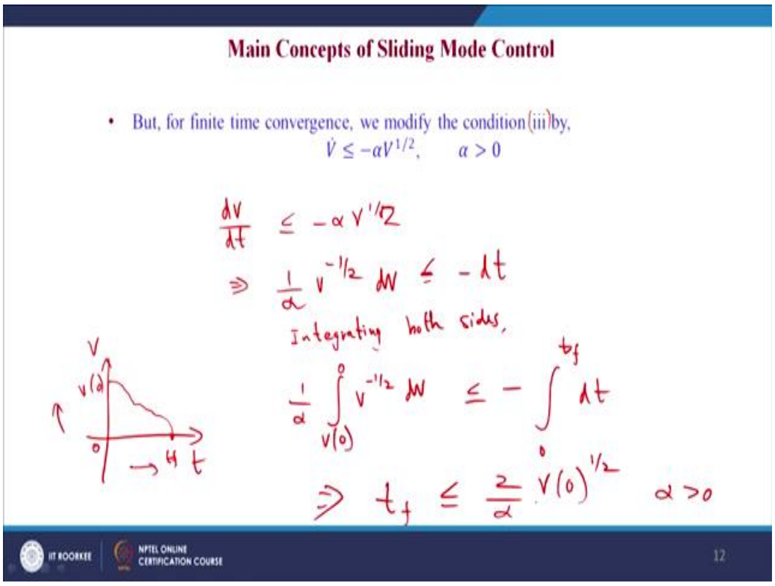


But, for finite time convergence we modify the condition 3 by

$\dot{V} \leq -\alpha V^{1/2}$. Where $\alpha$ is a positive constant, positive value. Thus, instead of $\dot{V} \leq 0$ we take this condition so, that we conform finite time convergence. That is $\dot{V} \leq -\alpha V^{1/2}$ will lead to finite time convergence. So, we have the derivation here in such a way that

$\frac{dV}{dt} \leq -\alpha V^{1/2}$ which implies I am just changing $\frac{1}{\alpha} V^{-1/2} dv \leq -dt$. So, integrating both sides gives

$$\frac{1}{\alpha} \int_{v(0)}^{0} V^{-1/2} dv \leq -\int_{0}^{t_f} dt$$

This integration has been taken by this x axis is t and y axis is the $v$ value. So, $v$ value initially is $v(0)$. Finally, it is going to be 0 and t finally, say $t_f$ so, that is value from here it comes down to 0 value at time $t_f$, goes this way and it goes this way. So, the initial value of $v$ is $v(0)$ and final value is 0. Similarly, t varies from 0 to $t_f$ that is the thing; which leads to after simplifying this $t_f \leq \frac{2}{\alpha} v(0)^{1/2}$, where $\alpha > 0$ . So, we see that $t_f$ is not infinity whereas; it is a value which is a finite value which implies that $t_f$ is the finite value so, that $v$ gets converged in that finite time.

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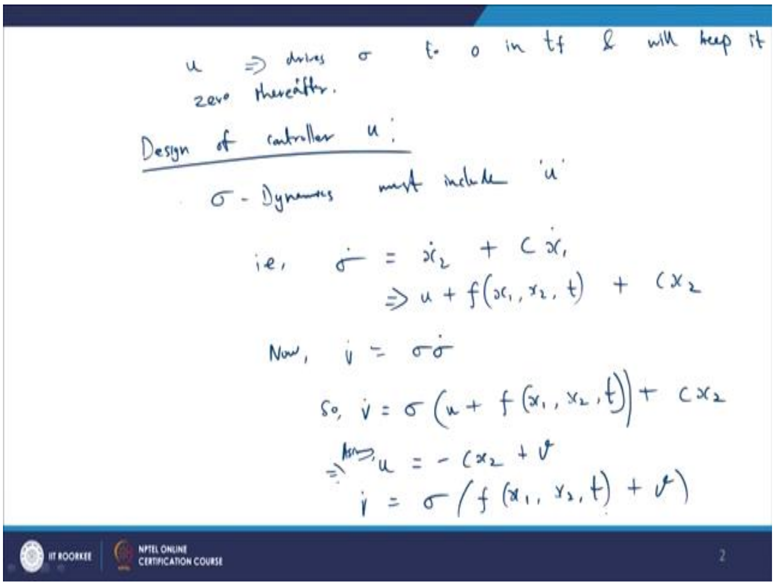


