Robotics and Control: Theory and Practice
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Lecture – 15
Stability of a Dynamical System

So, this lecture is on the Stability of Dynamical Systems. So, in the previous lecture we have seen how to write the dynamic equation of a robot manipulator.

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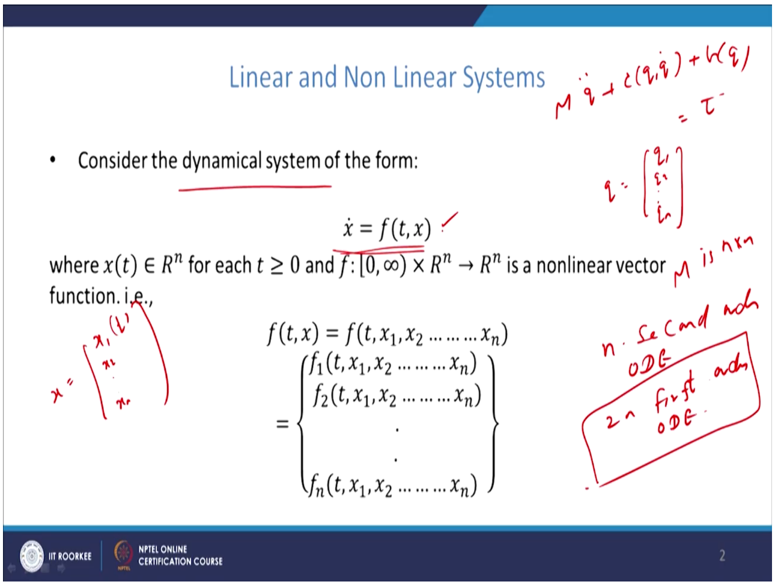

So, we have seen that the dynamic equation can be written as M q double dot plus C q q dot plus G of q equal to tau, where q is a vector contains q 1 q 2 q n. These are the joint variables and M is the inertia matrix n cross n matrix etcetera. Now, from this we can see that this can be written as n second order ordinary differential equation, because it contains the second derivative of q 1 q 2 q n. So, we get n second order ordinary differential equation, which can be converted into 2 n first order ordinary differential equation.

So, in that context we can say that this is a dynamical system. So, what is a dynamical system? The definition is any first order differential equation written in this particular form is called a dynamical system. So, here x is a vector x contains x 1 x 2 x n they are functions of t and therefore, f also should be a vector f 1 f 2 f n and it is a function of t and x 1 x 2 x n. So, dynamical system containing n first order differential equation is represented by x dot equal to f of t x.

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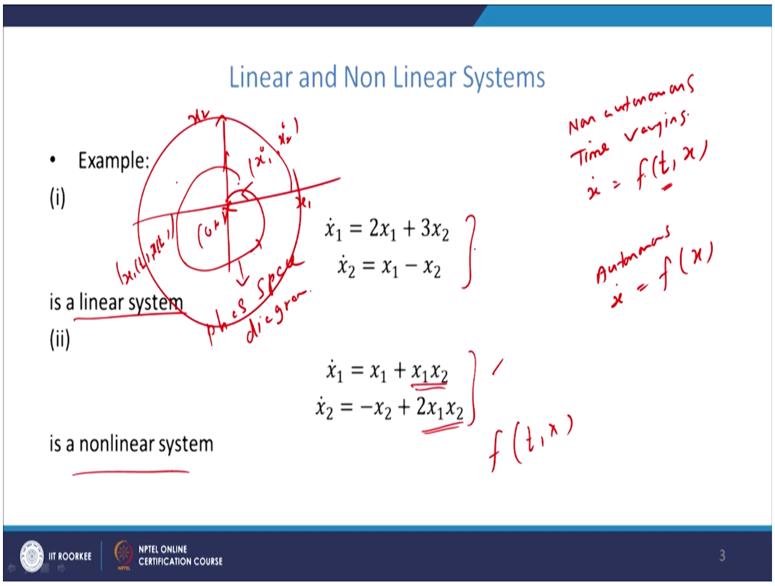

So, this can be classified into various types one is the non autonomous or time varying system. So, that is x dot equal to f of t x here, if t appears explicitly in the equation, then we say that it is time varying. And, autonomous means it is time invariant, it is x dot equal to f of x where t does not appear explicitly in the given system of equation. See, even though the all the systems are varying with respect to time x is a function of time.

So, it is keep on changing, but time varying time invariant is classified only based on whether t appears in the equation explicitly or not. So, for example, here we take x 1 dot equal to 2 x 1 plus 3 x 2 x 2 dot is this. So, this is simply time invariant system or autonomous system. So, this is another autonomous system, we can also classify it as linear, because the right hand side functions are simply linear functions and it can be classified as non-linear.

Because, here the right hand side we have non-linear expressions like the product of the variables or if square of the variables etcetera appears, in the function f of t x then it is called a non-linear system.

