Robotics and Control: Theory and Practice
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Lecture – 12
Dynamics of Manipulator

Welcome to this lecture on Dynamics of Manipulators. In this lecture, we have we shall see how to formulate the dynamics equations of various robot manipulators. In the previous lectures, we have seen how to formulate the kinematics equation using the DH procedure. So, kinematics equation is basically on the relation between the position and orientation of the end effector of a robot manipulator and the joint variables.

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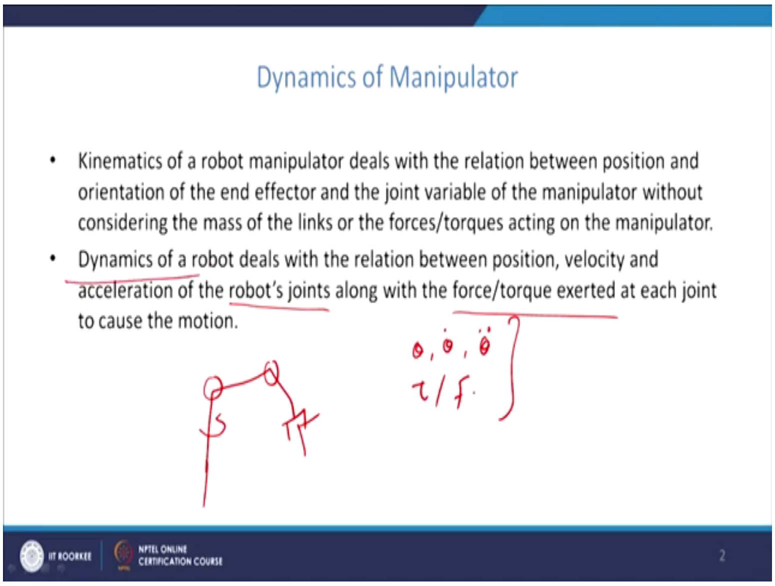

So, it will not take into account the mass or the force torque of exerted on the robot manipulators for causing a particular motion. So, for example, if you have the robot manipulator so, the position and orientation of the robot manipulator and the relation between various joint angles gives the kinematics equation. The dynamic equation is a relation between the position, velocity, acceleration of the robot joints and the force and torque applied on the actuators of the robot manipulators to cause a motion of the robot manipulator.

So, here when we take force torque etcetera naturally, we take into account the mass of various links and the acceleration etcetera of the robot manipulator along with the position orientation velocity and acceleration of the robot joint. So, dynamic equation is basically a relation between the theta, theta dot theta double dot where theta is the joint angles of the robot manipulator; the vector representing the joint angles and the torque or the force acting at every joint of the robot manipulator. So, this relation is the dynamics of the robot.

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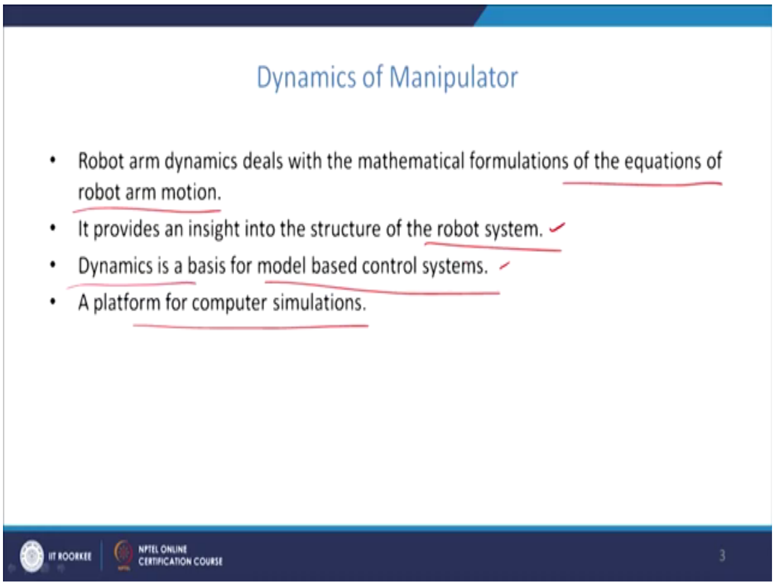

So, here so, the dynamics is nothing, but the equation of motion of the manipulator robot manipulator and the it provides the insight into the structure of the robot system. So, for example, if a robot is having n joints then there will be a system of n second order differential equation and the structure of the equation varies whether a particular joint is a revolute joint or prismatic joint etcetera.