We have to design a controller u. Controller u has to be designed now. So, that this drives $\sigma$ variable to 0 and infinite time that is, it drives we need to design a controller u so, that it drives the variable $\sigma$ to 0 in finite time and we will keep it in 0 thereafter. And, we will keep the variable in 0 thereafter. So, now, we will focus in the design of controller u. First of all the $\sigma$ dynamics must include the control law u that is; what we have the $\sigma$ dynamics is? $\dot{\sigma} = \dot{x}_2 + c\dot{x}_1$ which implies $u = f(x_1, x_2, t) + cx_2$. Now, we know that $\dot{V} = \sigma\dot{\sigma}$. So, $\dot{V} = \sigma\big(u + f(x_1, x_2, t)\big) + cx_2$. So, this is the thing and which implies after getting $cx_2$ getting cancelled so, we have $u = -cx_2 + v$ where $v$ is a new variable ok. So, we assume $u = -cx_2 + v$ so, we get $\dot{V} = \sigma(f(x_1, x_2, t) + v)$.

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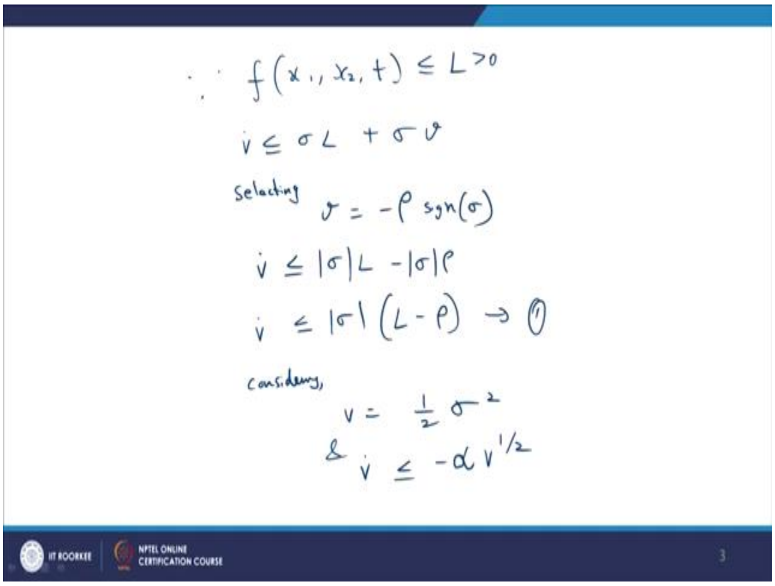


Since, $f(x_1, x_2, t) \leq L > 0$, which is a positive value. So, we have $\dot{v} \leq \sigma L + \sigma v$. Selecting the new variable $v = -\rho sign(\sigma)$, we have $\dot{v} \leq |\sigma| L - |\sigma| \rho$. Thus,

$\dot{v} \leq |\sigma|(L - \rho)$ say equation 1.

Considering now, considering $V = \frac{1}{2}\sigma^2$ and $\dot{V} \leq -\alpha V^{1/2}$.

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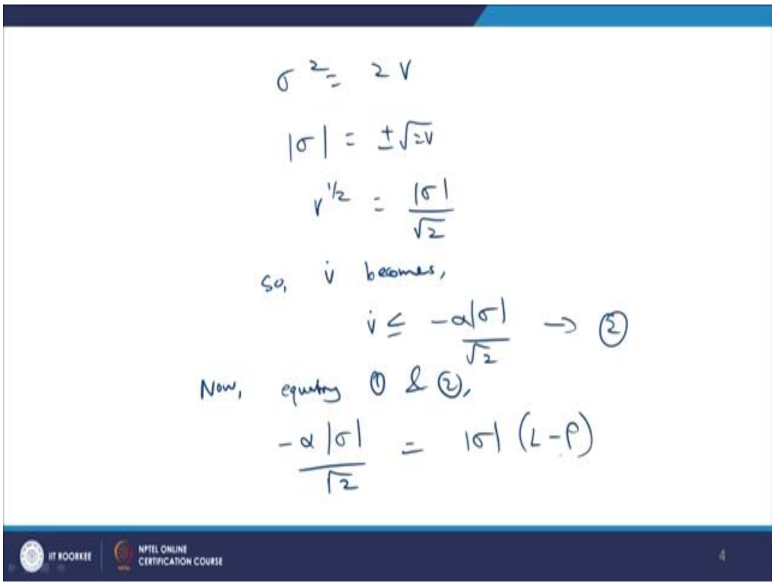
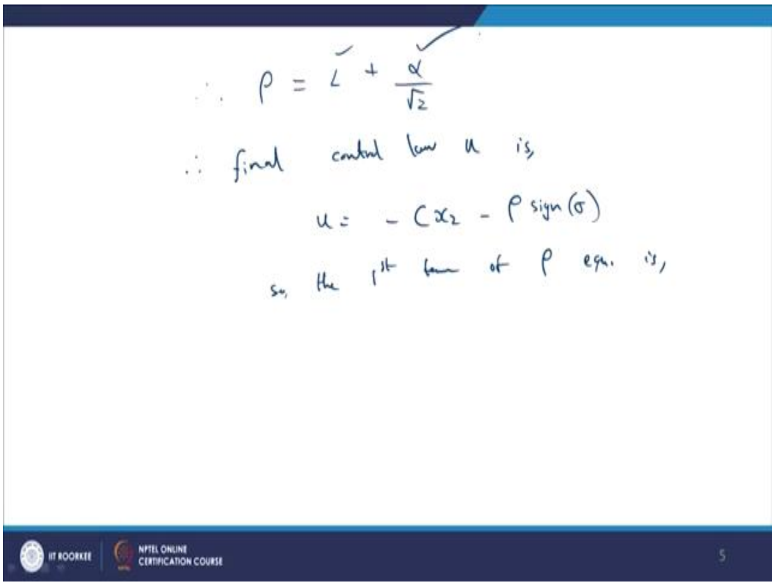



