Robotics and Control: Theory and Practice

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Lecture -14
Manipulator Dynamics Multiple Degree of Freedom

This is the continuation of the previous lecture. In this lecture we will see how to write the dynamic equation of n degree of freedom manipulator.

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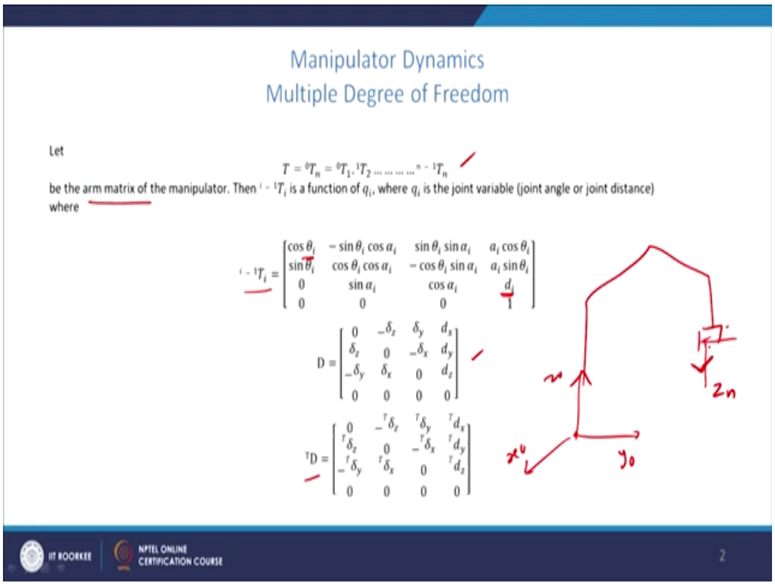

So, the last lecture we have seen how to write the dynamic equation of a two degree of freedom manipulator with uniformly distributed mass. So, here we consider a general n degree of freedom manipulator, whose arm matrix is written as 0 T n which is the product of

0 T 1, 1 T 2, etcetera. So, in general this i minus 1 T i, it represent the coordinate frame of the i th joint with respect to the i minus 1th coordinate freedom so.

So, if you consider a general robot manipulator with n degree of freedom, so, the so, if you consider the 0th frame as x 0, y 0, z 0 that is the base coordinate frame and xn, yn, zn is the end effector coordinate frame then, 0 Tn it represent the n the tool frame, end effector frame with respect to the base frame given by this and then it is already known from the dh algorithm that i minus 1 T i can be written like this.

So, if the i th joint is a revolute joint then theta i it represent the joint angle it is a variable in case i th joint is a prismatic joint then theta is a constant and d i is a variable in that case. Now, the differential transformation; d represent a matrix which contains dx, dy, dz and del x, del y, del z. Which are the translational and rotational velocities of a particular joint with respect to the base frame?

And, here d super fix T it represent the translational and rotational velocity with respect to its own the current frame the T frame itself. So, the Jacobean lecture we have seen the relation between d and dt through the arm matrix.

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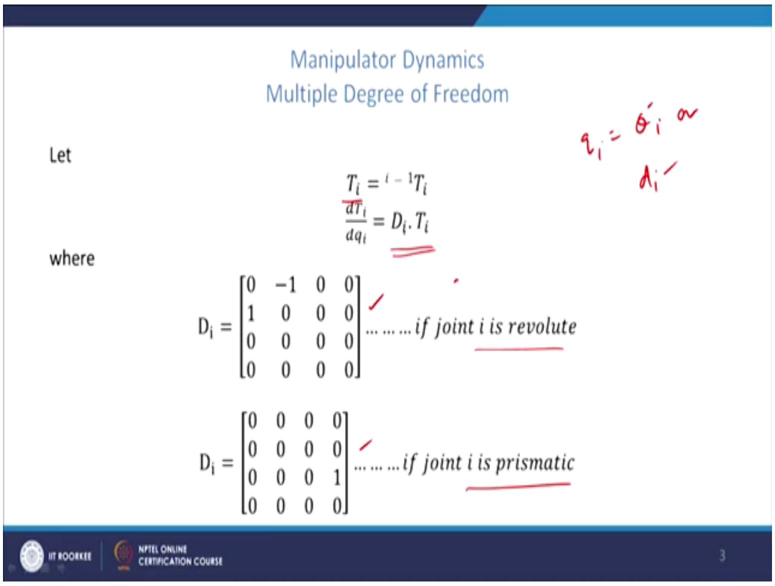

So, now if you denote i minus 1 T i as T suffix i then the derivative with respect to the variable qi because, the i th frame is a function of qi.

