

**Robotics and Control: Theory and Practice**  
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**Lecture – 12**  
**Dynamics of Manipulator**

Welcome to this lecture on Dynamics of Manipulators. In this lecture, we have we shall see how to formulate the dynamics equations of various robot manipulators. In the previous lectures, we have seen how to formulate the kinematics equation using the DH procedure. So, kinematics equation is basically on the relation between the position and orientation of the end effector of a robot manipulator and the joint variables.

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**Dynamics of Manipulator**

- Kinematics of a robot manipulator deals with the relation between position and orientation of the end effector and the joint variable of the manipulator without considering the mass of the links or the forces/torques acting on the manipulator.
- Dynamics of a robot deals with the relation between position, velocity and acceleration of the robot's joints along with the force/torque exerted at each joint to cause the motion.

The slide includes a diagram of a 2-link robot arm with joints labeled  $\theta_1$  and  $\theta_2$ . To the right of the diagram, there is a handwritten note in red ink:  $\theta_1, \dot{\theta}_1, \ddot{\theta}_1$  and  $\tau_1, f_1$ .

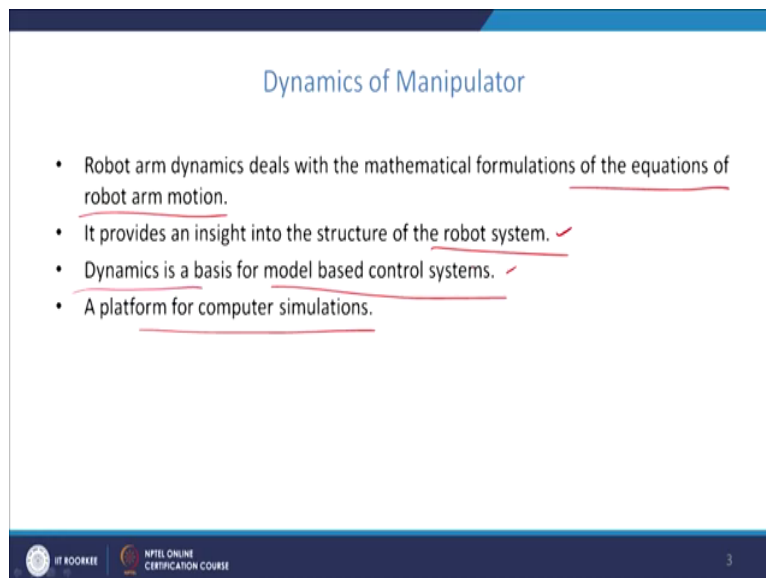
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So, it will not take into account the mass or the force torque of exerted on the robot manipulators for causing a particular motion. So, for example, if you have the robot

manipulator so, the position and orientation of the robot manipulator and the relation between various joint angles gives the kinematics equation. The dynamic equation is a relation between the position, velocity, acceleration of the robot joints and the force and torque applied on the actuators of the robot manipulators to cause a motion of the robot manipulator.

So, here when we take force torque etcetera naturally, we take into account the mass of various links and the acceleration etcetera of the robot manipulator along with the position orientation velocity and acceleration of the robot joint. So, dynamic equation is basically a relation between the  $\theta$ ,  $\dot{\theta}$ ,  $\ddot{\theta}$  where  $\theta$  is the joint angles of the robot manipulator; the vector representing the joint angles and the torque or the force acting at every joint of the robot manipulator. So, this relation is the dynamics of the robot.

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The slide is titled "Dynamics of Manipulator" in blue text. It contains four bullet points, each underlined in red. The first point is "Robot arm dynamics deals with the mathematical formulations of the equations of robot arm motion." The second point is "It provides an insight into the structure of the robot system." followed by a red checkmark. The third point is "Dynamics is a basis for model based control systems." followed by a red checkmark. The fourth point is "A platform for computer simulations." The slide footer includes the IIT Kharagpur logo, the text "IIT KHARAGPUR", the NPTEL logo, and the text "NPTEL ONLINE CERTIFICATION COURSE". The number "3" is in the bottom right corner.

