Robotics and Control: Theory and Practice
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Lecture - 03
Coordinate Frames and Homogeneous Transformations – II

Hello viewers, welcome to this lecture on Coordinate Frames and Homogenous Transformations. So, this the second lecture on Coordinate Frames. So, in this lecture, we will give the relation for a general rotation.

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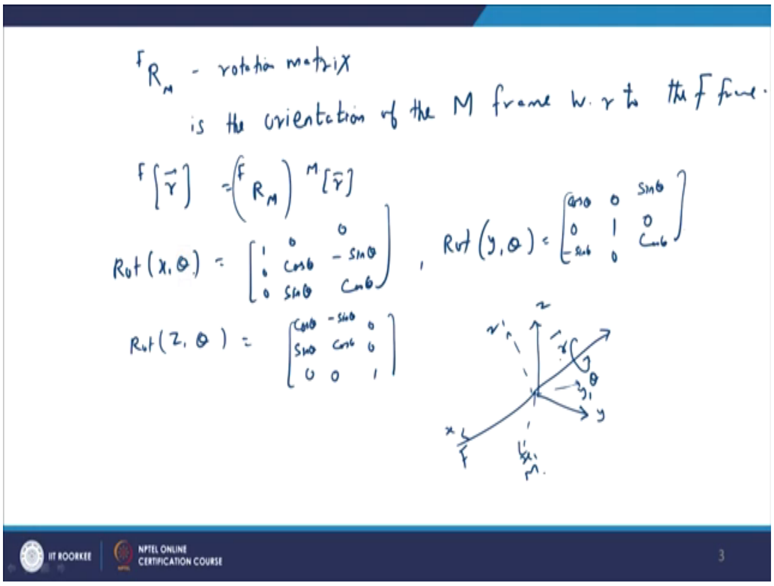

So, in the last lecture the notation ${}^{F}R_{M}$ was introduced which represent a rotation matrix; it represent a rotation matrix and it gives orientation of the M frame with respect to the F frame. So, it gives the relation between a coordinate the coordinate of a point r and the coordinate of the same point with respect to M how they are related. So, this ${}^{F}R_{M}$ represent the matrix relating the M frame with respect to F frame.

And we have also introduced the fundamental rotation that is if we rotate the F frame about its x axis by an angle θ, it is denoted by $[1\ 0\ 0; 0\ \cos\theta - \sin\theta; 0 \sin\theta \cos\theta]$. So, similarly the rotation matrix about the y axis by an angle θ, it can be written as $[\cos\theta\ 0\ \sin\theta; 0\ 1\ 0; -\sin\theta\ 0 \cos\theta]$ and the rotation about the z axis by an angle θ is given by $[\cos\theta - \sin\theta\ 0; \sin\theta \cos\theta\ 0; 0\ 0\ 1]$.

So, this three are called the fundamental rotation and in this lecture, we will see if the fixed frame if a frame F is rotated, these are the three axis x y z axis and if it is rotated with respect to a general axis, let us say general vector r by an angle θ, then the new coordinate system will be obtained; we call it as say x_1, y_1 and z_1. So, the old frame is F and the new frame is M. So, what will be the relation between F and M if the rotation is about any general axis r by an angle θ. So, that is what we will see in this lecture.

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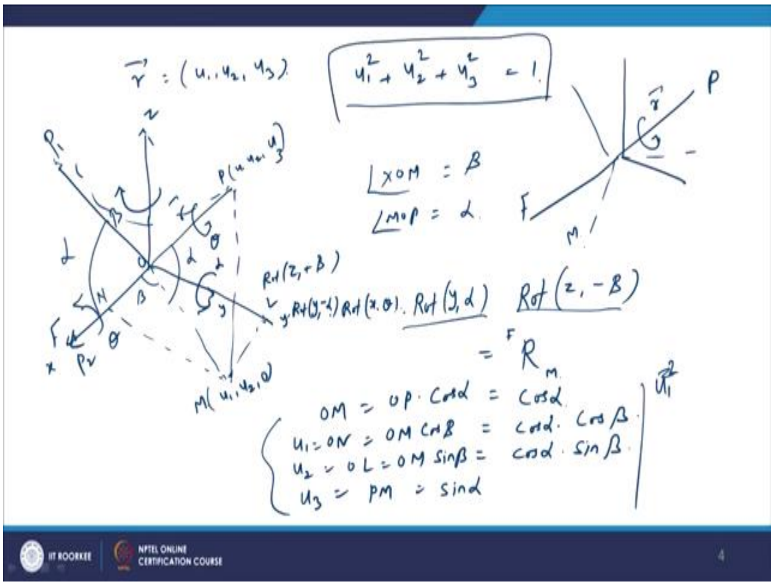