Where qi is either theta i or di depending on whether the i th frame is a revolute joint or prismatic joint. So, the derivative with respect to the i th variable qi the partial derivative is given by D suffix i into Ti. Where D suffix i is this matrix if it is a revolute joint or this matrix if it is a prismatic joint so, the partial derivative with respect to the variable is given by di multiplied by the matrix D suffix i.

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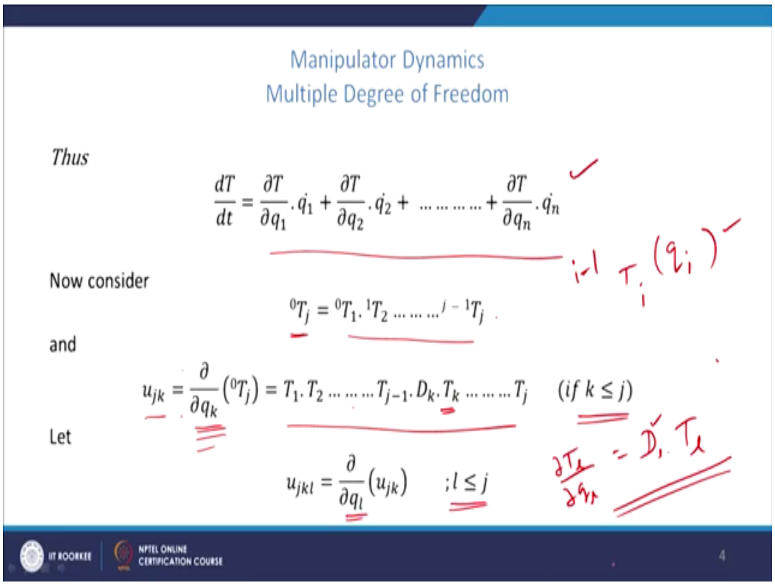

So, the total derivative dT by d small t is given by the standard chain rule; del T by del q 1 into q 1 dot plus del T by del q 2 into d q 2 by dt etcetera, is the total derivative of the matrix T where del T by del q a is given in the previous slide. Now, if you consider 0 T j the j th coordinate frame with respect to the base frame which is given by this product and if you denote u j k means, the partial derivative of 0 T j matrix with respect to the variable q k. Because, 0 t j it contains 0 T 1, 1 T 2 j minus 1 T j.

So, when we differentiate 0 T j with respect to a variable q k; where k is less than or equal to j we get because, that derivative is effective only with the matrix T k and the precious slide we have seen that the partial derivative of T k with respect to qk is given by D k multiplied by T k so, by substituting that we the get the derivative to be like this in this way.

Now, if we further differentiate ujk with respect to ql, where l is say less than or equal to j again then, we can denote it by ujkl the notation ujkl means, we differentiate u ujk with respect to ql that is. So, it is very clear that once again if you differentiate with this term del by del ql then wherever T suffix l appears that derivative will be effective in that place. Whether it is after T k or before T k wherever T l appears the derivative should be effective on that T l the derivative will be D suffix l into s o, this is del T l by del q suffix l.

So, depending on whether it is revolute or prismatic this matrix D can be selected and substituted.

