Robotics and Control: Theory and Practice
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Lecture – 02
Coordinate Frames and Homogeneous Transformations – I

Hello viewers, welcome this lecture on Coordinate Frames. In this lecture, we shall see the how the position and orientation of a coordinate frame with respect to another coordinate frame can be expressed in the form of a matrix.

So, this concept is very important in robotics because it is very essential to know the position and orientation of a robot with respect to its environment or with respect to various object associated with it to perform a particular task. So, first before going into those details about robotics we will concentrate on coordinate frames. So, first let us introduce some basic notations and definitions.

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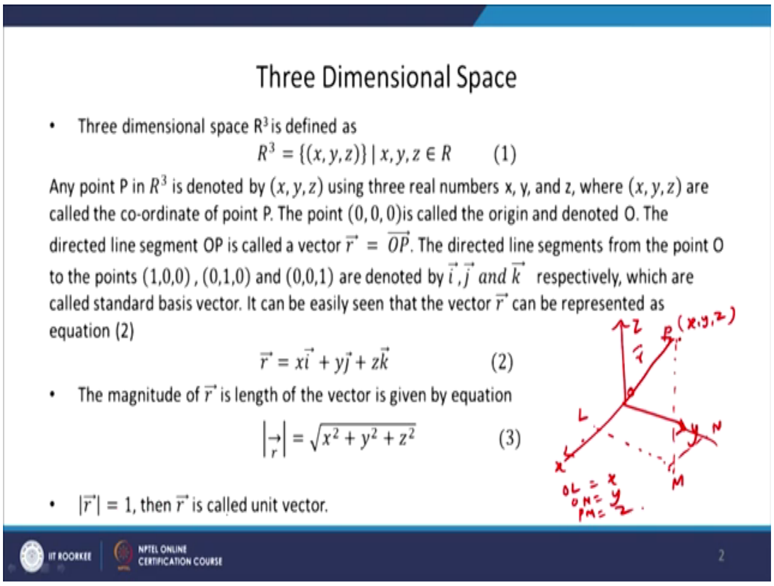

So, Three dimensional space R^3 is defined as

$$R^3 = \{(x, y, z)\} \mid x, y, z \in R \qquad (1)$$

And, we can interpret this as if you draw a perpendicular to the x, y plane it meets the x, y plane at M and if you draw perpendicular to the x axis from M it let us say it meet at L and if you draw perpendicular to y axis it meets at N. So, we can say that the OL is x, ON is y and PM is z. So, any point P is represented by a coordinate x, y, z with respect to this particular frame.

Any point P in R^3 is denoted by (x, y, z) using three real numbers x, y, and z, where (x, y, z) are called the co-ordinate of point P. The point $(0,\ 0,\ 0)$is called the origin and denoted O. The directed line segment OP is called a vector $\vec{r} = \overrightarrow{OP}$. The directed line segments from the point O to the points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ are denoted by $\vec{\imath}, \vec{\jmath}\ and\ \vec{k}$ respectively, which are called standard basis vector. It can be easily seen that the vector $\vec{r}$ can be represented as equation (2)

$$\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k} \qquad (2)$$

The magnitude of $\vec{r}$ is length of the vector is given by equation

$$\left|\vec{r}\right| = \sqrt{x^2 + y^2 + z^2} \qquad (3)$$

$|\vec{r}| = 1$, then $\vec{r}$ is called unit vector.

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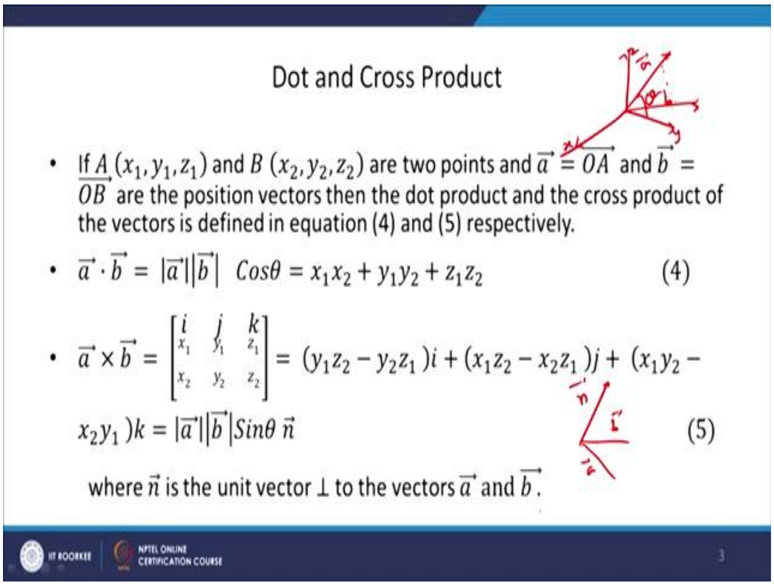

So, now let us see meaning of dot product and cross product.