### Dynamics of Manipulator

- Robot arm dynamics deals with the mathematical formulations of the equations of robot arm motion.
- It provides an insight into the structure of the robot system. ✓
- Dynamics is a basis for model based control systems. ✓
- A platform for computer simulations.

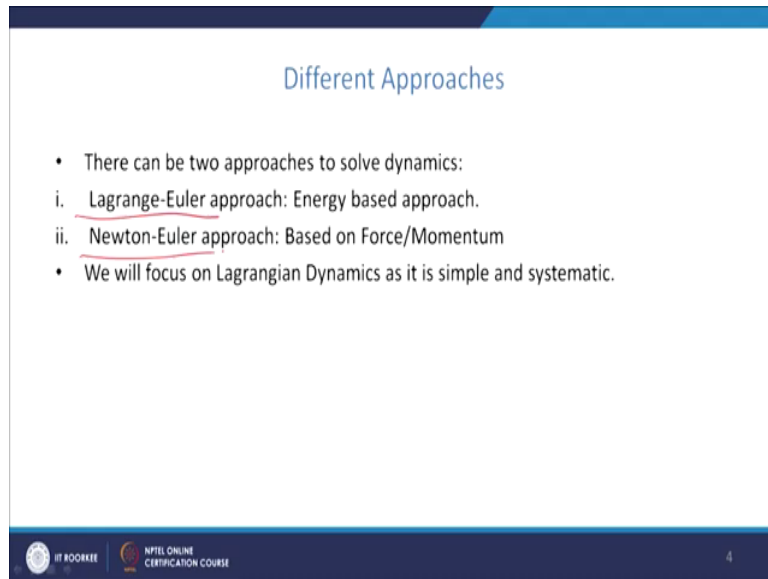
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So, here so, the dynamics is nothing, but the equation of motion of the manipulator robot manipulator and the it provides the insight into the structure of the robot system. So, for example, if a robot is having  $n$  joints then there will be a system of  $n$  second order differential equation and the structure of the equation varies whether a particular joint is a revolute joint or prismatic joint etcetera.

So, the equation of motion or the dynamic equation indicates the structure of the robotic system and the dynamic equation is very useful in constructing the control of a robot manipulator for obtaining a particular motion of the manipulator. So, the desired motion of the robot manipulator can be obtained by constructing a suitable control based on the equation of motion of the manipulator.

And using that, we can simulate the robot motion using these mathematical formulation and the control of the manipulator even before performing the actual experiment. So, this will save a lot of costs of performing an experiment and first reviewing the controller design. So, the simulation will be very useful in improving the controller design for a desired motion of the robot manipulator.

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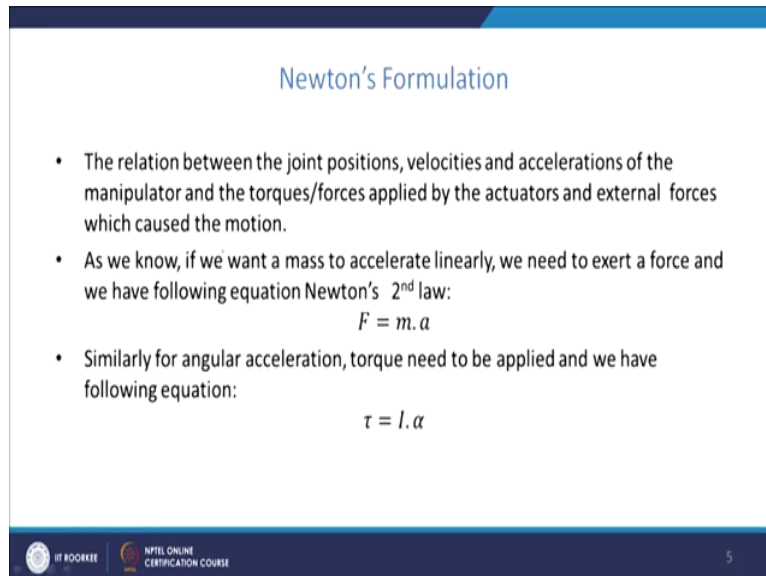
Different Approaches

- There can be two approaches to solve dynamics:
  - i. Lagrange-Euler approach: Energy based approach.
  - ii. Newton-Euler approach: Based on Force/Momentum
- We will focus on Lagrangian Dynamics as it is simple and systematic.