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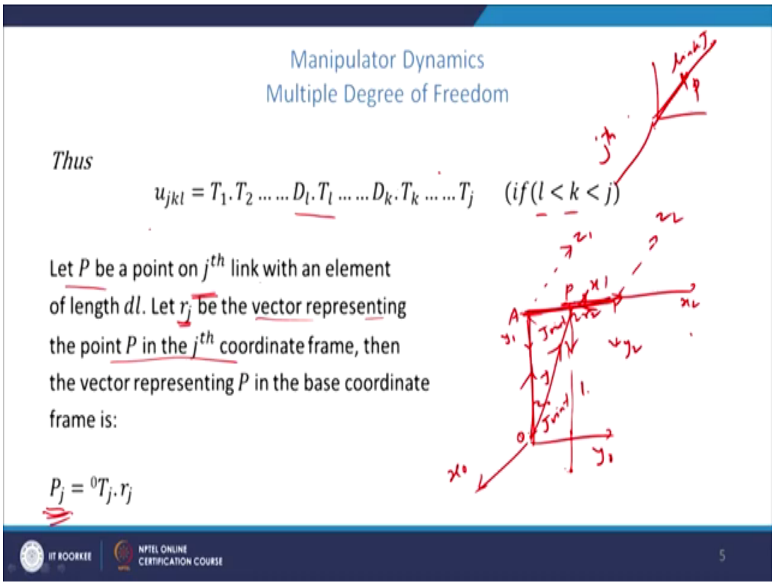

So, so for example, if you have ujkl it is if j l is less than k this D l T l will appear before D k T k or if l is greater than k then it will appear after this T k. So, in all the cases we can write ujkl notation to be the partial derivative. Now, let us assume that P be a point on the j th link with respect to j th link. So, if you consider the so, for example, let us say this is the first link this is the second link and the coordinates are x 0, y 0, z 0 and if you assume this is the rotation with respect to z 1, the revolute joint and we can fix the x 1 y 1.

So, if we consider this is the point o this is the point a and so, we have two coordinate frames x 0, y 0, z 0 and here we can fix x 1 y 1 z 1. Now, if I consider this is the first joint this is joint one and this is joint two. So, if I write the joint one with respect to the first frame.

Similarly, j th joint with respect to j th frame it means, the link is for example, here this is the first link and the first coordinate frame is z 1, x 1, y 1, etcetera that can be fixed. So, with respect to x 1 y 1 z 1 the link one is considered. Similarly, if you have another joint for example, here z 2 and x 2, y 2, etcetera can be fixed. So, with respect to x 2, y 2, z 2 the link 2 is returned. So, we take a point P general point P on the link 2 with respect to x 2, y 2, z 2.

Similarly, here according to this let P be a point on the j th link with respect to the j th coordinate frame. So, that is the meaning here. So, now, let rj denote the vector representing the point P in the j th coordinate frame. So, if you take a particular point P on the second frame and write the vector with respect to x 2, y 2, z 2 it is this vector.

This is the r 2 vector we can say that with respect to the x 2, y 2, z 2 frame for a fixed for a particular point on the second frame. Similarly, if you take a point P on the j th link with respect to j th frame it is coordinate vector we call it as r suffix j here. Now, the same point the point P if you write with respect to the base frame that is called P j the point with respect to base frame. So, for if the if r j is the coordinate a vector with respect to the jth coordinate frame then if we multiply in the left side by 0 T j it represent the same point with respect to the base coordinate frame. So, P j represent the point P with respect to the base frame.

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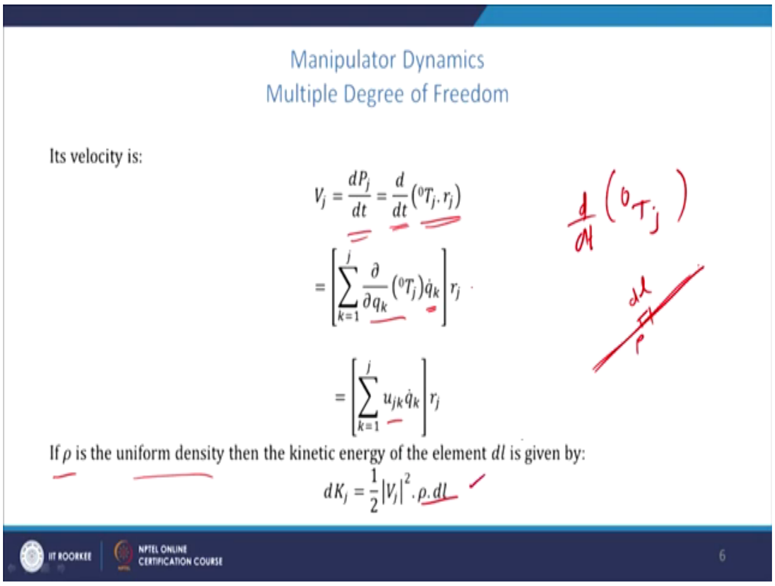