If A (x_1, y_1, z_1) and B (x_2, y_2, z_2) are two points and $\vec{a} = \overrightarrow{OA}$ and $\vec{b} = \overrightarrow{OB}$ are the position vectors then the dot product and the cross product of the vectors is defined in equation (4) and (5) respectively.

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cdot cos\theta = x_1x_2 + y_1y_2 + z_1z_2 \qquad (4)$$

$$\vec{a} \times \vec{b} = \begin{bmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix}$$

$$= (y_1z_2 - y_2z_1)i + (x_1z_2 - x_2z_1)j + (x_1y_2 - x_2y_1)k$$

$$= |\vec{a}||\vec{b}|Sin\theta\, \vec{n} \qquad (5)$$

where $\vec{n}$ is the unit vector ⊥ to the vectors $\vec{a}$ and $\vec{b}$.

So, here angle θ represent the angle between the two vectors.

. So, it also says that if we take $|\vec{a}|$ then $|\vec{a}|.\cos\theta$ is the projection of the vector on the . $\vec{b}$ and $|\vec{b}|$ is the length of the vector. So, the product of the two is the dot product between these two.

We can also observe that if $|\vec{b}|$ is 1, then a dot b gives the projection of the vector a on the unit vector b. The cross product a cross b is expressed or defined by the determinant as given in this expression. So, when we ex when we expand this determinant we will get the i component is y 1 z 2 minus y 2 z 1 plus j component k component as given here.

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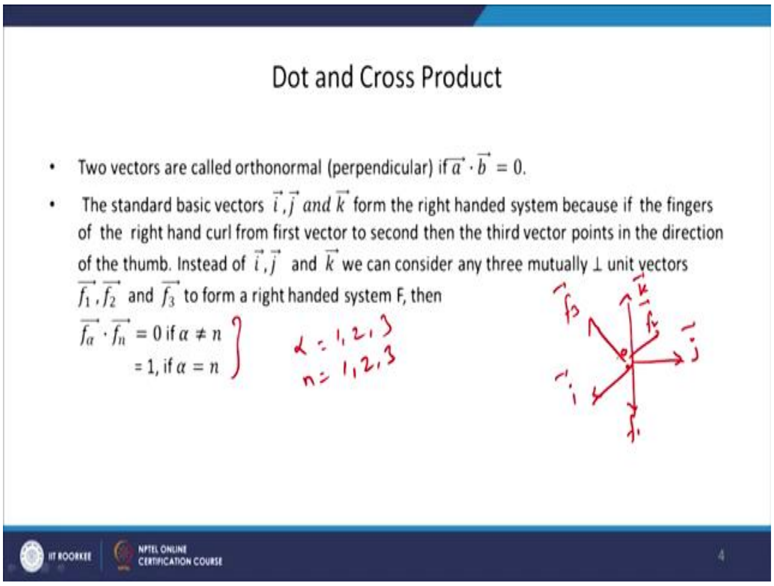

Two vectors are called orthonormal (perpendicular) if $\vec{a} \cdot \vec{b} = 0$.

The standard basic vectors $\vec{\imath}, \vec{\jmath}$ *and* $\vec{k}$ form the right handed system because if the fingers of the right hand curl from first vector to second then the third vector points in the direction of the thumb. Instead of $\vec{\imath}, \vec{\jmath}$ and $\vec{k}$ we can consider any three mutually ⊥ unit vectors $\vec{f_1}, \vec{f_2}$ and $\vec{f_3}$ to form a right handed system F, then