We get from the V value we get a $\sigma^2 = \pm\sqrt{2V}$ which implies $= \frac{|\sigma|}{\sqrt{2}}$. Thus, we can say that

$V^{1/2} = \frac{|\sigma|}{\sqrt{2}}$. So, $\dot{V}$ becomes $\dot{V} \leq -\alpha \frac{|\sigma|}{\sqrt{2}}$ , which is say equation 2. Now, equating equation 1 and 2 $-\alpha \frac{|\sigma|}{\sqrt{2}} = |\sigma|(L - \rho)$. (Refer Slide Time: 18:52)




Therefore, the control gain $\rho$ of the discontinuance controller is given by $\rho = L + \frac{|\sigma|}{\sqrt{2}}$ which is nothing, but the gain of the discontinuous control part. Therefore, final control law $u = -cx_2 - \rho sign(\sigma)$. $\rho = L + \frac{|\sigma|}{\sqrt{2}}$ . This is the constant, L is a constant, $\alpha$ is a constant so, $\rho$ will become a constant. So, the first term so, the first term of rho equation is responsible for the disturbance and the second term is responsible for finite time convergence.

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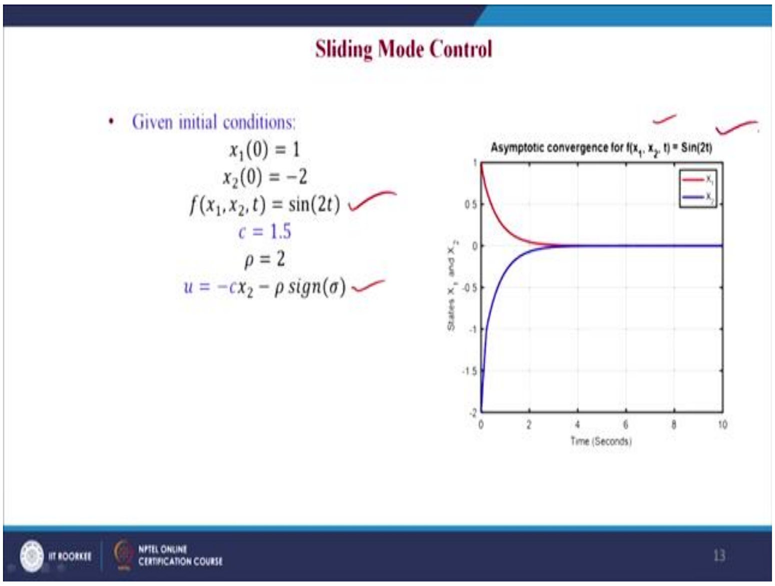


Now, let us see that through simulations having designed the controller we see through simulations what happens with the initial conditions on the values of $c$ and $\rho$ taken by 1.5 and 2. Thus, with the same example of the single dimension motion of the mass with states $x_1$ and $x_2$with the disturbance $\sin(2t)$, how asymptotic convergence of these state variables are obtained. With this control law which is $u = -cx_2 - \rho sign(\sigma)$, we get the asymptotic stability of the states as seen in the figure.