So, the equation of motion or the dynamic equation indicates the structure of the robotic system and the dynamic equation is very useful in constructing the control of a robot manipulator for obtaining a particular motion of the manipulator. So, the desired motion of the robot manipulator can be obtained by constructing a suitable control based on the equation of motion of the manipulator.

And using that, we can simulate the robot motion using these mathematical formulation and the control of the manipulator even before performing the actual experiment. So, this will save a lot of costs of performing an experiment and first reviewing the controller design. So, the simulation will be very useful in improving the controller design for a desired motion of the robot manipulator.

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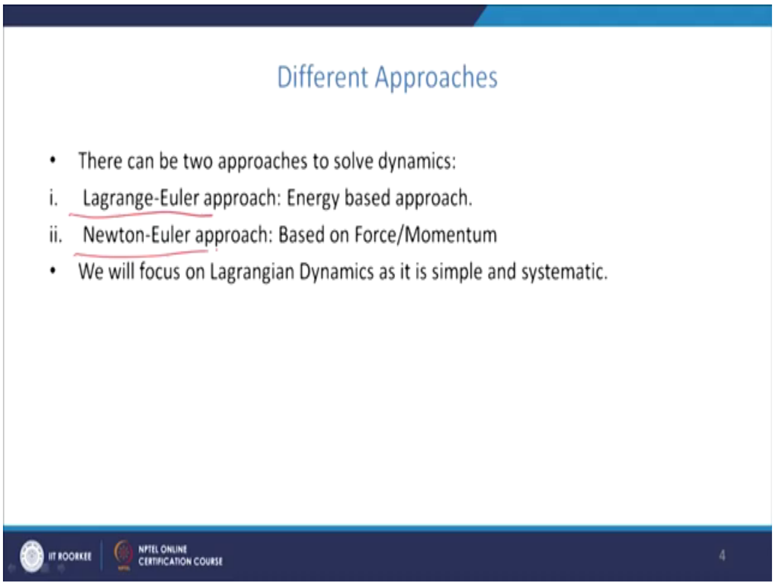

So, so to formulate the dynamic equation there are two approaches one is the Euler Lagrange approach which is energy based approach and another is the Newton Euler approach and it is based on the force and momentum of the manipulator. So, here in this lecture, we will concentrate on the Lagrange method; Euler Lagrange method.

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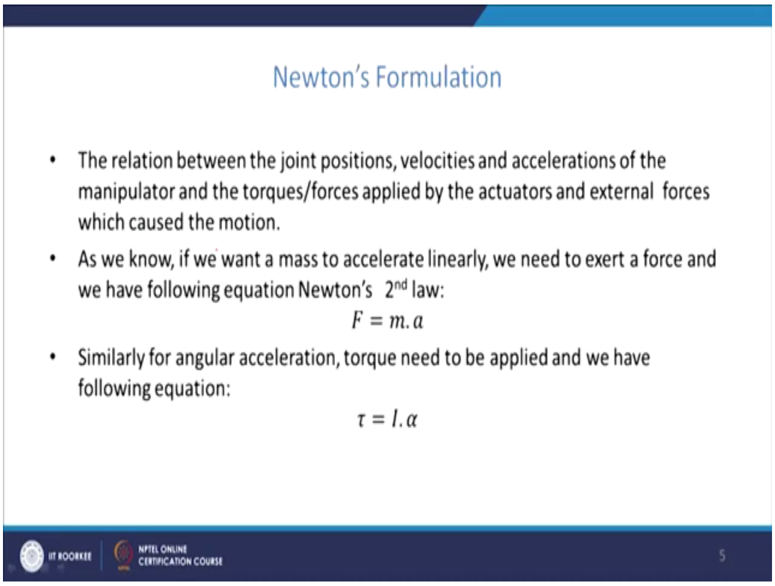

So, the Newton method is basically the second law of motion Newton's law of motion which is the force equal to mass into acceleration. If it is a linear motion if the joint is a prismatic joint, we can apply the linear motion equation and the if a motion is a revolute one then the torque causing that revolving motion is equal to the inertia into the angular acceleration. So, basing based on this two formulations, one can construct the equation of motion or the dynamic equation of a robot manipulator.