$$\vec{f_\alpha} \cdot \vec{f_n} = 0 \text{ if } \alpha \neq n$$

$$= 1, \text{ if } \alpha = n$$

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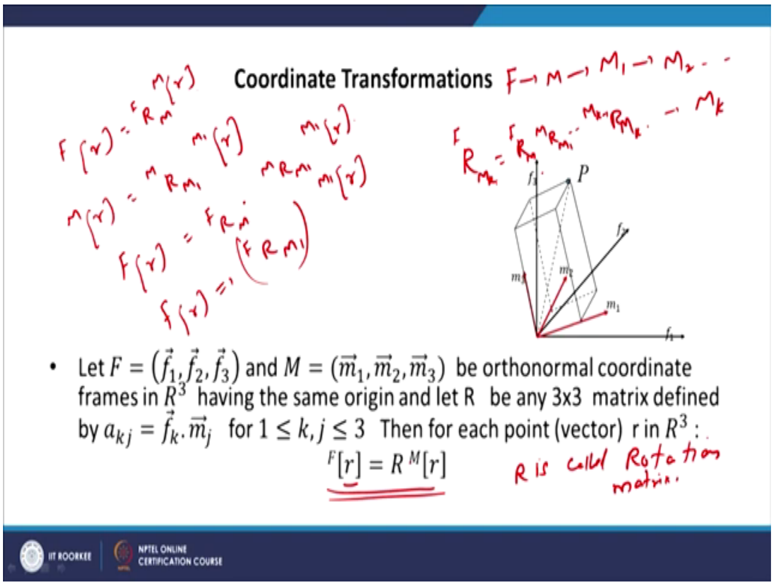

Now, we consider how to represent the position and orientation of one coordinate frame with respect to another coordinate frame.

Let $F = (\vec{f}_1, \vec{f}_2, \vec{f}_3)$ and $M = (\vec{m}_1, \vec{m}_2, \vec{m}_3)$ be orthonormal coordinate frames in R^3 having the same origin and let R be any 3x3 matrix defined by $a_{kj} = \vec{f}_k.\vec{m}_j$ for $1 \leq k, j \leq 3$ Then for each point (vector) r in R^3 :

$$F[r] = R\,M[r]$$

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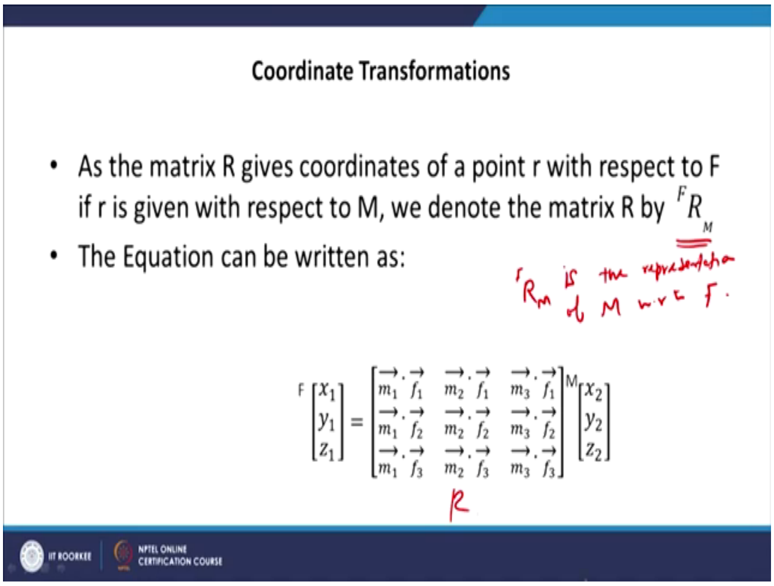

As the matrix R gives coordinates of a point r with respect to F if r is given with respect to M, we denote the matrix R by ${}^{F}R_{M}$

The equation can be written as:

$$ {}^{F}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \overrightarrow{m_1}\cdot\overrightarrow{f_1} & \overrightarrow{m_2}\cdot\overrightarrow{f_1} & \overrightarrow{m_3}\cdot\overrightarrow{f_1} \\ \overrightarrow{m_1}\cdot\overrightarrow{f_2} & \overrightarrow{m_2}\cdot\overrightarrow{f_2} & \overrightarrow{m_3}\cdot\overrightarrow{f_2} \\ \overrightarrow{m_1}\cdot\overrightarrow{f_3} & \overrightarrow{m_2}\cdot\overrightarrow{f_3} & \overrightarrow{m_3}\cdot\overrightarrow{f_3} \end{bmatrix} {}^{M}\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} $$

So, ultimately we get the relation x_1, y_1, z_1 is the coordinate of the vector R with respect to F frame and x_2, y_2, z_2 is the coordinate with respect to m frame then they are related by this expression where R is this matrix given by this expression.