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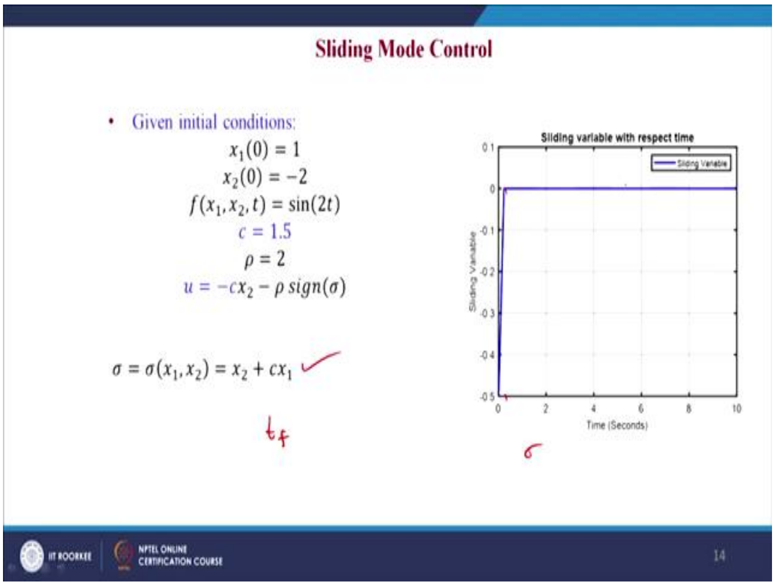

Next what happens to the state to the variable that is a sliding variable? How it is converging in the time domain? We are seeing in finite time convergence, as you can see that it is getting converged in the finite time to 0. The sliding variable sigma is converged to the equilibrium point 0 at the finite time $t_f$ which is much close to 0.25 seconds as can be seen here in this figure.

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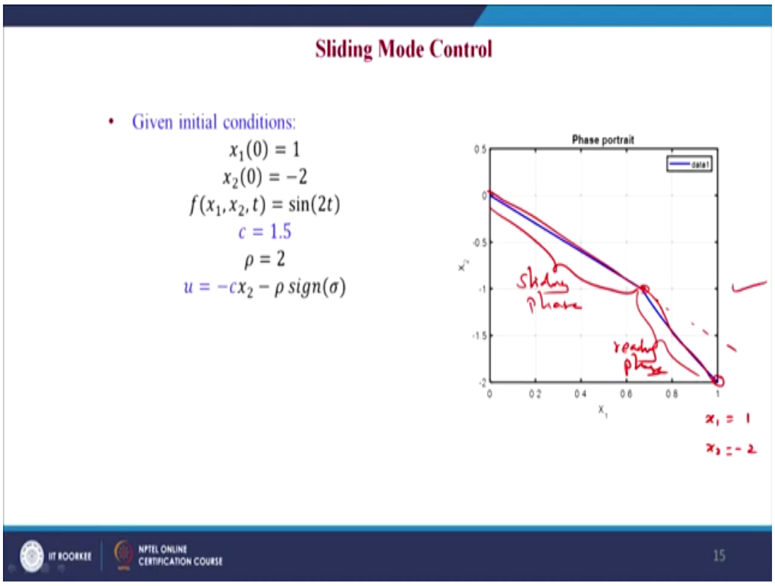


Next we are seeing that convergence of the state variables from the initial case $x_1(0) = 1\ and\ x_2(0) = -2$, we have from started from this trajectory that is the reaching trajectory after reaching it goes to the sliding surface, that is here this phase is called reaching phase of the system states. And, this phase is called sliding phase of the system states.

Once this sliding surface is like this it continues. So, once the system state trajectory reaches the sliding surface it, the sliding surface is designed in such a way that the system state trajectory will be reaching the equilibrium point. That is a theory behind this sliding mode control.

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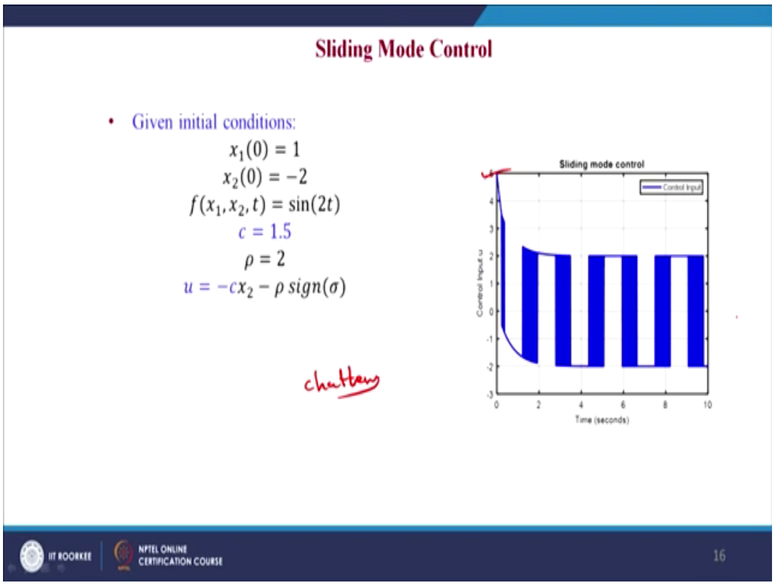