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So, so to formulate the dynamic equation there are two approaches one is the Euler Lagrange approach which is energy based approach and another is the Newton Euler approach and it is based on the force and momentum of the manipulator. So, here in this lecture, we will concentrate on the Lagrange method; Euler Lagrange method.

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The slide is titled "Newton's Formulation" in blue text. It contains two bullet points. The first bullet point describes the relation between joint positions, velocities, and accelerations of a manipulator and the torques/forces applied by actuators and external forces. The second bullet point states that to accelerate a mass linearly, a force must be exerted, and it provides Newton's 2nd law equation  $F = m \cdot a$ . It then states that for angular acceleration, torque must be applied, and it provides the equation  $\tau = I \cdot \alpha$ . The slide footer includes the IIT Kharagpur logo, the text "NPTEL ONLINE CERTIFICATION COURSE", and the number "5".

### Newton's Formulation

- The relation between the joint positions, velocities and accelerations of the manipulator and the torques/forces applied by the actuators and external forces which caused the motion.
- As we know, if we want a mass to accelerate linearly, we need to exert a force and we have following equation Newton's 2<sup>nd</sup> law:  
$$F = m \cdot a$$
- Similarly for angular acceleration, torque need to be applied and we have following equation:  
$$\tau = I \cdot \alpha$$

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So, the Newton method is basically the second law of motion Newton's law of motion which is the force equal to mass into acceleration. If it is a linear motion if the joint is a prismatic joint, we can apply the linear motion equation and the if a motion is a revolute one then the torque causing that revolving motion is equal to the inertia into the angular acceleration. So, basing based on this two formulations, one can construct the equation of motion or the dynamic equation of a robot manipulator.

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### Lagrangian Formulation



- Based on differentiation of Energy terms w.r.t. system's variables and time.
- If  $(q_1, q_2, \dots, q_n)$  and  $(\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n)$  are the joint position and velocity variables (generalized coordinates) of a robot manipulator respectively, then its kinetic energy  $K$  is a function of  $\dot{q}_i$  and potential energy  $P$  is a function of  $q_i$ .
- Lagrangian is defined as:
 
$$L = K - P$$

where  $L$  is lagrangian,  $K$  and  $P$  are kinetic and potential energies of the system respectively.

- Then we have following generalized second order ordinary differential equations for both linear and rotational motions:
 
$$F_i = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_i} \right) - \frac{\partial L}{\partial x_i} \dots \dots (1)$$

$$\tau_i = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_i} \right) - \frac{\partial L}{\partial \theta_i} \dots \dots (2)$$

where  $F$  is summation of all external forces for linear motion,  $\tau$  is summation of all torques in rotational motion,  $q$  is  $\theta$  in case of revolute joint and  $x$  in case of prismatic joint denotes system variables.



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But another procedure is the Euler Lagrange method it is based on the Lagrangian  $L$  which is the difference between the kinetic energy and the potential energy of a body in motion. Any rigid body can be analyzed using this procedure in our case it is the robot manipulator.

So, while constructing the dynamic equation, we have to construct the kinetic energy and potential energy and find the Lagrangian by this formula, then the equation of motion is given by the system of equation given in one and two. So, if a joint is a prismatic joint, then force is applied at that particular joint that is  $F_i$ . And if a joint is a revolute one, the torque is applied at that joint.