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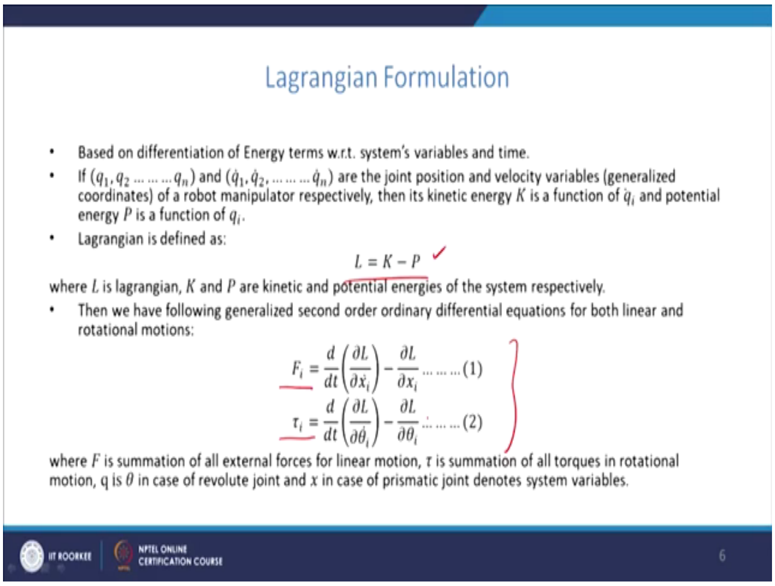

But another procedure is the Euler Lagrange method it is based on the Lagrangian L which is the difference between the kinetic energy and the potential energy of a body in motion. Any rigid body can be analyzed using this procedure in our case it is the robot manipulator.

So, while constructing the dynamic equation, we have to construct the kinetic energy and potential energy and find the Lagrangian by this formula, then the equation of motion is given by the system of equation given in one and two. So, if a joint is a prismatic joint, then force is applied at that particular joint that is F i. And if a joint is a revolute one, the torque is applied at that joint.

So, the equation of motion is based on this d by d t of del L by del x i dot minus del L by del x i. If it is a linear motion and d by d t of del L by del theta i dot minus del L by del theta i where theta is the angle the joint angle. So, this we will see using various examples how to construct the dynamic equation of robot manipulators.

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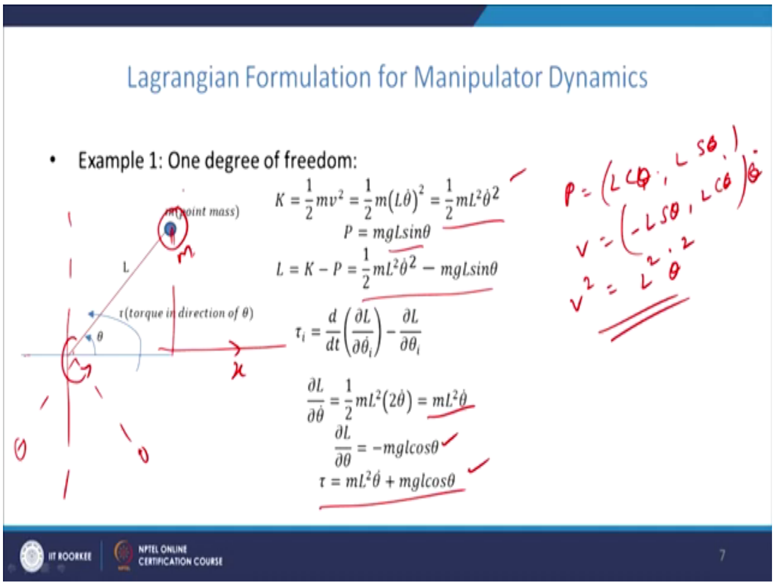

So, the first simplest case is the one arm manipulator. So, single arm manipulator or it is a controlled pendulum, we can say. So, where the torque is applied at the joint like this and the pendulum moves in the various directions. So, the motion of the pendulum we can analyze using this formula.

So, the kinetic energy is half into mass into velocity square. So, here it is an assumption that the mass is concentrated at a point only at the end of the pendulum and the remaining mass is negligible. So, m is concentrated at the end of the pendulum that is the assumption.

So, the point P at the end of the manipulator, it is given by the formula that is the point P is equal to the length is L here, L into cos theta 1; cos theta here there is angle is theta. So, the x distance is L cos theta and the y distances is L sin theta and when we differentiate this thing, the velocity is minus L sin theta into theta dot and L cos theta. So, theta dot is the common thing.

So, we can see that the velocity square is there modulus square of the vector. So, that gives L square and L theta naught square. So, this is it is also half into mass into V square is given by half into m into L square theta dot square it, is the kinetic energy.

The potential energy is mass into gravity into the height. So, we are assuming that the line x axis is the line of 0 potential and the height from that line is L sin theta this one. So, m g h gives the potential energy. So, the Lagrangian L is K minus P is given by this one.