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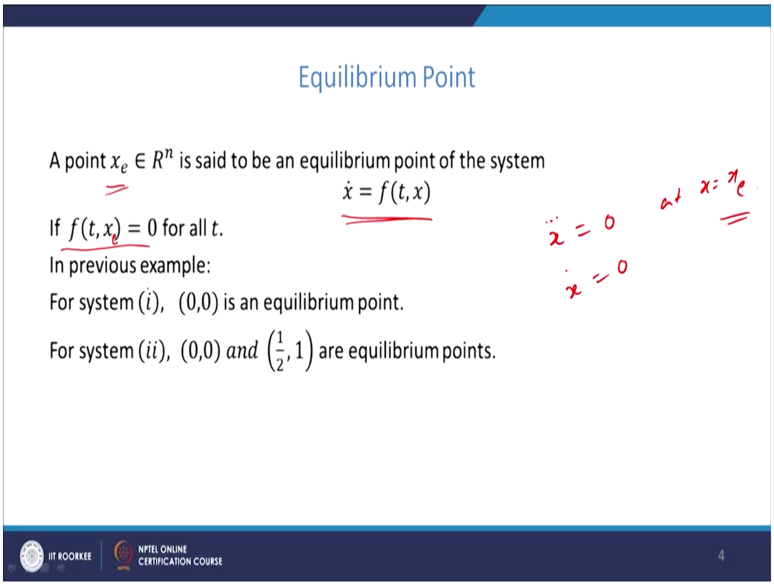

So, let us consider a system of this form, the equilibrium point of a system is defined as the point at which f of t x equal to 0 for all t value. So, xe the point is called an equilibrium point if f of t x suffix e here is equal to 0 for all t; that means, that x dot becomes 0 at the point x equal to x e. In other words the point x x e it does not move as a function of time. Once the solution is at the point x e, then it will not move further because x dot the velocity is 0.

So, the previous example if you see that if you equate the right hand side to 0, the 2 x 1 plus 3 x 2 is 0 and x 1 minus x 2 is 0 and if you solve we will get 0 comma 0 as the solution. In the second case if we equate the right hand side to 0 we get 0 0 as 1 solution and 1 by 2 1 as another. So, there are two equilibrium points for the system. So, in general for a given dynamical system there can be several equilibrium points.

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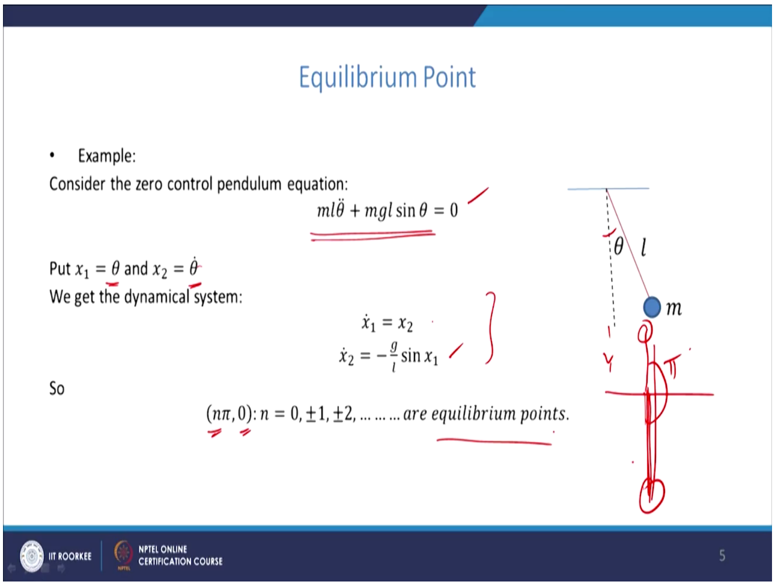

Now, for example, consider the equation of the pendulum simple pendulum with length l and mass m. So, we have already seen the dynamic equation of a pendulum can be written like this, where theta is measured from the downward vertical line to the pendulum itself this is the angle theta.

So, m l theta double dot plus mg l sin theta equal to 0 is the equation of motion of this pendulum. So, this can be converted into 2 first order differential equation, putting x 1 equal to theta and x 2 is theta dot the angle and angular velocity. These are the 2 variables x 1 and x 2, we will get x 1 dot is equal to x 2 and x 2 dot it means theta double dot is given by minus m g by ml in to sin theta.

So, from this second from the equation itself, we can get to the theta double dot that becomes the right hand side of the second equation. So, this is the system of equation, which falls in the definition of a dynamical system. Now, if we equate the right hand side to 0 we get x 2 is equal to 0 and sin x 1 is equal to 0 or x 1 is nothing, but n in to pi for different values of n.

So, these are the various positions which we obtain. For example, when n equal to 0; that means, theta x 1 is 0 and x 2 is 0 x 1 is theta x 2 is theta dot. So, it means that the pendulum is in this particular position. And, the velocity is also 0, it will not move from this position, when n equal to 1 it means it is pi theta is pi and velocity is 0. So, it means the pendulum will be moved pointing upward direction the angle is pi here.

And, for 2 again it comes to the same position and it stays at this position with 0 velocity etcetera. So, we can see that n pi is the angle and the 0 is the angular velocity. So, these positions are called the equilibrium positions.