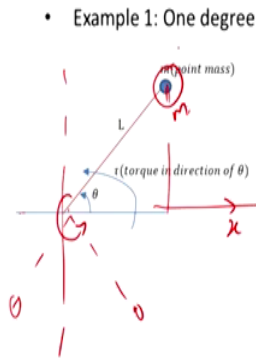
So, the equation of motion is based on this  $\frac{d}{dt}$  of  $\frac{\partial L}{\partial \dot{x}_i}$  minus  $\frac{\partial L}{\partial x_i}$ . If it is a linear motion and  $\frac{d}{dt}$  of  $\frac{\partial L}{\partial \dot{\theta}_i}$  minus  $\frac{\partial L}{\partial \theta_i}$

where theta is the angle the joint angle. So, this we will see using various examples how to construct the dynamic equation of robot manipulators.

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**Lagrangian Formulation for Manipulator Dynamics**

- Example 1: One degree of freedom:



$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(L\dot{\theta})^2 = \frac{1}{2}mL^2\dot{\theta}^2$$

$$P = mgL\sin\theta$$

$$L = K - P = \frac{1}{2}mL^2\dot{\theta}^2 - mgL\sin\theta$$

$$\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i}$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}mL^2(2\dot{\theta}) = mL^2\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mgL\cos\theta$$

$$\tau = mL^2\ddot{\theta} + mgL\cos\theta$$

*Handwritten notes on the right:*

$$P = (L\cos\theta, L\sin\theta)$$

$$V = (-L\dot{\theta}\sin\theta, L\dot{\theta}\cos\theta)$$

$$V^2 = L^2\dot{\theta}^2$$

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So, the first simplest case is the one arm manipulator. So, single arm manipulator or it is a controlled pendulum, we can say. So, where the torque is applied at the joint like this and the pendulum moves in the various directions. So, the motion of the pendulum we can analyze using this formula.

So, the kinetic energy is half into mass into velocity square. So, here it is an assumption that the mass is concentrated at a point only at the end of the pendulum and the remaining mass is negligible. So, m is concentrated at the end of the pendulum that is the assumption.

So, the point P at the end of the manipulator, it is given by the formula that is the point P is equal to the length is L here, L into cos theta; cos theta here there is angle is theta. So, the x distance is L cos theta and the y distances is L sin theta and when we differentiate this thing, the velocity is minus L sin theta into theta dot and L cos theta. So, theta dot is the common thing.

So, we can see that the velocity square is there modulus square of the vector. So, that gives L square and L theta naught square. So, this is it is also half into mass into V square is given by half into m into L square theta dot square it, is the kinetic energy.

The potential energy is mass into gravity into the height. So, we are assuming that the line x axis is the line of 0 potential and the height from that line is L sin theta this one. So, m g h gives the potential energy. So, the Lagrangian L is K minus P is given by this one.

Now applying the formula the Euler Lagrange formula has given here the second equation because it is a revolute joint, we will get the torque is equal to the torque applied at this joint is equal to this expression del L by del theta dot is calculated from here. It is m L square theta dot and del L by del theta is calculated from here it is m g L cos theta is given here.

And therefore, when we differentiate once again we will get the m L square theta double dot plus this expression minus del L by del theta is here plus m g L cos theta equal to torque. So, we get the equation of motion of the pendulum or we can say that it is a one arm manipulator robot manipulator.



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### Lagrangian Formulation for Manipulator Dynamics

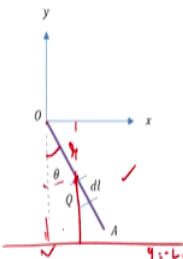
- Example 2: One degree of freedom (Uniformly Distributed mass)
- Let  $OA = L_1$  and  $\rho$  be the Uniform density of the rod.

Then mass of the rod is:  
 $M = \rho \cdot L_1$



Let  $Q$  be a point on the rod at  
 A distance  $l$  from  $O$ .

Then the coordinated of  $Q$  are:  
 $(l \sin \theta, -l \cos \theta)$ .