Now applying the formula the Euler Lagrange formula has given here the second equation because it is a revolute joint, we will get the torque is equal to the torque applied at this joint is equal to this expression del L by del theta dot is calculated from here. It is m L square theta dot and del L by del theta is calculated from here it is m g L cos theta is given here.

And therefore, when we differentiate once again we will get the m L square theta double dot plus this expression minus del L by del theta is here plus m g L cos theta equal to torque. So, we get the equation of motion of the pendulum or we can say that it is a one arm manipulator robot manipulator.

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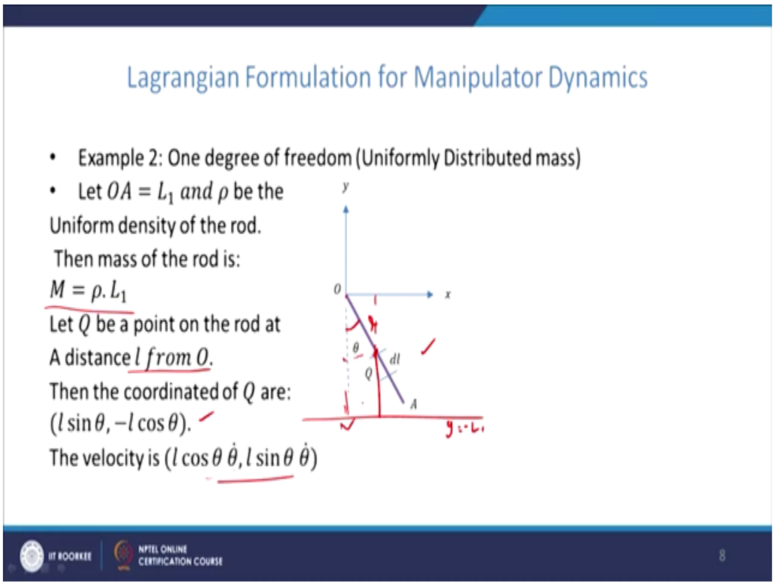

Now, we assume instead of assuming that the mass is concentrated at the end of the pendulum, we assume that the mass is uniformly distributed. So, while constructing the dynamic equation, it is we can fix the any of the axis for measuring the angle. So, in this example we have measured the angle from the x axis up to the pendulum and, but if he take the line vertical line the negative y axis as the a 0 degree angle and then from there we measure the angle up to the pendulum, then also we can construct the dynamic equation in the similar manner.

So, here the mass is uniformly distributed. So, first what we should do is it we take a point Q on this pendulum which is at an distance l from the point O from the fulcrum. So, we take the point Q and which is at a distance L from the point O here. The length of this pendulum is L 1 and rho is the uniform density of the pendulum. So, the total mass is rho density into the total length L 1 that is m.

Now, we calculate the coordinate of the point Q based on the angle given here. So, the x value is L sin theta and the y value is a negative value L cos theta minus L cos theta. So, the velocity is the time derivative of the position. So, this L cos theta into theta dot d theta by d t and L sin theta into theta dot that is for particular point Q.