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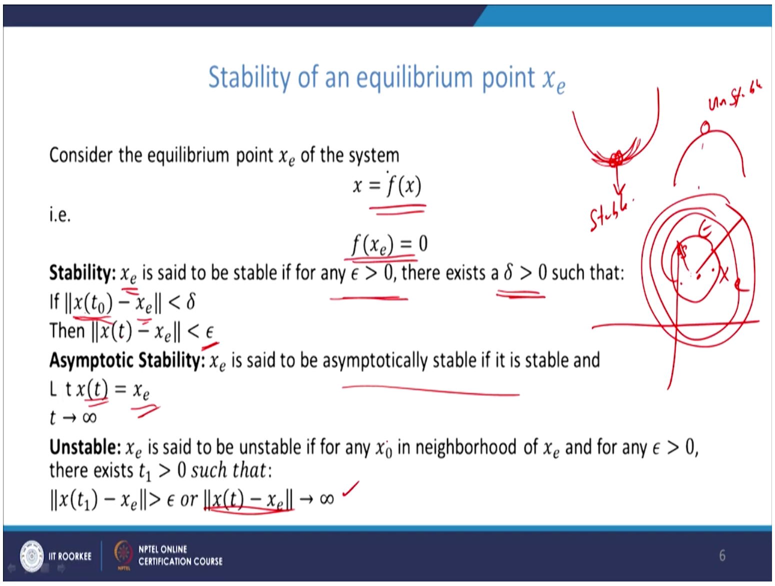

So, in general if we say, autonomous system that is time invariant then the point x e is said to be the equilibrium point if f of x e equal to 0. Now, the definition the equilibrium points can be stable or unstable or asymptotically stable. So, for example, if we consider a system this example, if you take if there is a vessel of this type and a ball is there, then if you disturb this it will oscillate for some time and then come back to this stop at this point.

So, here we can say that it is asymptotically stable if it stops, if it keep on moving from left to right in this way, then it is only stable. Similarly, if you take this point here it is the equilibrium point, it can stay at this place, but if you slightly disturb it will never come back to the same point.

So, we can say that it is unstable this situation is unstable. So, this we can say that it is stable. And, so, in general for a dynamical system for example, if we consider a system like this x 1 dot and x 2 dot system, it has 2 variables x 1 is a variable and x 2 is another variable. And, if it the equilibrium point is 0 comma 0 for this system.

Now, if we disturb this equilibrium point and take a initial condition as let us say x 1 0 and x 2 0 as initial condition and solve this equation, then we will get a for different value of t we will get x 1 of t and x 2 of t. Because, x 1 and x 2 or function of t we solved this equation as usual as we solved in dynamical system, then we get a expression for x 1 of t and x 2 of t which will give a curve in the x 1 x 2 axis.

So, this is called the phase space diagram. When the solve system of a dynamical system and then plot it in the variable in the space x 1 x 2 space it is called this phase space diagram. And, so, if this solution is always within a bounded region for all values of t, then it is called stable, but if the solution it comes as t tends to infinity, if it reaches the if the limiting point is 0 comma 0, then that 0.00 is called asymptotically stable.

And, unstable means for a given initial condition, if the solution x of t it goes away to infinity then it is called unstable situation. So, for a given dynamical system we can easily analyze, whether it is stable or unstable or asymptotically stable by using various techniques. So, in this lecture we will not analyze or show all the procedures for finding the stability of the system, but we will concentrate on only one type of procedure that is called the Lyapunov theory.

So, here the definition of the stability the stability is the equilibrium point x e is said to be stable, if for any epsilon there exist a delta such that, if we take a initial condition x at time t naught very close to x e, that is the distance between the x e point and the initial point is less than delta, then the solution x of t the for all t is at a epsilon distance from x e.

So, the meaning is if you take a equilibrium point x e, then if epsilon is given a radius, epsilon is given center is the equilibrium point and radius of the circle is epsilon, then we can find a delta. So, we can find another circle around this equilibrium point with delta radius, such that whenever we take a initial condition within this delta circle.

And solve the system of equation, we get the solution always inside the epsilon circle x of t minus x e it is modulus. The distance between that point x t and x e is less than epsilon. So, it is always inside this. It means that the solution will not go away from the equilibrium point for any time t, it is called asymptotically stable, if it is a stable that previous definition.

In addition to that if limit t tends to infinity of x t it approaches the point the limiting point is x e. And, it is unstable if it goes away from the equilibrium point x e, that is the distance between x t and x e keep on increasing as t increases and then it is called unstable.

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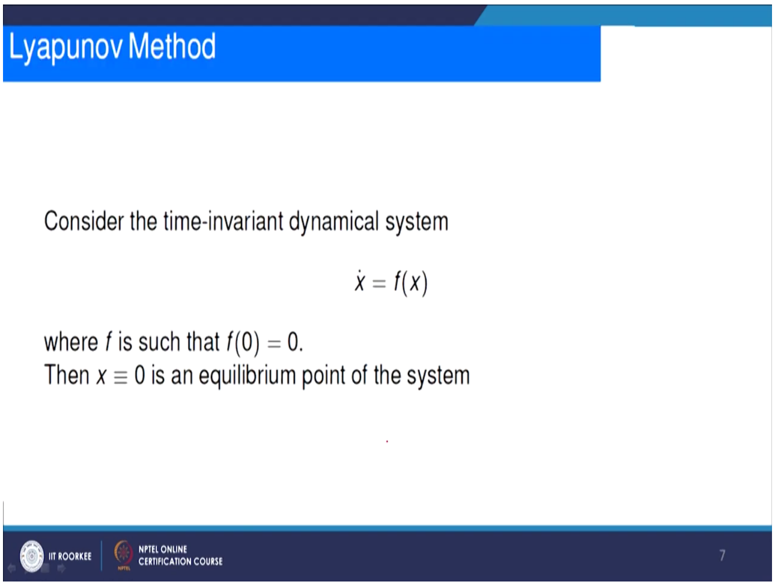

So, now we will come to the Lyapunov method, we consider the autonomous system the time invariant system. And, we assume that f of 0 equal to 0; that means, 0 x equal to 0 is the equilibrium point.