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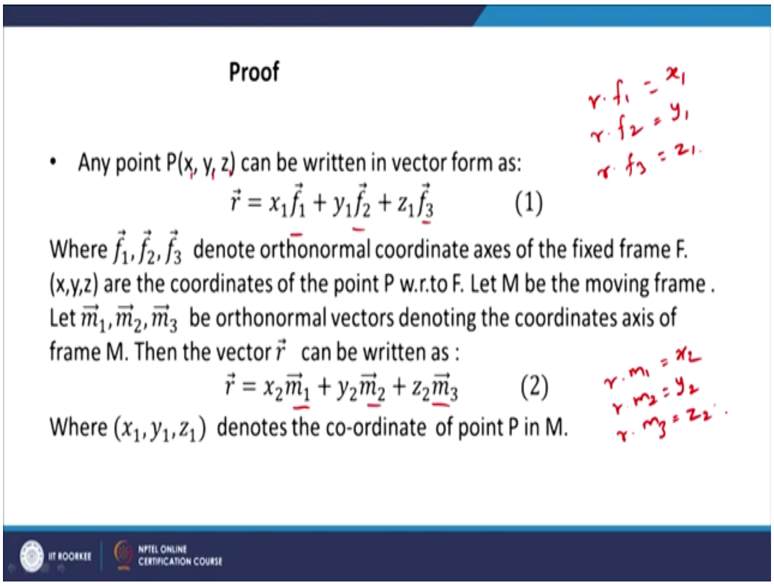

So, how to prove this relation:

Any point P(x, y, z) can be written in vector form as:

$$\vec{r} = x_1\vec{f_1} + y_1\vec{f_2} + z_1\vec{f_3}$$

Where $\vec{f_1}, \vec{f_2}, \vec{f_3}$ denote orthonormal coordinate axes of the fixed frame F. (x,y,z) are the coordinates of the point P w.r.to F. Let M be the moving frame . Let $\vec{m}_1, \vec{m}_2, \vec{m}_3$ be orthonormal vectors denoting the coordinates axis of frame M. Then the vector $\vec{r}$ can be written as :

$$\vec{r} = x_2\vec{m}_1 + y_2\vec{m}_2 + z_2\vec{m}_3$$

Where (x_1, y_1, z_1) denotes the co-ordinate of point P in M.

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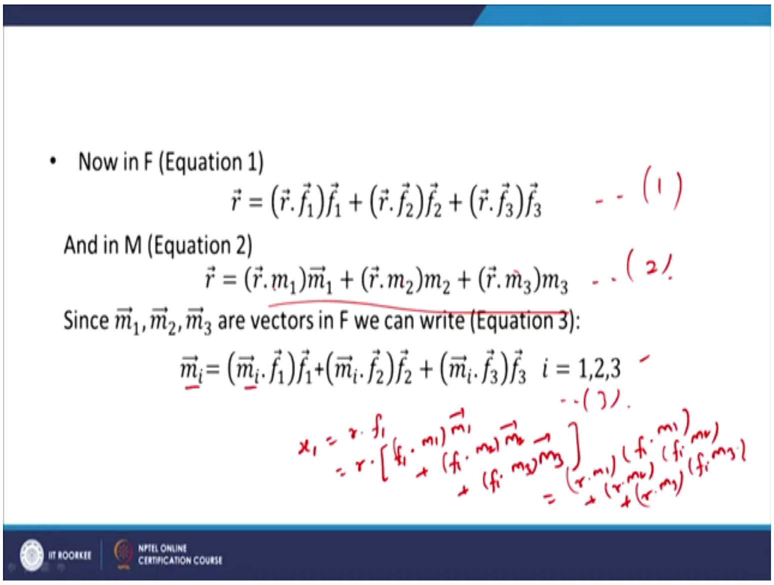

Now in F (Equation 1)

$$\vec{r} = (\vec{r}.\vec{f_1})\vec{f_1} + (\vec{r}.\vec{f_2})\vec{f_2} + (\vec{r}.\vec{f_3})\vec{f_3}$$

And in M (Equation 2)

$$\vec{r} = (\vec{r}.m_1)\vec{m}_1 + (\vec{r}.m_2)m_2 + (\vec{r}.m_3)m_3$$

Since $\vec{m}_1, \vec{m}_2, \vec{m}_3$ are vectors in F we can write (Equation 3):

$$\vec{m}_i = (\vec{m}_i.\vec{f_1})\vec{f_1} + (\vec{m}_i.\vec{f_2})\vec{f_2} + (\vec{m}_i.\vec{f_3})\vec{f_3} \quad i = 1,2,3$$