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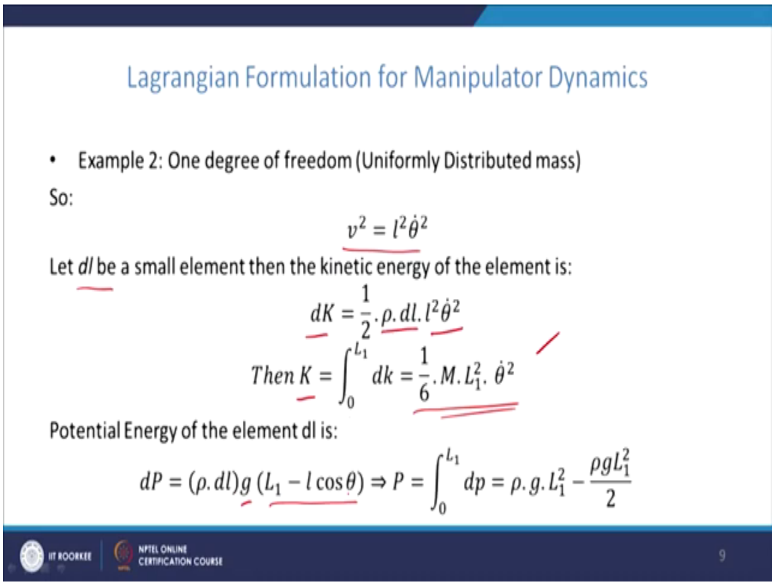
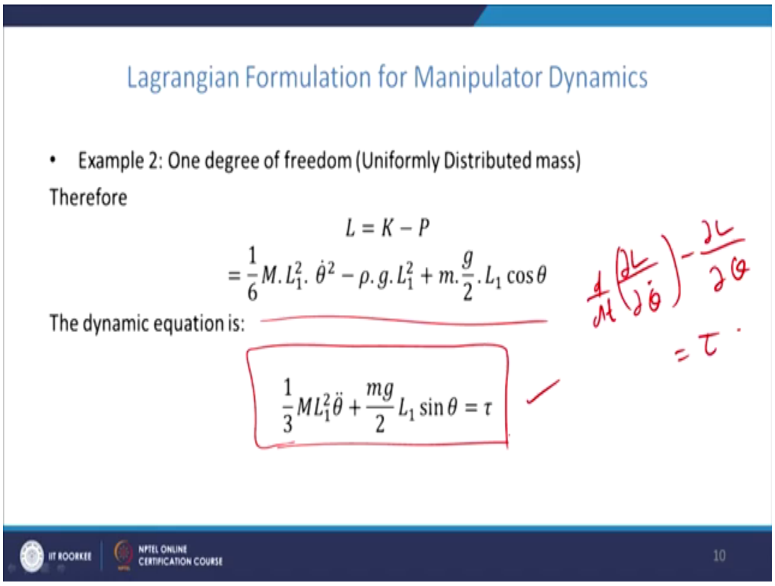


Now, if you take a small element of length dL on this pendulum so, at that point Q the velocity given by the velocity V square the modulus of the velocity vectors square is small l square theta dot square.

Now, if you take a small element of length dl the kinetic energy of that particular element of the length dl is denoted by dK. It is half into density in to the small length dl, l square theta dot square. Now, if you integrate the whole thing, then we will get to the total kinetic energy of the entire pendulum. So, we have to integrate with respect to L, L is varying from 0 to 1; the total length of the pendulum.

So, if it integrate, we will get 1 by 6 m L 1 square theta dot square, it is the kinetic energy. Similarly the potential energy is the mass is rho into dl gravity is g and the height from that point. So, the point here is Q, we assume that the L 1 line that is y is equal to minus L 1 that line is the point the line of 0 potential energy. So, the height from that point is this one. So, it is nothing, but L 1 minus this height that is l cos theta small l cos theta. So, the height mass into gravity into the height it is the L 1 minus small l cos theta. Now, if you integrate from 0 to L 1, we get the total potential energy that is given by this expression. (Refer Slide Time: 16:26)



Now, we can construct the Lagrangian which is K minus P. And we get this expression as the Lagrangian by applying the formula d by d t of del L by del theta dot minus del L by del theta equal to the torque, we get the equation of motion of the pendulum with the uniform distribution of mass here. So, the previous one if the mass is concentrated at a single point, we get the equation of motion like this and if the mass is distributed uniformly, we get the equation. So, this is more practical in many real life robot manipulators, the mass is uniformly distributed. So, one can rely on the second example for more practical purposes.

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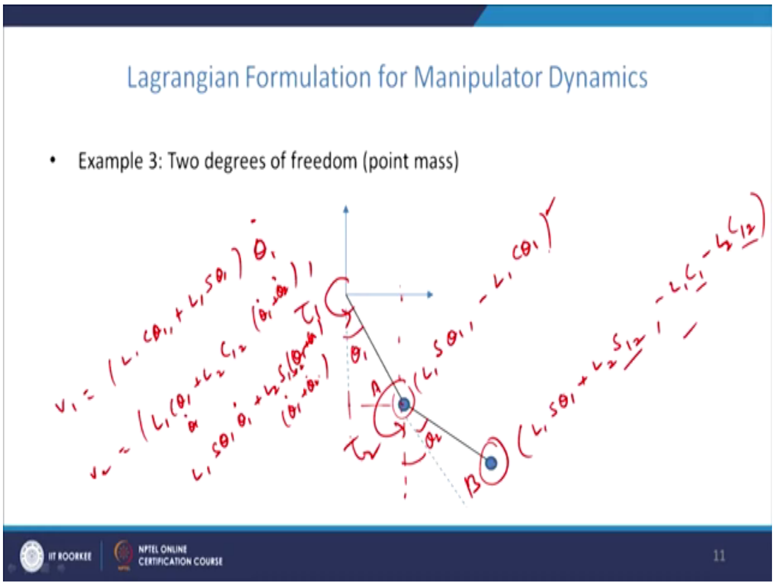

So, similarly here the double pendulum or a robot arm with two links here. So, the torque is applied at two joints here this is torque 1 and torque 2, both the joints are revolute joints here the.