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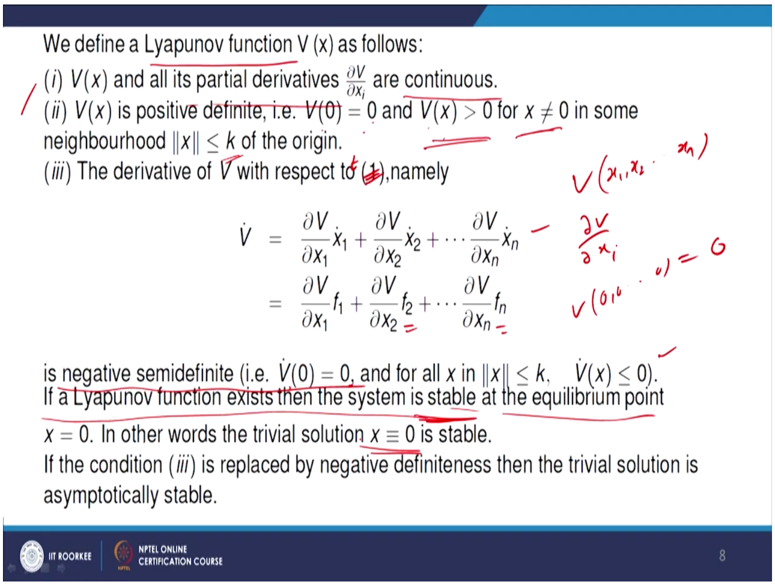

So, to analyze whether the equilibrium point is stable or unstable or asymptotically stable, we use this following theorem the Lyapunov function. What is a Lyapunov function? It is a function V is a function of the variables x 1, x 2, x n, where because we know that x is a vector in the dynamical system containing x 1 x 2 x n.

So, V is a function of x 1 x 2 x n and it is partial derivative, that is del V by del x i for all i they are all continuous functions. And, it satisfies the positive definite condition that is at 0 V at 0 0 etcetera 0, the value is 0, it is a real valued function. First thing is x is a vector, but V of x is a real value. So, V of 0 is 0 and V of any other point x 1 x 2 x n it is positive, whenever it is not the origin. So, V of x is 0 only at x equal to 0.

So, it this may be for all value of x or it may be for a neighborhood of the 0that is in a circle of radius k if this property is there then also we can take that function. Now, the derivative of

V with respect to t so, dv by dt using chain rule it can be written as del V by del x 1 into x 1 dot etcetera, this and x 1 dot is f 1 x 2 dot is f 2 etcetera.

Now, we see that there is a relation between the V function and the dynamical system. So, through this chain rule, because xi dot is fi is used here. So, the derivative dv by dt, that should be negative semi definite, negative semi definite means a positive definite means it is 0 at 0 and it is positive, for all other point x. Negative definite means the function is 0 at 0 and it should be strictly negative for all other points. Negative semi definite means it is it should be 0 at the 0.0 and it is less than or equal to 0 for all other points.

So, when x is not equal to 0 the function V dot x may be 0 also for some nonzero points. So, such a function is called a Lyapunov function. If you are able to find this type of function that function is called a Lyapunov function. So, the theorem says that if a Lyapunov function exists, then the state the equilibrium point x equal to 0 is stable this is given here.

In other words the trivial solution x equal to 0 the equilibrium point x equal to 0 is stable. The point x equal to 0 is called a trivial solution, because it directly satisfies the given equation. If, I put x equal to 0 it satisfy both sides the derivative and f of 0 is 0. So, x equal to 0 is the trivial solution for the system.

Now, if the condition two, if it is if the condition is replaced this condition is replaced by negative definite, instead of negative semi definite, if you write negative definite, it means dv by dt is strictly negative less than 0, whenever x is not equal to 0, then the system is asymptotically stable.

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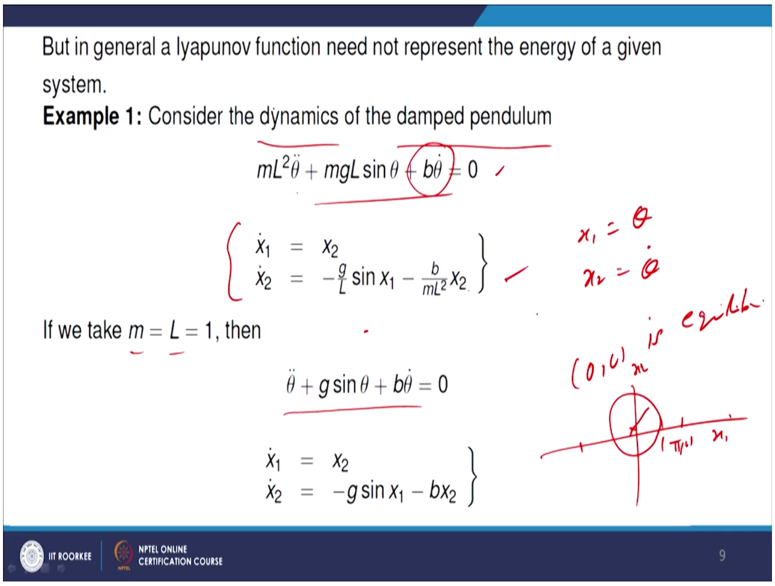