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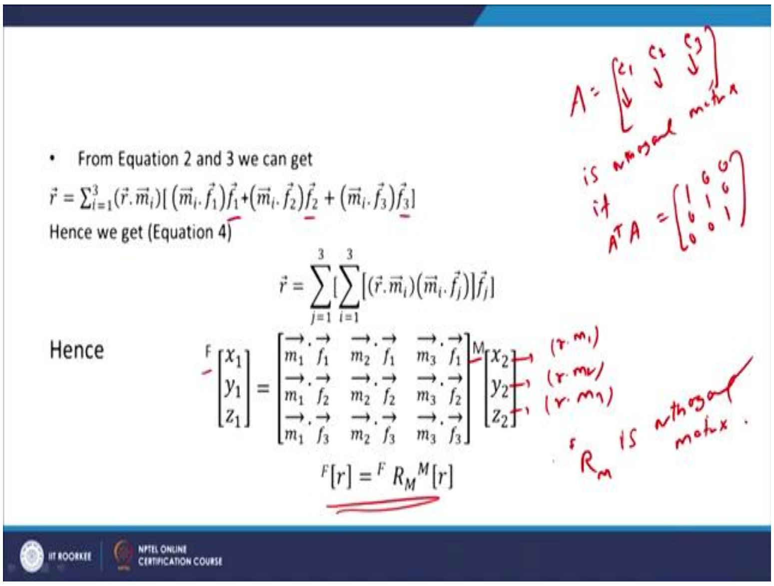

From Equation 2 and 3 we can get

$$\vec{r} = \sum_{i=1}^{3}(\vec{r}.\vec{m}_i)[\,(\vec{m}_i.\vec{f_1})\vec{f_1} + (\vec{m}_i.\vec{f_2})\vec{f_2} + (\vec{m}_i.\vec{f_3})\vec{f_3}]$$

Hence we get (Equation 4)

$$\vec{r} = \sum_{j=1}^{3}[\sum_{i=1}^{3}[(\vec{r}.\vec{m}_i)(\vec{m}_i.\vec{f_j})]\vec{f_j}]$$

Hence

$$
{}^{F}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \overrightarrow{m_1}\cdot\overrightarrow{f_1} & \overrightarrow{m_2}\cdot\overrightarrow{f_1} & \overrightarrow{m_3}\cdot\overrightarrow{f_1} \\ \overrightarrow{m_1}\cdot\overrightarrow{f_2} & \overrightarrow{m_2}\cdot\overrightarrow{f_2} & \overrightarrow{m_3}\cdot\overrightarrow{f_2} \\ \overrightarrow{m_1}\cdot\overrightarrow{f_3} & \overrightarrow{m_2}\cdot\overrightarrow{f_3} & \overrightarrow{m_3}\cdot\overrightarrow{f_3} \end{bmatrix} {}^{M}\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}
$$

$$
{}^{F}[r] = {}^{F}R_{M}\,{}^{M}[r]
$$

.

So, if we observe the first column of this matrix ${}^{F}R_{M}$, the first column is nothing, but the coordinate of the vector m_1 with respect to f_1, f_2, f_3 coordinate frame. Similarly, the second column represent the coordinate of the m_2 vector written with respect to the f_1, f_2, f_3 frame.

So, the three columns are simply the coordinate of the m_1 vector, m_2 vector, m_3 vector and we know that this three vectors are mutually perpendicular and the length is 1. So, the matrix represented by this expression is a orthogonal matrix. What is an orthogonal matrix? If A is an a is a matrix such that the columns let us say c_1, c_2, c_3 the columns are such that the length of each column is 1 and the dot product between any two column is 0, then that matrix is called orthogonal matrix it is orthogonal matrix.

If we take for example, the $A^T A$ gives you the identity 1 0 0; 0 1 0; 0 0 1. So, it can be easily seen if a transpose is the inverse of the matrix a then it is an orthogonal matrix. So, the same thing happens here because the three columns are mutually perpendicular and they are of length 1. So, the vector ${}^{F}R_{M}$ between any two frame F frame and M frame it is a orthogonal matrix. But the converse generally is not true.

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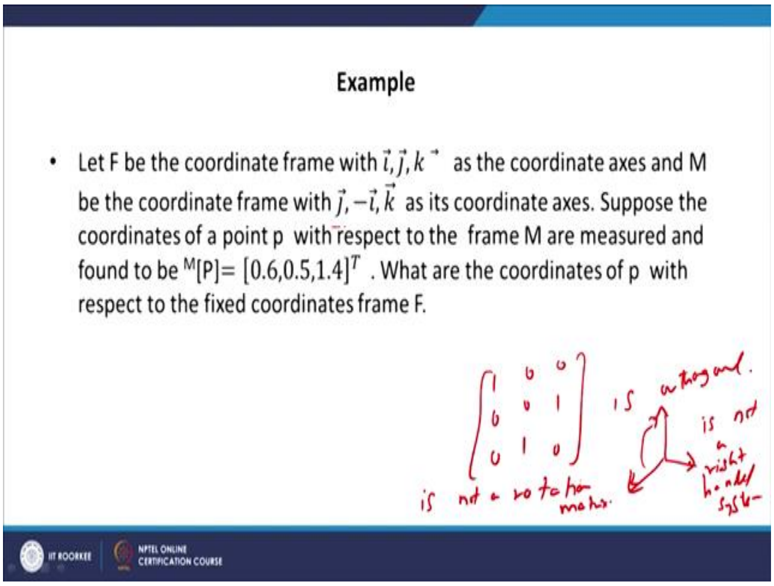