The velocity is  $(l \cos \theta \dot{\theta}, l \sin \theta \dot{\theta})$



The diagram shows a 2D Cartesian coordinate system with x and y axes. A rod of length  $L_1$  is pivoted at the origin  $O$ . The rod is at an angle  $\theta$  from the positive x-axis. A point  $Q$  is marked on the rod at a distance  $l$  from the origin. A small differential element  $dl$  is also indicated on the rod. The end of the rod is labeled  $A$ . The total length  $L_1$  is marked along the x-axis.



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Now, we assume instead of assuming that the mass is concentrated at the end of the pendulum, we assume that the mass is uniformly distributed. So, while constructing the dynamic equation, it is we can fix the any of the axis for measuring the angle. So, in this example we have measured the angle from the x axis up to the pendulum and, but if he take the line vertical line the negative y axis as the a 0 degree angle and then from there we measure the angle up to the pendulum, then also we can construct the dynamic equation in the similar manner.

So, here the mass is uniformly distributed. So, first what we should do is it we take a point  $Q$  on this pendulum which is at an distance  $l$  from the point  $O$  from the fulcrum. So, we take the point  $Q$  and which is at a distance  $L$  from the point  $O$  here. The length of this pendulum is  $L$

and  $\rho$  is the uniform density of the pendulum. So, the total mass is  $\rho$  density into the total length  $L$  that is  $m$ .

Now, we calculate the coordinate of the point Q based on the angle given here. So, the x value is  $L \sin \theta$  and the y value is a negative value  $L \cos \theta$  minus  $L \cos \theta$ . So, the velocity is the time derivative of the position. So, this  $L \cos \theta$  into  $\dot{\theta}$  and  $L \sin \theta$  into  $\dot{\theta}$  that is for particular point Q.

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**Lagrangian Formulation for Manipulator Dynamics**

- Example 2: One degree of freedom (Uniformly Distributed mass)

So:

$$v^2 = l^2 \dot{\theta}^2$$



Let  $dl$  be a small element then the kinetic energy of the element is:

$$dK = \frac{1}{2} \cdot \rho \cdot dl \cdot l^2 \dot{\theta}^2$$

Then  $K = \int_0^{L_1} dk = \frac{1}{6} \cdot M \cdot L_1^2 \cdot \dot{\theta}^2$

Potential Energy of the element  $dl$  is:

$$dP = (\rho \cdot dl)g (L_1 - l \cos \theta) \Rightarrow P = \int_0^{L_1} dp = \rho \cdot g \cdot L_1^2 - \frac{\rho g L_1^2}{2}$$

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Now, if you take a small element of length  $dL$  on this pendulum so, at that point Q the velocity given by the velocity  $V$  square the modulus of the velocity vectors square is small  $l$  square  $\dot{\theta}$  square.

Now, if you take a small element of length  $dl$  the kinetic energy of that particular element of the length  $dl$  is denoted by  $dK$ . It is half into density into the small length  $dl$ ,  $l$  square theta dot square. Now, if you integrate the whole thing, then we will get to the total kinetic energy of the entire pendulum. So, we have to integrate with respect to  $L$ ,  $L$  is varying from 0 to  $l$ ; the total length of the pendulum.

So, if it integrate, we will get  $\frac{1}{6} m L \dot{\theta}^2$ , it is the kinetic energy. Similarly the potential energy is the mass is  $\rho$  into  $dl$  gravity is  $g$  and the height from that point. So, the point here is  $Q$ , we assume that the  $L=0$  line that is  $y$  is equal to minus  $L$  that line is the point the line of 0 potential energy. So, the height from that point is this one. So, it is nothing, but  $L$  minus this height that is  $l \cos \theta$  small  $l \cos \theta$ . So, the height mass into gravity into the height it is the  $L$  minus small  $l \cos \theta$ . Now, if you integrate from 0 to  $L$ , we get the total potential energy that is given by this expression. (Refer Slide Time: 16:26)

**Lagrangian Formulation for Manipulator Dynamics**

- Example 2: One degree of freedom (Uniformly Distributed mass)

Therefore

$$L = K - P$$



$$= \frac{1}{6} M \cdot L_1^2 \cdot \dot{\theta}^2 - \rho \cdot g \cdot L_1^2 + m \cdot \frac{g}{2} \cdot L_1 \cos \theta$$