So, let us assume that the general vector given is a unit vector; we call it as u_1, u_2, u_3. So, the unit vector so, this sum of the square of the coordinate should be equal to 1 and it is rotated by an angle θ. So, if we consider this unit vector u_1, u_2, u_3 the projection of this on the x y plane, it is $u_1, u_2, 0$. So, this point is P OP is 1 and this point we call it as M and if it is projected on the x axis and y axis, let us call it as M N and this is L. So, O N is u_1 and O L is u_2 and T M is u 3 that is the value here.

So, if you assume that this angle XOM this angle, we call it as let us say β and MOP we call this angle as α.

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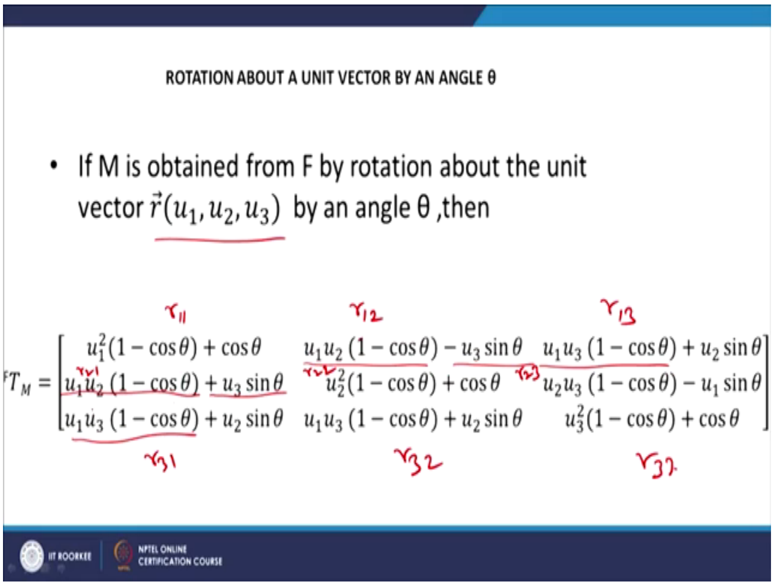

And the rotation matrix is given by this particular formula a 3 by 3 matrix given by this expression.

$$^{F}T_M = \begin{bmatrix} u_1^2(1-\cos\theta)+\cos\theta & u_1u_2\,(1-\cos\theta)-u_3\sin\theta & u_1u_3\,(1-\cos\theta)+u_2\sin\theta \\ u_1u_2\,(1-\cos\theta)+u_3\sin\theta & u_2^2(1-\cos\theta)+\cos\theta & u_2u_3\,(1-\cos\theta)-u_1\sin\theta \\ u_1u_3\,(1-\cos\theta)+u_2\sin\theta & u_1u_3\,(1-\cos\theta)+u_2\sin\theta & u_3^2(1-\cos\theta)+\cos\theta \end{bmatrix}$$

Now, it can be easily verified that if u_1, u_2, u_3 is 1 0 0 that is the x axis, then we will get the fundamental rotation; rotation of x y and angle θ can be obtained as given earlier. So, tOProve the rotation matrix as shown in the previous slides, we adopt the following procedure. We will rotate vector OP about the z axis by an angle β like this. So, it will be fixed in the x z plane from its original position OP it has come to the position, let us say OP1 and observe that this movement is moving from y direction to x direction in the negative direction. So, it means that we are performing a rotation matrix about the z axis by an angle β, but it is in the negative direction that is y is moving towards x so, it rotation of about the z axis by an angle minus θ.

Now, OP1 will move towards the x axis. So, it means the movement is z towards x. So, it is a positive direction by an angle α so, by this angle α. So, we are going to multiply that rotation about the y axis by positive angle α like this. So, earlier we have see that if we perform the operation rotation or any other type of operation with respect to the fixed frame, then we have to multiply those rotation matrix in the left side. So, after bringing that P OP1 to this axis so, let us call it as P2. So, the OP1 has become OP2 vector here. So, it is coinciding with the x axis of the fixed frame.