So, these statements is very useful in analyzing various problems. For example, here if you consider a dynamical system of this form m L square theta double dot plus mgL sin theta, plus b theta dot equal to 0. So, this is also the equation of a simple pendulum, but here the friction air friction etcetera the damping term is taken into account, it is written as b theta dot for some constant b. So, it is a rough mathematical model of a damped pendulum, simple pendulum.

So, this is converted into 2 first order equation X 1 equal to theta X 2 equal to theta dot. So, we can easily see that X 1 dot is X 2 X 2 dot is this. And, if we equate the right hand side to 0 we get 0 0 is the equilibrium point, because first we put X 2 equal to 0, then we get sin X 1 equal to 0. So, X 0 0 is a equilibrium point. So, now, if you put a particular value for m and L these are constant then we get a simple equation like this.

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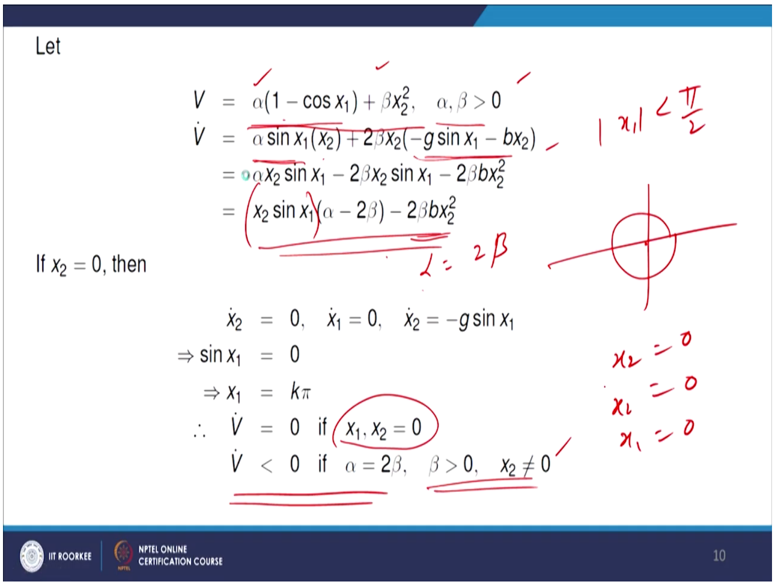

And, so, see here in addition to 0 0 there are several other equilibrium points, but we restrict ourselves to 0 0 only, that is 0 0 as we have already seen we can have pi comma 0 2 pi comma 0 all of them are equilibrium point, but in a neighborhood, let us say a small radius less than, pi if you take in that neighborhood we can get a Lyapunov function like this.

So, if you consider because according to the previous theorem, if you are able to find a Lyapunov function satisfying the properties, it is stable asymptotically stable etcetera. So, let us try to find a Lyapunov function. Let V be a function of X 1 X 2 given by this expression alpha times 1 minus cos X 1 plus beta times X 2 square alpha beta or positive numbers.

So, it is a positive definite function provided X 1 is small value ok. It should not be equal to this thing pi here we have seen that. So, if you take X 1 less than pi by 2. So, if X 1 is pi by 2;

obviously, 1 minus 1 is 0 we do not want such situation, it should be always positive, because V should be a positive definite function.

So, both the terms should be a positive value. So, if x 1 modulus of x 1 is less than pi by 2 if you take any value in that neighborhood of 0 0. A radius of less then pi by 2, we get always the value of V to be strictly greater than 0. And, if you differentiate with respect to t we get this second term alpha times cos X we get to the derivative is sin X 1 into d X 1 by dt that is X 2.

Here, 2 times beta X 2 in to X 2 dot X 2 dot is given by this. So, now, if you collect the terms finally, we get this expression X 2 into sin X 1 of these quantity multiplied by alpha times, this is a different minus 2 beta times b X 2 square. So, now, if you select this term alpha because alpha and beta we can easily select to be 2 positive values.

So, if alpha equal to 2 beta this term is becoming 0 and the term second term is always negative value ok. So, we get V dot is less than or equal to 0. Now, if X 2 is 0 X 2 dot is also 0 if x 2 equal to 0 x 2 dot will be equal to 0. So, that will automatically imply that sin x 1 is 0 that will imply x 1 is also equal to 0.

So, V dot is strictly less than 0, we will get V dot is equal to 0 only when X 1 and X 2 both are equal to 0 and V dot is strictly less than 0, if we take alpha equal to 2 beta and beta greater than 0 and X 2 is not equal to 0. So, it implies that the system is asymptotically stable, the damped pendulum is asymptotically stable. In other words the oscillation will finally, stop to the 0.0 0 as t becomes larger and larger. So, that is a meaning of physical meaning of this expression.