Every rotation matrix is a orthogonal matrix as given here, but every orthogonal matrix need not be a rotational matrix. Here for example, if we take 1 0 0; 0 0 1 and 0 1 0 it is orthogonal by the definition $A^T A$ is identity. But, here we can see that this will not form the first column represent the x axis for example, 1 0 0; the second column represent the z 0 0 1; third column is 0 1 0 this one.

So, if we rotate from the first to the second column then the thumb should be pointing towards the third one, but it is pointing in the opposite direction. It will not form a right hand system, it is not a rotation matrix. So, is not a rotation matrix. So, for example, if F is a coordinate frame with i, j, k as the coordinate axis and M is another coordinate frame with the first axis as j, second is minus i and third is k. This will form a right hand system.

Let F be the coordinate frame with $\vec{i}, \vec{j}, \vec{k}$ as the coordinate axes and M be the coordinate frame with $\vec{j}, -\vec{i}, \vec{k}$ as its coordinate axes. Suppose the coordinates of a point p with respect to the frame M are measured and found to be $^{M}[P] = [0.6, 0.5, 1.4]^T$. What are the coordinates of p with respect to the fixed coordinates frame F.

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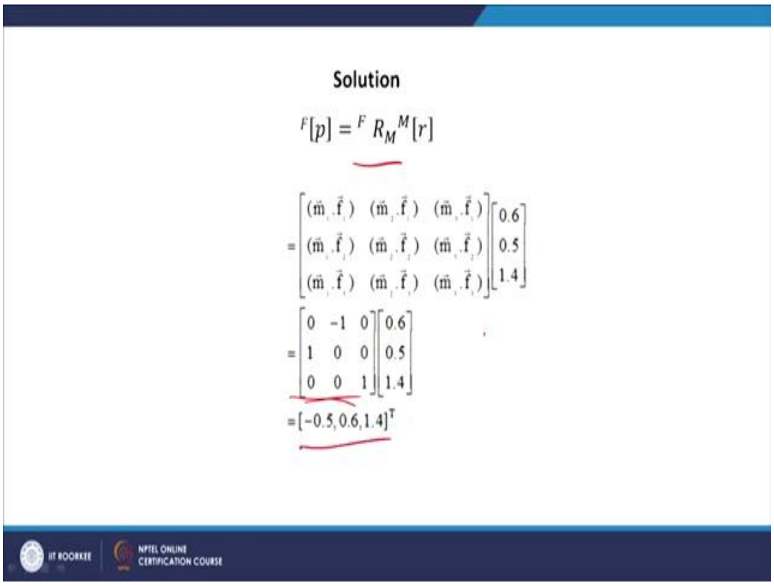

$$
= \begin{bmatrix} (\vec{m}_1.\vec{f}_1) & (\vec{m}_2.\vec{f}_1) & (\vec{m}_3.\vec{f}_1) \\ (\vec{m}_1.\vec{f}_2) & (\vec{m}_2.\vec{f}_2) & (\vec{m}_3.\vec{f}_2) \\ (\vec{m}_1.\vec{f}_3) & (\vec{m}_2.\vec{f}_3) & (\vec{m}_3.\vec{f}_3) \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.5 \\ 1.4 \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.5 \\ 1.4 \end{bmatrix}
$$

$$
= [-0.5, 0.6, 1.4]^T
$$

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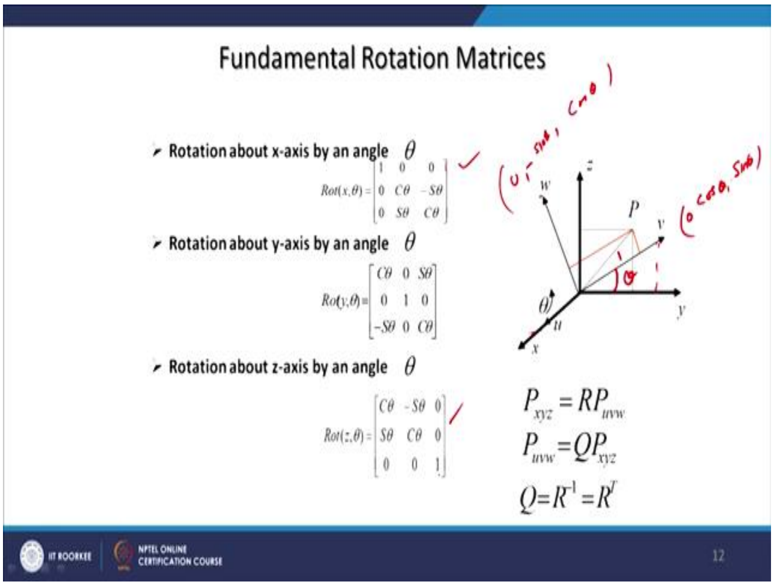