Now, we will perform the required operation because we wanted to rotate OP by an angle θ. So, instead of rotating it at its original position, we brought it to the x axis and then here we will rotate it by an angle θ in the positive direction θ. So, and now because this O x is the x axis of the fixed frame we have to multiply that rotation in the left side x by an angle θ.

So, now we want to take this OP vector to its original place, then only the meaning will be completed is not it the requirement is completed. So, the OP2 will be taken back to the OP1 position by making a rotation about the y axis in the opposite way. So, we have to multiply the left side by a rotation matrix about the y axis by an angle minus α in the left side. So, it has come to OP1 positive. Now it should be taken back to OP positive in the reverse way. So, we have to multiply by a rotation matrix about the z axis. So, I will write a rotation about z axis by an angle minus β where earlier minus β. So, here it is plus β because we are moving from x to y direction for the rotation.

So, this five matrix multiplication finally, will give the relation between the fixed frame and the m frame. Originally it is fixed frame after making a rotation, we will get a new frame. So, originally we got the F frame, this is our vector r vector. So, after making a positive rotation of this axis we will get a new coordinate frame, we call it as the M frame. So, the relation between F and M frame the M frame with respect to F frame is given by the product of this five matrices.

So, we know that is structure of all this fundamental rotation matrix as written in the previous slide rotation about x y z can be written like this. So, by substituting the appropriate angles in this and multiplying this five matrix, we will obtain a expression like this. So, an expression of this form in this way, but we have convert all this α β gamma which was mentioned in the previous slide as follows.

So, we can see here this expression OM is nothing, but the projection of OP on the x y plane. So, it is nothing, but OP $\cos\alpha$, but OP is of length 1 its a unit vector. So, it is simply $\cos\alpha$ OM is $\cos\alpha$ and then u 1 is O N the x coordinate of this point p is u 1 it is nothing, but OM cosβ that OM is cosα therefore, we get cosα into cos β. So, u_1 value is given by cosα.cosβ similarly u_2 is the y coordinate that is given by OM that is OL and it is OM sin β. So, it should be cosα.sinβ and z axis that is u_3 it is nothing, but p m this length and that is given by simply sinα.

So, now we can use this relation for u_1 u_2 u_3 cosα sinβ etcetera along with the relation $u_1^2 + u_2^2 + u_3^2 = 1$. So, in this product we will be getting for example, in this matrix we will get cosβ sinβ terms in the 3 by 3 matrix and in this expression we will get cosα sinα etcetera. So, when the product of all the five matrices comes, we you will be able to get these products cosα cosβ sinα sinβ etcetera. So, all of them can be written in terms of u_1 u_2 u_3 along with this relation.

So, a general rotation matrix is given now if we substitute here x axis in in the place of the unit vector, we will get the fundamental rotation about x axis similarly 0 1 0 will give fundamental rotation about y axis similarly for the z axis.

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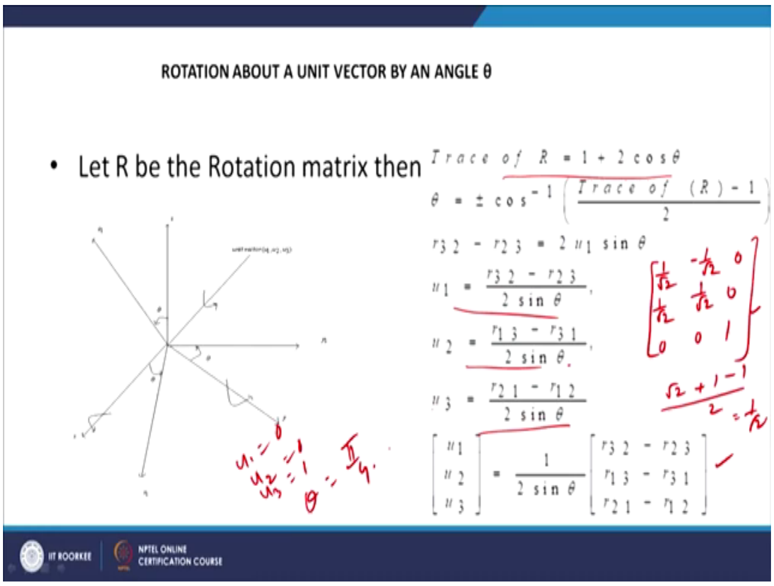