Now, what happens to the control input value? So, you can see that initially the control signal is with 5, who is the value and it can come as the time increases it has this value which is having certain chattering problem, chattering that is the control law is the one which stabilizes the system to converge to asymptotic stability. But, the control law is having chattering and the chattering can be addressed in the higher order sliding mode involving super twisting algorithm and other approaches.

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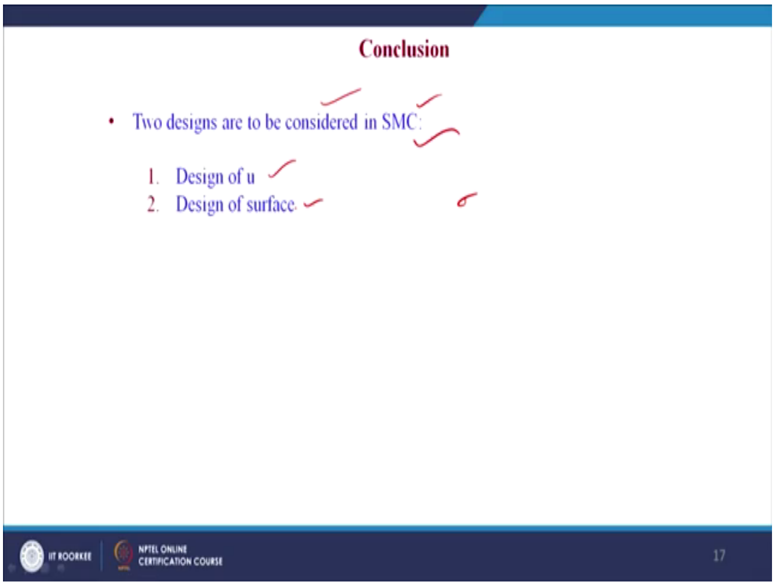

And finally, now, coming to the conclusions; we say that in this lecture we have started with the second order system and we have converged it asymptotically when the disturbance is not given. Once the disturbance is given, the conventional controller says for example, the state feedback controller brings the system to the bound of, a bounded domain in such a way that it cannot converge to 0. Whereas, in the designed sliding mode controller having two phases: one is the reaching phase and the sliding surface phase and sliding phase.

By these two phases this sliding mode controller confirms that the system states can be reaching the convergence point, reaching the stable point asymptotically by finite time convergence of the variable sigma which is a sliding variable. And, two designs are to be considered in the sliding mode controller. One is the design of the control law u so, that the system can be robust to the given input and design of the surface that is a sliding surface so, that the asymptotic stability of the system state variables is confirmed.

These are the two conditions to be considered while considering the sliding mode controller. So, in this lecture we have seen the sliding mode control basics, we started with the controlling single dimensional motion, single dimensional motion of a 2 degrees of freedom system, second order system. So, that the system under state feedback controller can be made to get asymptotically converged when the disturbance is 0, once the disturbance is given the system could bring it to the domain, bounded domain of the disturbance. So, that the asymptotic convergence is not possible with state feedback controller.

So, sliding mode controller has to be designed. So, that two design conditions has to be, have to be considered. One is the design of the control u so, that it can bring the system to the sliding phase like it should bring through reaching phase, it should bring the system to the sliding surface. Once the sliding surface is reached by the design of the sliding surface we could assure that the system states can be converged to the equilibrium point in asymptotic time, that is in as a time increases the system states will be converged to 0 by the design of the sliding surface.

So, this is as an example we can simply think about a simple example for the sliding mode control logic, it is in such a way that when you throw a ball in a open space the ball will not go to the particular equilibrium point. Because, we are throwing it from any initial point, but once we consider a sliding surface which is like a channel that will end up in the equilibrium point so, our main aim is to bring the ball to the initial point of the channel so, that the channel will take the ball to the converging equilibrium point; so, this is how.

The channel here in this sliding mode controller is basically the sliding surface where we denote it as sigma equal to 0, this is sliding surface that will bring the system trajectory to equilibrium point. So, our aim is to by reaching phase we bring the system state to the sliding surface initial point, once it reaches the sliding surface after completing the reaching phase then the sliding surface will make the system trajectory to get converged in the equilibrium point asymptotically. That is all about this lecture.

Thank you very much.