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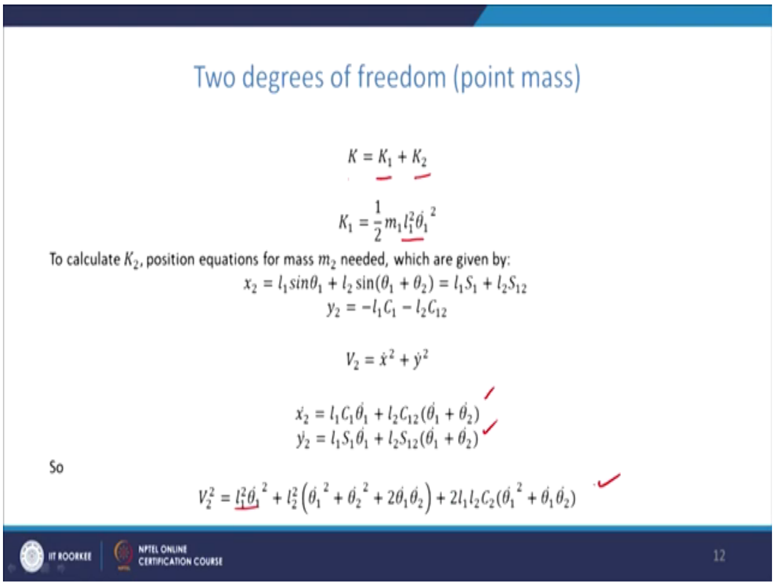

Here we are assuming that the mass of the first one is concentrated at this point, the mass of the second link is concentrated at this point; the other masses are negligible in the. So, for this type of robot manipulator the equation of motion can be constructed in the following way. So, again the same procedure we have to construct the kinetic energy. So, there are two links here. So, we have to calculate the kinetic energy of the first link K 1 and the second link K2, then add K 1 K2 that will give the total kinetic. So, here we can see that the first link is of link L 1 and its coordinate the coordinate of this points are L 1 here; sin theta one.

If you measure theta from this one theta 1 and then theta 2 here. So, L 1 L 1 sin theta 1 and the y coordinate is minus L 1 cos theta 1 and the coordinate of this point, we can call this point as A and this point is B. So, the coordinate of this point B is L 1 sin theta 1 plus L 2 sin theta 1 plus theta 2. The angle the total angle is K theta 1 and then theta 2 theta 1 plus theta. Similarly the y coordinate is minus L 1 cos theta 1 minus L 2o cos theta 1 plus theta 2.

So, now, once we have the coordinate, we can differentiate; we will get the velocity. The velocity of the point A, it is the vector representing the velocity is L 1 cos theta 1 and minus L 1 sin theta 1 plus here multiplied by theta 1 dot. Similarly, we can get the when we differentiate with respect to t we get the velocity 2 of velocity of the point B, the vector is here. We differentiate with respect to this, we will get L 1 cos theta 1e plus L 2 cos of theta 1 plus 2.

So, here we multiply by theta 1 dot and this is multiplied by theta 1 dot plus theta 2 dot and then the second one is component is L 1 sin theta 1 into theta 1 dot plus L 2 sin theta 1 plus two into theta 1 dot plus theta 2. Here so, different notations are used. Here sin of theta 1 plus theta 2 this expression. So, C12 means cos of theta 1 plus theta 2 C1 means cos of theta 1 etcetera.

So, I think the notations are mixed here sin theta 1 here sin theta 1 plus theta 2 sin S12 means. So, now, if you differentiate with respect to t, we get velocity of the point A and this second one will give the velocity of the point B.

Now, the modulus of V 1 square will give the velocity of velocity square of the point A that is nothing, but l 1 square theta 1 dot square and the velocity of the point two point B, we can easily see that from here. If you take the square of the first term and plus the square of the second term, we will get l 1 square into theta 1 dot square. So, we get like this similarly all other terms can be calculated this is the x component, this is y component. So, if you square this and square this and then add, we get the velocity square of the point B.