Now, the velocity of the same point means, we have to differentiate the vector P j with respect to time. So, d by dt of 0 T j into r j. So, we always keep it mind that r j is a fixed vector quantity and 0 T j is a function of q 1, q 2, up to q j here.

So, we have already seen that the derivative d by d t of 0 T j, how to write it is through the summation? In the previous slide we have seen already the derivative del by del q k of 0 T j for different value of k. So, that is substituted here because it is a total derivative in terms of the partial derivative we will write the summation k equal to 1 to j of del by del q k of this into the d q k by d t. So, it is the same formula as given here the total derivative this thing; instead of n we have j here that is the only difference. So, multiplied by r j so, by using the notation u j k we can write this can be replaced by u j k. Now, the point P one point P is considered, if the link is with uniform mass distribution then at a point P if we take a length dl a small element of length dl is taken.

So, the if rho is the uniform density then the kinetic energy of a element of length dl is given by half into mass is rho into dl into velocity square, the velocity of that element is V and its square modulus of velocity can be taken. So, the kinetic energy of the small element of mass can be calculated by this formula.

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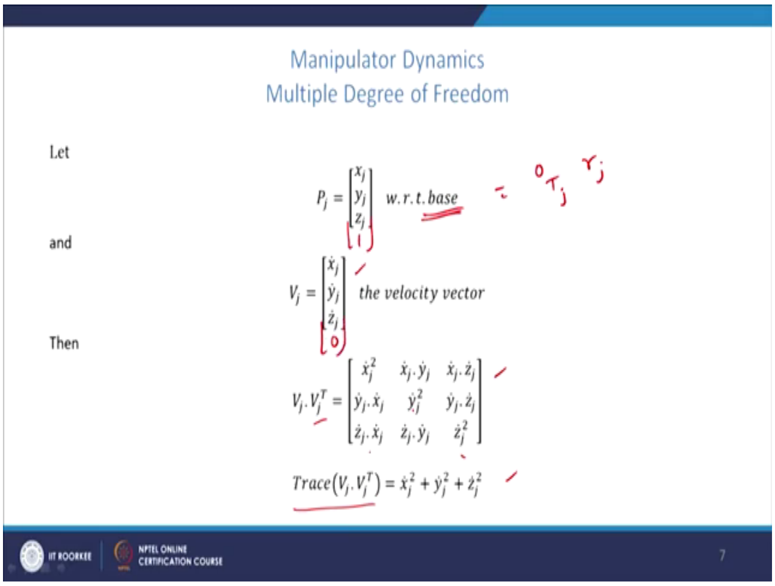

So, how to calculate that? We take the point P to be x j y j z j with respect to the base and V j is given by x its derivative is x j dot y j dot z j dot. And, V j is a column vector V j transpose is rho vector the same thing.

If you multiply V j and V j transpose we get x i x j dot square, x j y j dot and x j dot z j dot, etcetera. So, directly multiplying V j and V j transpose is this. Now the trace of this matrix is the sum of the diagonal elements that is x j dot square plus y j dot square z j dot square. That is the velocity modulus of the velocity square. So, in the place of modulus of velocity square we replace by the it is trace of V j V j transpose.