So, now we will see this structure is useful in identifying the amount of rotation and about which axis a rotation is performed. So, for example, a arbitrary rotation matrix is given. So, earlier we have seen that a rotation matrix is a orthogonal matrix in addition to that it should form a right handed system. So, if you give a rotation matrix, then how to identify the axis of rotation as well as the amount of rotation. So, for that it will be very useful.

$$Trace\ of\ R = 1 + 2\cos\theta$$

$$\theta = \pm\cos^{-1}\left(\frac{Trace\ of\ (R) - 1}{2}\right)$$

$$r_{32} - r_{23} = 2u_1 \sin\theta$$

$$u_1 = \frac{r_{32} - r_{23}}{2\sin\theta},$$

$$u_2 = \frac{r_{13} - r_{31}}{2\sin\theta},$$

$$u_3 = \frac{r_{21} - r_{12}}{2\sin\theta}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

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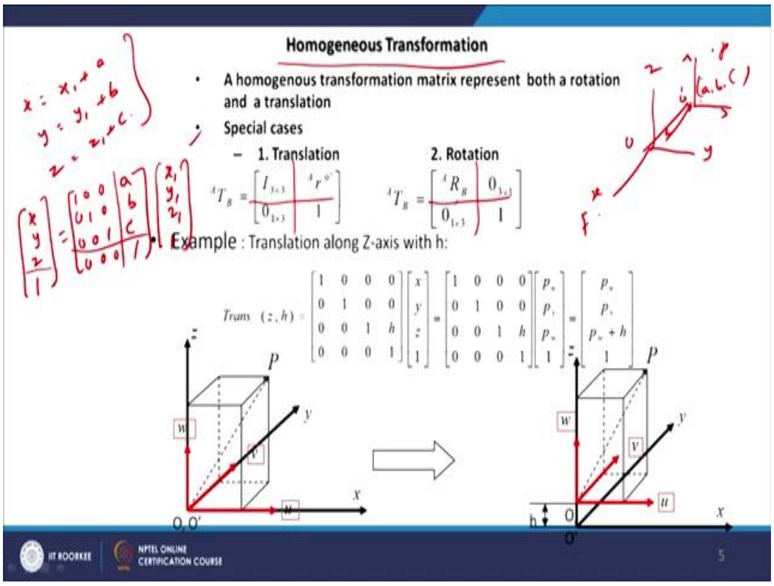

So, now we consider the homogeneous transformation. So, so far it was considered only about the rotation, rotation about x y z axis or rotation about any general axis, but if the coordinate frame is translated that is a fixed frame F is given and if the origin is shifted to some other position without disturbing the directions of the three axis x y z axis are kept parallel all the time and only origin is shifted to a different position in that case the motion is called a translation.

So, the translation the new translated origin will have a coordinate a b c let us say with respect to the old frame. So, in that case we can write the point any point p with respect to the u coordinate frame, if you mention it as x_1, y_1, z_1 and the same point with respect to old coordinate its x y z, then we will get the relation:

$$x = x_1 + a$$

$$y = y_1 + b$$

$$z = z_1 + c$$

So, that is very clear because the x y z old one and the new ones are parallel to each other only the origin is shifted to a point a b c. So, we can conclude that the old value of x is the new value of x plus how much it is shifted the origin is shifted etcetera.

And now we cannot write it in the form of a matrix. As in the case of rotation matrix we cannot write the relation of the translation in the form of a 3 by 3 matrix, but if you slightly modify the structure of representing a vector with three numbers then we will be able to perform the matrix relation. So, instead of representing the vector x y z with three numbers, if I write a dummy variable 1. So, with four numbers if I represent where the fourth number is a dummy value 1, then we can write the relation between x and x_1, y_1, z_1 in the following way. This can be written as 1 0 0 a; 0 1 0 b; 0 0 1 c. So, we can write the relation between x y z, x_1, y_1, z_1 in this manner using a matrix whereas, for a rotation a 3 by 3 matrix itself is sufficient to indicate.

So, to combine both rotation and translation in one particular structure we device a 4 by 4 matrix structure which is called the homogeneous transformation. So, a homogeneous transformation matrix represents both rotation and translation in the following way. So, as previously seen the rotation matrix is R and because, we are having a dummy variable 1 as the fourth value in the vector. So, we have to introduce the fourth row also as 0 0 0 1 for the rotation matrix and the fourth column is 0 0 0 1. So, that should be introduced the fourth row and fourth column are introduced for a rotation matrix.