Now, the fundamental rotation matrices. Fundamental rotation matrix is as follows. If we rotate about the x axis by an angle θ then we can see that the x axis will remain the same, but y and z axis will be shifted to this position. So, we can see that the length 1 because this length is 1 here we get the projection of this is $\cos\theta$. So, the x axis will not change, the y value has become this expression the y coordinate has come to this expression. Here it is $\cos\theta$, y value and z value is $\sin\theta$.

And, here we can see that it is x value is the same because it is in the y z coordinate, z axis has shifted to this position w, it is in the y-z plane. So, the coordinate of this w point is x is 0, y value is $-\sin\theta$ and z value is $\cos\theta$. So, if you write this in the columns the first column is 1, 0, 0 only because the axis is the same for the first initial frame and after rotation also the x coordinate is 1, 0, 0 and the y axis is given by 0, $\cos\theta$ and $\sin\theta$. So, that is in the second column, the third column is 0, $-\sin\theta$ and $\cos\theta$ that is given here. So, this represent the fundamental rotation about the x axis by an angle theta.

Similarly, if we rotate about the y axis by an angle θ the y axis will not change. So, the second column is as it is 0, 1, 0, but the x and z axis will change it is position like this thing $\cos\theta$, 0 , $-\sin\theta$ will be the coordinate of the x axis after rotation and z axis after rotation the coordinates are $\sin\theta$, 0, $\cos\theta$ the unit vector in the z direction. So, similarly the rotation about z axis can be written in this expression. So, these three rotations are called the fundamental rotations. Now, using this fundamental rotation we can obtain any general rotation.

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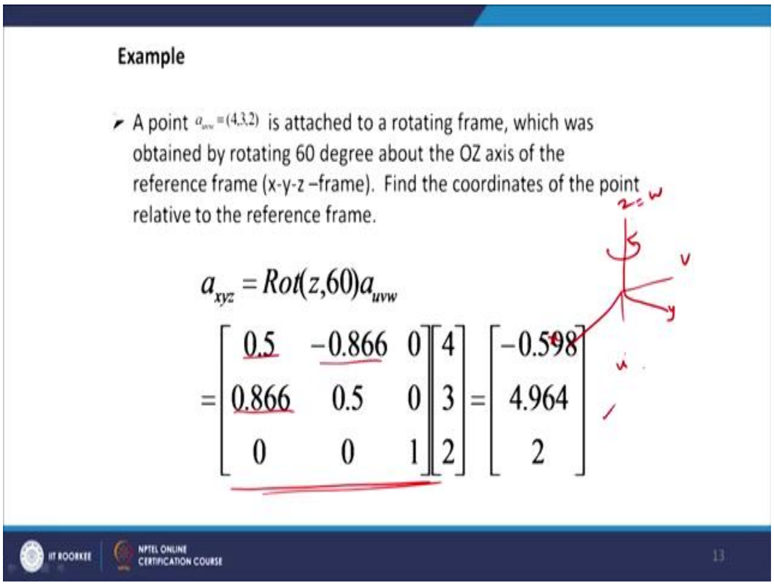

So, here we see another example a point alpha uvw is given. So, we have two coordinate system xyz coordinate system, after rotating it has become uvw coordinate system. So, here it was obtained by rotating 60 degrees about the OZ axis. So, we have x, y, z axis and if you rotate 60 degrees about the z axis we get the new coordinate frame like this we call it as uvw frame because we are rotating about z axis, w is same as z axis.

So, now, the point in the uvw frame is given as 4, 3, 2 and what will be the point with respect to the original xyz frame. So, we can make use of the rotation matrix because it is rotation with respect to z the fundamental rotation matrix can be obtained.