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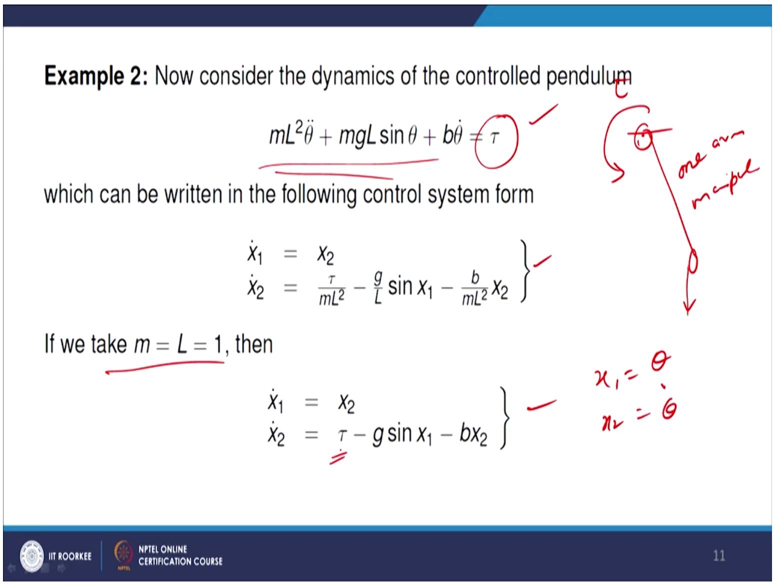

So, similarly we consider the controlled damped pendulum. So, we have a pendulum the previous example this tau is not there it was 0, but here we have the torque applied at this joint. So, with this torque as well as with the force of gravity the pendulum is moving.

So, we can consider this as a single arm robot manipulator one arm manipulator there is only one link here one joint. So, this is the dynamic equation of the one arm manipulator or we can also call it as controlled pendulum. Now, converting it into the two first order differential equation X 1 dot is X 2 X 2 dot is as usual x 1 means theta x 2 means theta dot, we get this expression. And for this constant value we get this.

Now, the question is how to find this torque, because torque is not given to us and there is no equilibrium point also here, because we do not know, what is tau? We want to find a control for a specific problem. So, what is that specific problem? If, tau is not there then automatically the equilibrium point is 0 comma 0 or n pi comma 0 etcetera, but if tau is there we can fix a target.

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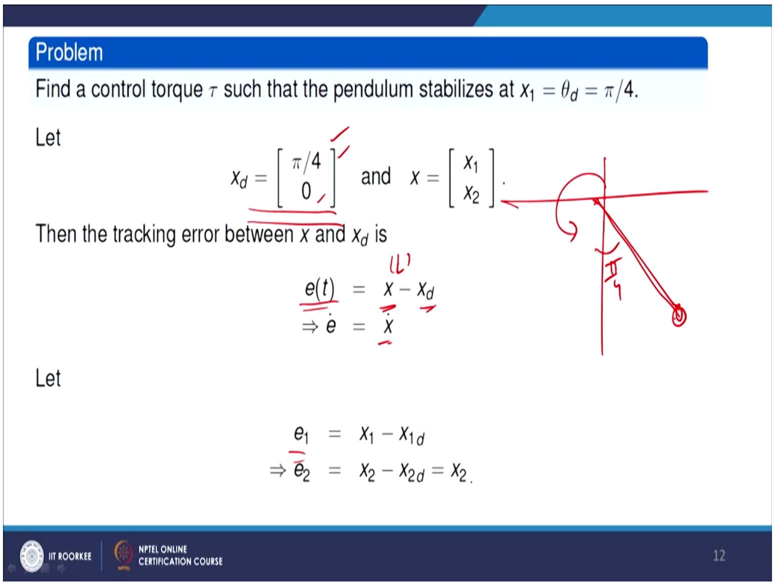

For example, let us assume that we want to stop the pendulum in this particular position, that is the pendulum angle should be 45 degrees pi by 4 and the velocity should be 0. So, we want to stop the pendulum or the single arm manipulator at this particular position. So, it becomes the equilibrium point in that case, because what is equilibrium point a point where the system stops with 0 velocity it never moves afterwards.

So, the pi by 4 is the angle and then 0 is the velocity means we want to stop the pendulum at this position. For that what should be the torque applied on the pendulum to fix it constantly at this position. So, now we can calculate the error if X is X of t is the desired if the current position, when the pendulum is moving and X desired is this position we want to stop it here. Then, the error at any time t is X of t minus X d desire this vector.

So, now if you differentiate with respect to t X of t is X dot yes. Now, we can get because there are 2 components for this vector X, we call the first component as X 1 minus X 1 desired is this and second component is X 2 minus X 2 desired is 0. So, it is simply X 2 itself.