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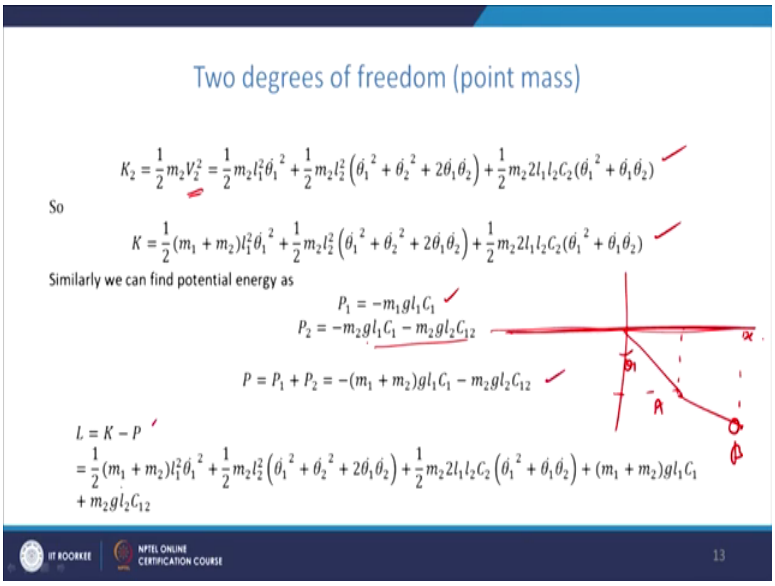

So, now the kinetic energy of the second link is half m 2 the mass of the second link into the velocity of the point B velocity square of the point B. So, it is given by this expression. So, the total kinetic energy is the kinetic energy of the first link plus kinetic energy of second link that is given by summing the two terms. Similarly the potential energy so, we get to the first two potential energy is given by this expression a P 1 is minus mass into gravity into l 1 cos theta 1. This particular formula the 0 potential line is taken as the x axis itself.

So, the height the point A is here, the height from this particular line is minus l 1 cos theta 1 theta is this value. So, minus l 1 cos theta 1 is the height. So, the potential energy with respect to this x axis is given by this. Similarly the potential energy of the point B with respect to the x axis is given by m 2 into gravity into the height is given by this. So, now the total potential energy is calculated, the Lagrangian is calculated and then using the equation for using the equation.

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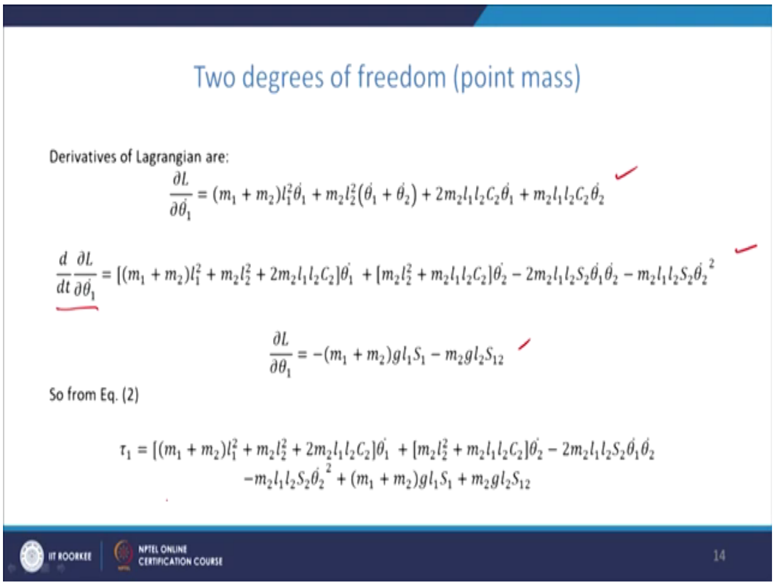

We have to calculate the del L by del theta 1 dot and then del L by del theta 2 dot for the second equation. So, first del L by del theta 1 dot directly we can differentiate with respect to theta 1 dot only. So, this term is to be differentiated, this is to be differentiated.

So, wherever theta 1 is appearing we have to differentiate those particular terms and then we have to differentiate it with respect to t d by dt of this thing. We get this expression and del L by del theta 1. So, it is independent of theta wherever only theta 1 appears. So, here it will be 0 the derivative with respect to theta 1, the first term is 0 and the second term is also 0.

So, theta 1 appears in this term cos theta 1. So, we have to differentiate this term and here cos of theta 1 plus theta 2 that can be differentiated. So, we get to the d L by del theta 1 to be like this.