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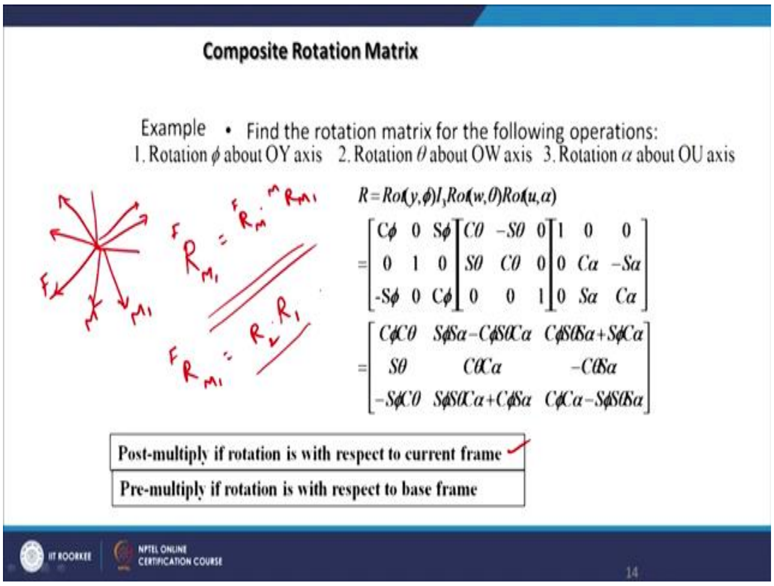

Now, the composite rotation matrix. So, here we note that if you are performing successively various rotation operations. So, if you have initially a fixed frame here, frame with coordinate frame f_1, f_2, f_3 then we make some kind of rotation and then we obtain a new coordinate frame. So, let us say m_1, m_2, m_3 frame. So, we call it as M frame and then we again make a rotation with respect to M frame and we obtain a new frame. So, let us say M frame and it is another frame M_1 frame with three coordinate says coordinate axis.

So, first we had a frame F that is fixed after that we made a rotation we got M again we made a rotation with respect to M frame then we obtained a new frame M_1. So, what will be the relation between the F frame and then M_1 frame? So, the orientation of M_1 frame with respect to F frame is called ${}^{F}R_{M_1}$. So, that is obtained by just multiplying ${}^{F}R_{M}$ and then ${}^{M}R_{M_1}$. The orientation of M_1 with respect to M frame and then we pre multiply the orientation of M with respect to F frame. So, that will give the representation of M_1 with respect to F frame.

But, the other scenario is if first we have the fixed frame F and then we make a rotation with respect to the F frame and we got a new frame M. Now, we make again a rotation with respect to F frame itself with respect to F frame and we obtain the M_1 frame. So, this is different from the previous step. We are making in the previous we are doing the rotation with respect to the current frame whatever is the present frame we are making rotation with respect to that frame itself. So, then we have to multiply in the right hand side that is post multiply.

But, if we make rotation with respect to the base frame every time and obtaining a new frame then the relation between the first F frame and the last frame is given by the. So, if we make for example, the rotation 1 sum operation with respect to F frame and obtain the M frame. And, then if you make another rotation with respect to the base frame and obtain a new frame M_1 then the relation between M_1 frame and F frame is obtained by pre multiplying the operation successively. So, in this result we have shown that we have R with respect to F frame it is given by $^{F}R_{M}$ and R with respect to M frame.

Now, if we make a rotation with respect to M frame and obtain a M_1 frame, then the relation between M frame and M_1 frame is obtained and the coordinate of a point in M_1 frame is related by the coordinate of the point in M frame is given by this expression by the same theorem replacing F and M by M and M_1. Now, if you substitute this R_M from the second equation we get. So, the coordinate of the vector r with respect to F frame is $^{F}R_{M}$. So, this implies that r with respect to F and R with respect to M 1 they are related by $^{F}R_{M_1}$. So, $^{F}R_{M_1}$ is nothing, but $^{F}R_{M}$ multiplied by $^{M}R_{M_1}$

So, similarly if we make a rotation with respect to M_1 then we have to multiply in the right hand side. So, in general we have if we get from F if we take M and from M we get M_1 and then M_2 etcetera M_k after k steps we come to a coordinate frame M_k, then what is the relation between F and M_k that is given by $^{F}R_{M_1}$, $^{F}R_{M}$ and $^{M}R_{M}$ etcetera $^{M_{k-1}}R_{k}$. So, by successively multiplying in the right hand side we get the relation. This is for the rotation with respect to the current frame.

Similarly, we can show if we do the rotation every time with respect to the base frame, then we can obtain we have to pre-multiply the rotation at every step. So, this concept will be very useful in understanding various coordinate frame related concepts in robotics. So, that we will see in the coming lectures.

Thank you.