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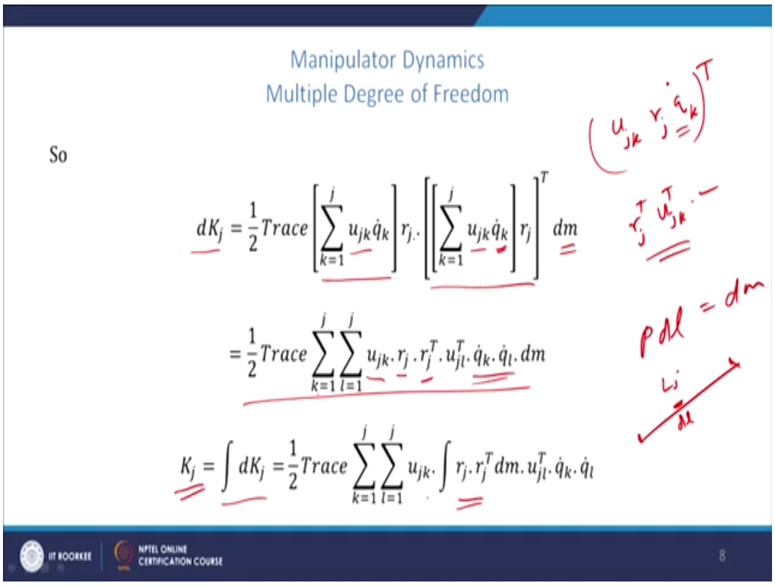

So, the kinetic energy of the small element d k is half into trace of this is V j into this whole thing is V j element and then its transpose V j transpose. Now, if you multiply two summations here it varies from 1 to j so, there are double summation so, we replace one variable we call it as k and another variable is called as L here.

So, k varies from 1 to j and L varies from 1 to j and if we take transpose of this matrix because r is a vector and here u is a matrix q is a number, it is a joint variable it q dot is the velocity so, it is a number here so, u when we take the transpose of the product of two matrices, here it is u j k and r j multiplied by this q k dot is there if you take transpose of this one and because, this is simply a real value we take the transpose of these two it is r j transpose u j k transpose.

So, by using that transpose property we can write this product as double summation k equal to 1 to j and L equal to 1 to j of u j k is this and here r j is as it is then we get rj transpose, as we are using this one then multiplied by u j l transpose the transpose of this one, multiplied by these two real values q k dot q l dot; multiplied by this small mass here. Because rho into dl it is the small element mass m.

Now, we can calculate the total kinetic energy of the jth link it is the integrally, the jth link has length let us say L j then we have to integrate from 0 to L j because, we have taken small element mass length dl and if we integrate from the total length from 0 to L we get the total kinetic energy.

So, we get the same formula can be integrated the integration is effective only on this r j and r j transpose because, r j is the vector in the jth coordinate frame; which is a constant vector similarly, r j transpose is a constant vector we multiply this two and then integrate from 0 to the length of the particular link.

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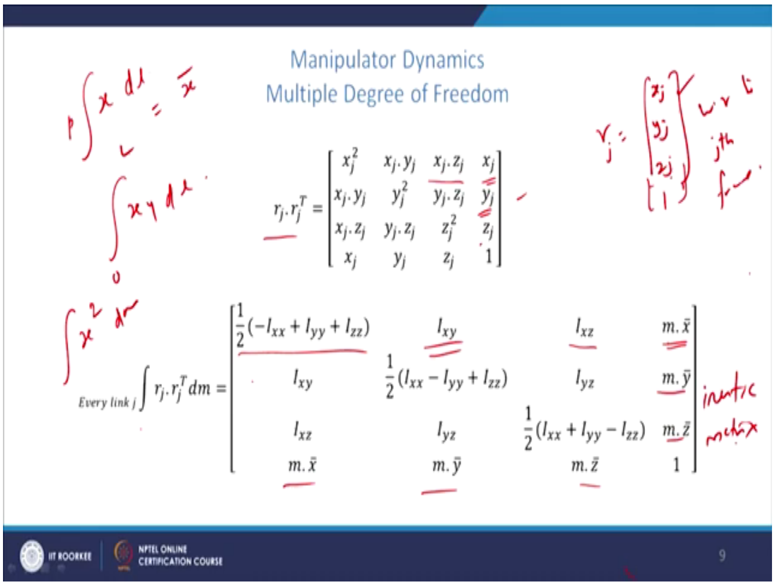

So, now, we can write r j r j transpose is given by this expression x j square if I write the vector r j to be x j y j z j then with respect to the jth frame. Now, if you multiply r j r j transpose we get this expression and if you integrate from 0 to the length of the the particular link we get this expression. So, for example, if we integrate x into y into the length, dl 0 to the total length for example, that is given by I x y the inertia with respect to the x y plane.