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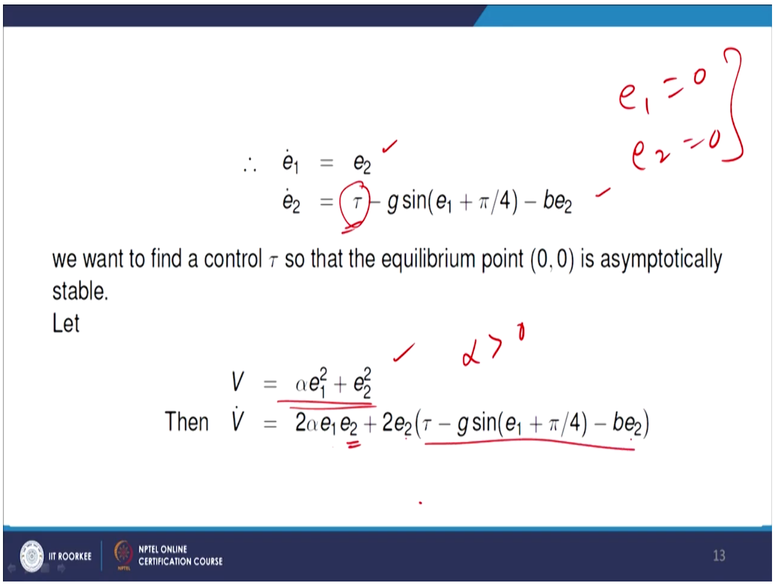

So, we get the equation e 1 dot equal to e 2 from the previous thing and e 2 dot is given by this 1. Now, how to find the torque we want to get e 1 equal to 0 and e 2 equal to 0 that is our aim. So; that means, 0 0 is the equilibrium point for this system we want to find. So, what is a suitable control or what is a suitable torque to be applied? For that let us assume a Lyapunov function, we are selecting a Lyapunov function V equal to alpha e 1 square plus e 2 square.

So, here we note that Lyapunov function there is no particular procedure for finding a Lyapunov function. Mostly, it is a trial and error method or in very few situations we have a standard procedure, which can be studied in a dynamical system course. So, here we are trying with a function like this, alpha e 1 square plus e 2 square because if alpha is positive it is a positive definite function.

Always it is 0 only at e 1 and e 2 0 and it is always positive if alpha is greater than 0. Now, the derivative is given by 2 alpha e 1 and e 1 dot is e 2 and it is derivative is 2 e 2 e 2 dot is given by this expression.

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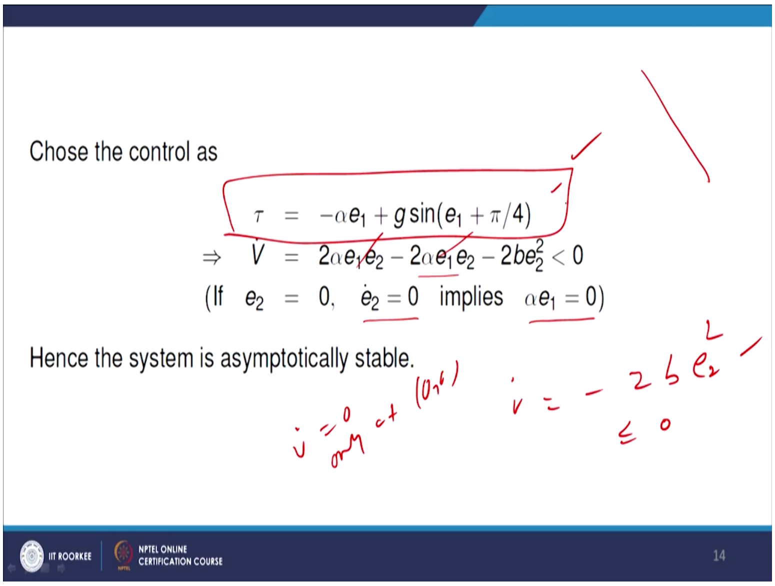

So, if you select this tau in such a way that it cancels out all the non-linear terms. Here, we can see that if tau can be selected as this term plus this term, it will cancel out some at least this term g times this one and b times theta 2. So, here it is selected like this tau is selected as minus alpha times e 1 plus g times sin e 1 plus pi by 4 is not it.

So, if you substitute it here we will get V dot is equal to the first term is already 2 alpha e 1 e 2, that is here and when we substitute this tau in this place the g etcetera will get cancelled. So, we get 2 e into minus b into e 2. So, we get minus 2 b e 2 square and with the alpha term we will get this one. So, this 2 gets canceled here, we get ultimately V dot equal to minus 2 b e 2 square which is less than or equal to 0. Now, we can easily note that if e 2 is equal to 0 it automatically implies that e 2 dot is 0 and from the equation we get e 1 is also equal to 0.

So, V dot will be 0 only at 0 comma 0 and it is always negative if e 2 is nonzero. So, if e 2 is nonzero automatically e 1 is also non zero, because of the proof as given here. So, this shows that the selection of the torque using this formula will make the system asymptotically stable or in other words it will make the pendulum or the 1 a manipulator to stop at a desired position as given in the goal here, that is we want to stop it at this 45 degree with 0 velocity. So, that can be achieved by giving this much of torque to the pendulum.