And for the translation matrix the first 3 by 3 matrix is the identity matrix and the fourth column represent the origin the new origin a b c that will come in the fourth column. So, so whenever we write a 4 by 4 matrix which represent a translation, it indicates that the fourth column represent the origin of the new coordinate frame which is shifted from the old frame. And the first three column of that transformation homogeneous transformation represent the rotation of the coordinate frames. So, it combines both of them position as well as orientation of a coordinate frame.

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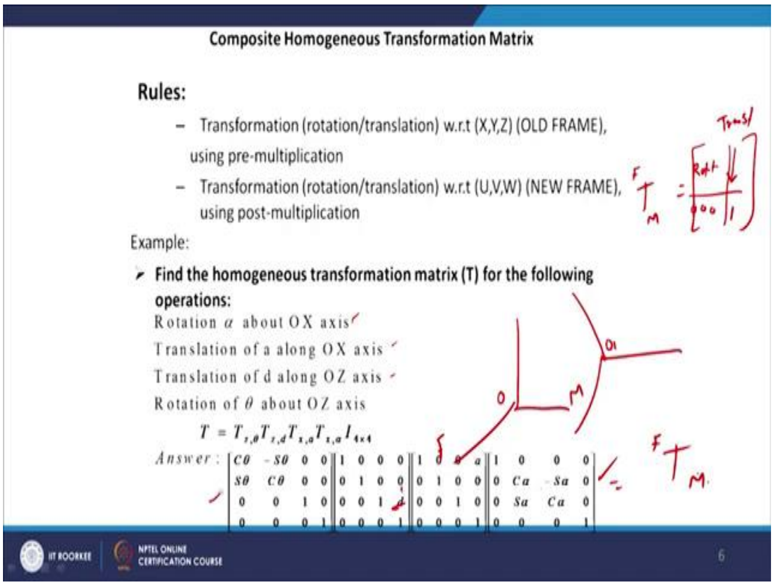

So, if we have if we have a coordinate frame F and it is shifted to some other position by making a rotation as well as a translation then its new position and orientation will be obtained for example, by this expression. Let us say o 1 is the new position of the origin and the orientation is given by this expression then we can write this F T M. If it is only rotation we were using r notation for a 3 by 3 matrix, but here because we are introducing a 4 by 4 matrix for the translation as well as rotation we use the notation T representing a transformation.

So, ${}^{F}T_M$ the homogeneous transformation is a 4 by 4 matrix in which the first 3 by 3 matrix the 3 by 3 block, it represent the rotation and the fourth column represent the translation and then the fourth row it is a dummy row to take care of the dummy variable 1 in the vector notation ok.

So,. So, this example shows that how successive trans translation and rotation can be handled when we are doing various types of motion from a fixed frame. First we have a fixed frame and then a rotation α about the x axis is performed, then a translation along the x axis is performed, then a translation along z axis is performed then rotation of an angle θ about the z axis is performed. So, if all this are performed with respect to the fixed frame base frame, here OX represent the base frame x axis. So, again OX represent the base frame x axis and OZ represent the z axis of the fixed frame. So, everything is done with respect to the fixed frame. So, the operation should be multiplied in the left side.

So, first is the rotation with respect to x axis is this one and then translation along the x axis that is a distance along the x axis and the therefore, y and z are 0 0 then a translation along z axis of the fixed frame. So, again we multiply in the left side o that is 0 0 d is the amount of translation we are performing then it rotation above the z axis is given by this matrix.

So, we multiply all this 4. So, after making these four types of promotion what will be the relation between the old frame and the new frame that is given by the product of these four matrix that is ${}^{F}T_{M}$. So, we can adapt this same procedure if you do the operations with respect to the current frame every time we obtain a new frame and with respect to the axis of that frame if you are doing the operations, we have multiply the right hand side and successive multiplication will give the relation between fixed frame and the moving frame. So, in either way we can obtain the relation between a given old frame and the current frame.

So, with this we come to the conclusion about the relation between various frames of references and which are represented in the form of a 4 by 4 matrix. So, this will be very useful in various robotics problems. So, in the next lecture we will see how this can be introduced in the robot kinematics and dynamics and various other concepts related to the robotics problem.

Thank you.