Similarly, if you integrate this it the is inertia with respect to x z and the because, normally in homogeneous transformation a vector is represented by 4 values, the first 3 represent the coordinate and the last one is a dummy variable 1, ok. So, that the first lecture we have seen how to write the coordinates. So, r j is so, here sorry, we have to see that everywhere we should write one here. So, that it is a homogeneous coordinate vector, ok.

If we differentiate 1 we will get 0, etcetera. So, we get here 0, 0, 0, etcetera. Here also the same thing r is represented by x y z and 1 here in the end yeah. So, when we do the integration we get inertia with respect to x y plane and here inertia with respect to x z plane, when we integrate this one the last one with respect to 0 to L dl this will give the mass the total mass of the link multiplied by d because, we know that integral x into dm multiplied by the uniform density this gives the value of the x x coordinate of the centre of mass of that particular link multiplied by m here.

So, we get this one integral of x into dm it represent mass into the centre of centre of mass x coordinate. Similarly, the integration of this will give mass into the centre of mass of the y coordinate of the centre of mass and integral L is the z coordinate of the centre of mass multiplied by m. So, the same thing is repeated here.

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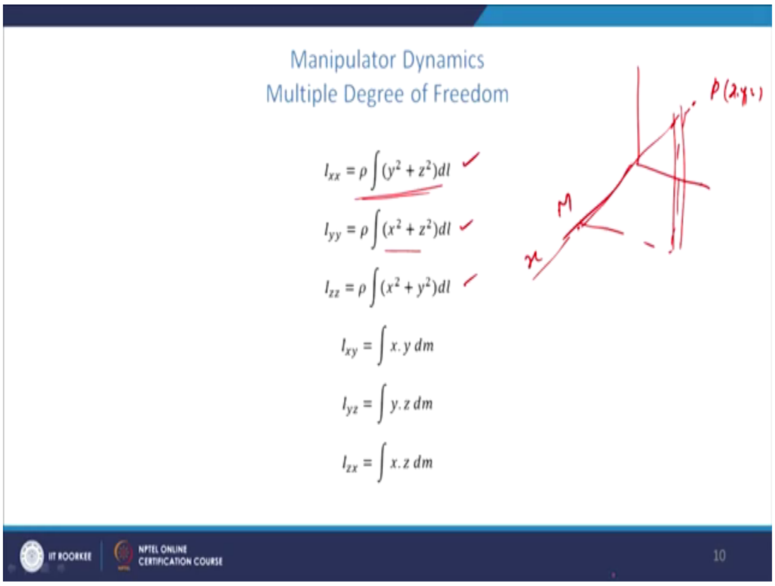

And, here also we can easily see the next slide we can see that i x x is nothing, but the inertia with respect to the X axis.

If you take any point P if you draw up a perpendicular so, this perpendicular to the X axis so, this is nothing, but y square plus z square, if the point P has x y z as the coordinate the length of this point; the if if we draw up a perpendicular to the X axis the length of the m is nothing, but y square plus z square length square. So, this gives the inertia with respect to the x coordinate and x square plus z square is the square of the distance from P to the Y axis. So, this gives the inertia with respect to Y axis and this is inertia with respect to the Z axis.

So, we can easily see from this one the calculation the integral of only x square if you take, into dm that is given by this expression if you substitute for i x x i y y i z z from this quantity divided by 2 will give only the first integral.

So, simply the integral of r j r j transpose is given by this matrix which is called the inertia matrix. This is a fixed one because, r j and r j transpose they are not going to change throughout the motion of the robot manipulator. So, once a robot manipulator is given for each joint each link j we can calculate the inertia matrix and it is a fixed one.