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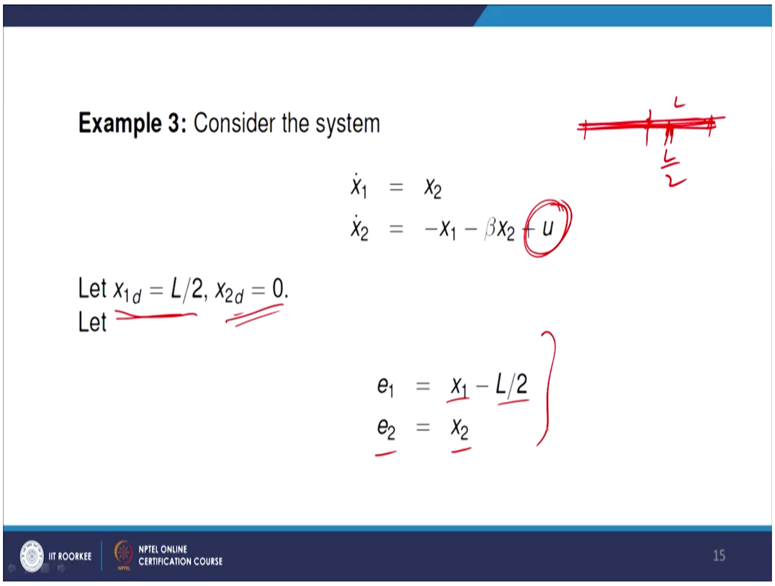

So, the third example is a simple one again if you consider a equation like this X 1 dot equal to X 2 X 2 dot equal to this represent a harmonic oscillator, if a mass is oscillating from left to right motion horizontal motion under a control u. And, if you are interested in the desired position, we want if the length is L here in both sides, this is L the length we want to stop that mass at L by 2 and the velocity should be 0, that is the meaning here the position is L by 2 and velocity is 0.

So, the mass is moving left and right it is a harmonic oscillator and under this control u we want to stop it here. So, how much of control should be applied on this mass. So, we can write the equation to be the error is actual value minus desired value. So, e 2 the error 2 is actual value X 2 desired value 0.

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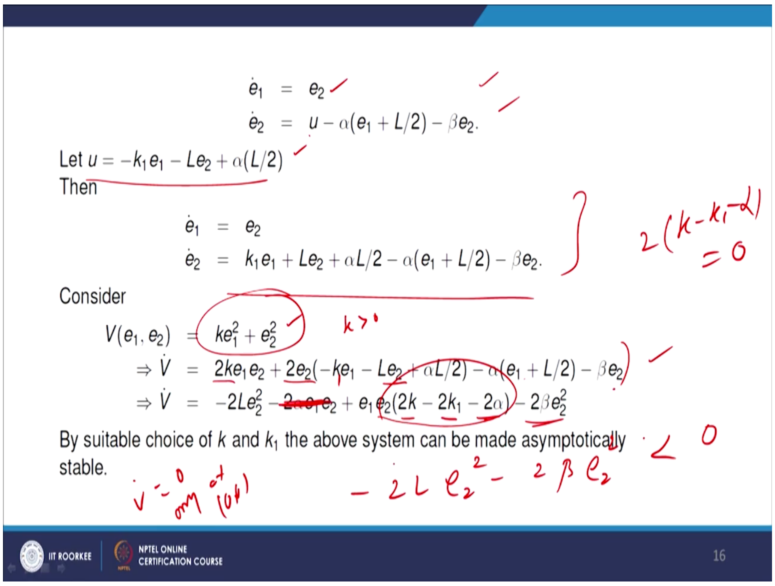

So, this is the error and if you differentiate the error term, we get the differential equation in terms of the error like this. And, if you select this control u in this particular manner as k 1 e 1 minus L e 2 plus alpha times L by 2. So, this is again a trial and error method type to see whether this control is working or not. So, if you substitute this control in this equation we get e 1 dot is equal to e 2 e 2 dot is given by this. And, now this is a dynamical system there is no the control is already substituted. So, it becomes only in terms of e 1 and e 2.

So, we see that this Lyapunov function where k is a variable which we have to select now, k is positive that much only we can say. If V e 1 e 2 is k e 1 square plus e 2 square it is a positive definite function. It is 0 only at e 1 and e 2 0 and it is always positive, the derivative V dot is given by this expression.

So, we can so, here there is something wrong here this term is not there. So, when we take V dot the derivative we get this expression. So, we get 2 L e 2 square from this expression L e 2 square. So, 2 L e 2 square is this and e 1 e 2 we get 2 k as 1 coefficient and minus 2 k this is k 1, because u is k 1 times.

So, minus 2 k 1 is this and minus 2 alpha is this 2 e 1 e 2 e 1 alpha this term is there. And, then product 2 e 2 here e 2 to beta times e 2 this bracket is up to this. So, now we see that if you select this 2 times k minus k 1 minus alpha equal to 0. If you make this particular term to be 0 this whole thing goes we get only minus 2 L e 2 square and here minus 2 beta e 2 square. So, it is negative definite we can see that if you put e 2 equal to 0 e 1 is automatically equal to 0.

So, this is strictly less than 0 for all e 1 e 2 not 0 0 and it is 0 only at 0 comma 0. So, V dot is 0 only at 0 comma 0 and it is strictly less than 0 for all other e 1 e 2. So, it is a negative definite this implies that the system is asymptotically stable. So, it means that we can stop this mass at this point using the control u as given here. So, this lecture shows that how we can control certain dynamical systems for a particular goal.

If you fix a goal to the Lyapunov theory can be used for controlling a robot manipulator to perform a particular work. So, that we will see in the coming lectures.

Thank you.