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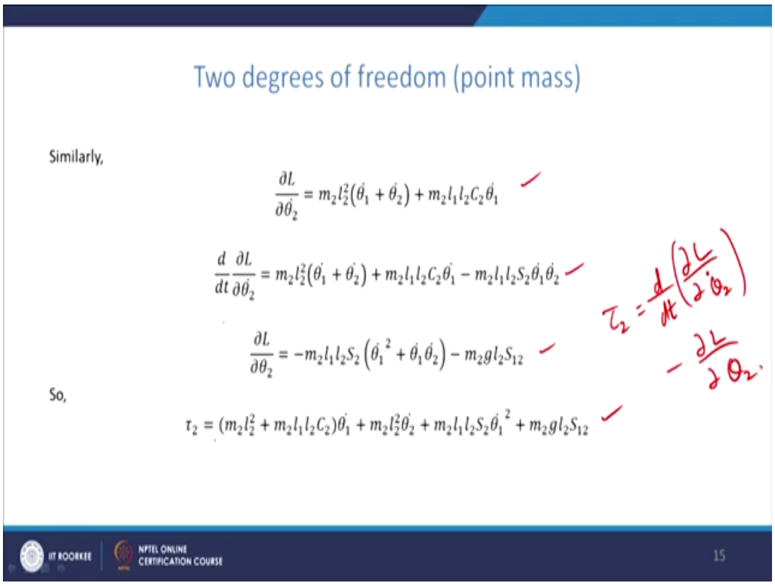

Similarly, for theta 2 del L by del theta dot is this and then its time derivative is given by this and del L by del theta 2 is given. So, we get the formula the torque 2 is d by dt of del L by del theta 2 dot minus del L by del theta 2. So, if you substitute these values, we will get the equation second equation tau 2.

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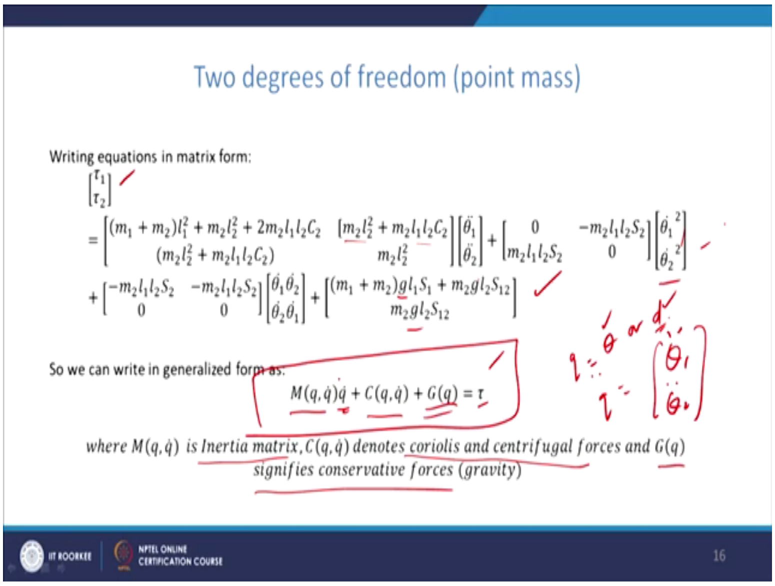

So, we observed that the equation of motion the tau 1 and tau 2 can be written in the matrix form like this combining the two equation. Here, the tau 1 equation is having components of theta 1 double dot. These are all theta 1 double dot and this is theta 2 double dot and remaining are theta 1 dot.

So, we have a theta 1 double dot, theta 2 double dot terms. Similarly in tau 2 also we have to theta 1 double dot theta 2 double dot terms. So, these terms can be collected and returned in the form of a matrix theta 1 double dot. So, collecting the second derivative terms as a matrix M separately the coefficients and the first derivative terms theta 1 dot and theta 2 dot etcetera. The second term is called C q q dot. Here q represent a general notation whether it is theta or d because we know that the theta represent the revolute angle joint distance. So, q is in general used for the variables for combining whether it is revolute or prismatic joint.

So, here q double dot means theta 1 double dot theta 2 double dot. And the C q q dot means the terms containing theta and theta dot all the terms whether it is linear or non-linear like this. We can just write collect all the terms having theta 1 and theta 1 dot in as a combination.

The third one is wherever the gravity is appearing G is appearing in this term that is called collect all such terms that is called the gravity term and tau is nothing, but this tau 1 and tau 2. So, here general dynamic equation will look like this. It is a two arm manipulator. Therefore, it is a M is a 2 by 2 matrix n link manipulator. The matrix M is the n by n matrix; M is called the inertia matrix and C is the Coriolis and centrifugal forces says, it contains those forces and then G contains the gravity term ok.

So, in this manner, we can construct the dynamics equation of the two arm manipulator, but in the coming lectures, we will see how to construct the dynamic equation for general manipulators.

Thank you.