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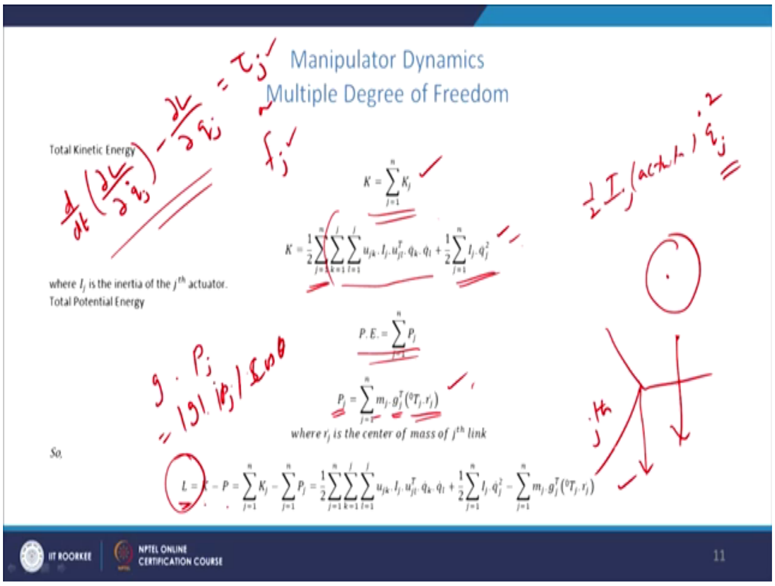

Now, the total kinetic energy is summation of all kinetic energies for the link 1 to n. So, we can simply write that the kinetic energy for the jth link is this and we are summing up all the so, it is j equal to 1 to n for all the links and the next two summation it gives the formula which we have given the kinetic energy of the jth link.

And this expression it is nothing, but half into the inertia of the jth actuator multiplied by q j dot square the velocity square so, if the actuator jth actuator is a revolute one if it is revolving with velocity q j dot then q j dot square multiplied by the inertia of that actuator it gives the kinetic energy of that particular actuator. So, now, if you add the kinetic energy of all the actuators we get this formula. So, total kinetic energy these are the link kinetic energy and then this is link kinetic energy and these are the actuator kinetic energy. Similarly, we take potential energy potential energy of each link can be calculated mass into gravity into the height.

So, if you take a particular point on the on the link this point so, the height to be calculated with respect to the base actually the we have to consider so, the base is this one and the jth link we let us say this one if P is the point we have to write the gravity vector will be pointing downwards and the height is from the point P to the projection of the point in the x y plane. So, we have to calculate so, the gravity with respect to the frame we have to write. So, the potential energy is given by mass into the gravity vector with respect to the jth frame.

Because, we are considering the potential energy of the jth frame so, if the jth frame is in different orientation let us say because, throughout the robot motion the frames are moving in different orientation, but the gravity is always pointing downwards. So, when we write the gravity vector in the jth frame then it will have 3 components, ok. It is a vector represented in the jth component it is a constant vector, so, mass into gravity into height the height is given by 0 T j into r j. So, the gravity vector dot product the P vector P j vector with respect to the base frame so, that gives the height, ok. So, this will give modulus of the gravity vector and this is modulus of P j into this thing cos of the angle between these two vectors.

So, it means it is the projection of the gravity vector on the perpendicular vector. So, it gives mass into gravity into height type of thing so, this gives the potential energy formula. Now, if you add if you write the Lagrangian that is kinetic energy minus potential energy of the entire system so, we can write we collect the kinetic energy, collect the potential energy and then take k minus P that is the L Lagrangian.

So, after this we can simply write the equation of motion that is d by d t of del l by del q j dot minus del l by del q j equal to tau j. If it is prismatic or f j if it is if it is revolute it is torque, if it is prismatic it is the force. So, the equation of motion can be written from the Lagrangian formula here after doing all the derivatives we can obtain the dynamic equation of this problem which is very lengthy process.

As it is seen here, there are too many summations and too many variables are there if it is a n link manipulator so, it is a really a very lengthy procedure to get the dynamic equation of the n arm robot manipulator. So, this lecture gives only the procedure to find the Lagrangian of the manipulator and the remaining can be calculated using this formula, ok.

Thank you.