The dynamic equation is:

$$\frac{1}{3} M L_1^2 \ddot{\theta} + \frac{mg}{2} L_1 \sin \theta = \tau$$

$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = \tau$

$= \tau$

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Now, we can construct the Lagrangian which is  $K$  minus  $P$ . And we get this expression as the Lagrangian by applying the formula  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = \tau$  equal to the torque, we get the equation of motion of the pendulum with the uniform distribution of mass here. So, the previous one if the mass is concentrated at a single point, we get the equation of motion like this and if the mass is distributed uniformly, we get the equation. So, this is more practical in many real life robot manipulators, the mass is uniformly distributed. So, one can rely on the second example for more practical purposes.

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Lagrangian Formulation for Manipulator Dynamics

- Example 3: Two degrees of freedom (point mass)

$v_1 = (L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2)) \dot{\theta}_1$   
 $v_2 = (L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2)) \dot{\theta}_1 + L_2 \sin(\theta_1 + \theta_2) \dot{\theta}_2$

$v_{1x} = -L_1 \sin(\theta_1) \dot{\theta}_1 - L_2 \sin(\theta_1 + \theta_2) \dot{\theta}_1$   
 $v_{1y} = L_1 \cos(\theta_1) \dot{\theta}_1 + L_2 \cos(\theta_1 + \theta_2) \dot{\theta}_1$   
 $v_{2x} = -L_2 \sin(\theta_1 + \theta_2) \dot{\theta}_1 + L_2 \cos(\theta_1 + \theta_2) \dot{\theta}_2$   
 $v_{2y} = L_2 \cos(\theta_1 + \theta_2) \dot{\theta}_1 + L_2 \sin(\theta_1 + \theta_2) \dot{\theta}_2$

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So, similarly here the double pendulum or a robot arm with two links here. So, the torque is applied at two joints here this is torque 1 and torque 2, both the joints are revolute joints here the.

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Two degrees of freedom (point mass)

$$K = K_1 + K_2$$

$$K_1 = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2$$

To calculate  $K_2$ , position equations for mass  $m_2$  needed, which are given by:

$$x_2 = l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2) = l_1 S_1 + l_2 S_{12}$$

$$y_2 = -l_1 C_1 - l_2 C_{12}$$

$$V_2 = \dot{x}^2 + \dot{y}^2$$

$$\dot{x}_2 = l_1 C_1 \dot{\theta}_1 + l_2 C_{12} (\dot{\theta}_1 + \dot{\theta}_2)$$

$$\dot{y}_2 = l_1 S_1 \dot{\theta}_1 + l_2 S_{12} (\dot{\theta}_1 + \dot{\theta}_2)$$

So

$$V_2^2 = l_1^2 \dot{\theta}_1^2 + l_2^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1 \dot{\theta}_2) + 2l_1 l_2 C_2 (\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2)$$

Here we are assuming that the mass of the first one is concentrated at this point, the mass of the second link is concentrated at this point; the other masses are negligible in the. So, for this type of robot manipulator the equation of motion can be constructed in the following way. So, again the same procedure we have to construct the kinetic energy. So, there are two links here. So, we have to calculate the kinetic energy of the first link  $K_1$  and the second link  $K_2$ , then add  $K_1 + K_2$  that will give the total kinetic. So, here we can see that the first link is of link  $L_1$  and its coordinate the coordinate of this points are  $L_1$  here;  $\sin \theta_1$ .

If you measure  $\theta_1$  from this one  $\theta_1$  and then  $\theta_2$  here. So,  $L_1 \sin \theta_1$  and the y coordinate is minus  $L_1 \cos \theta_1$  and the coordinate of this point, we can call this point as A and this point is B. So, the coordinate of this point B is  $L_1 \sin \theta_1$  plus  $L_2 \sin$

$\theta_1 + \theta_2$ . The angle the total angle is  $\theta_1 + \theta_2$ . Similarly the y coordinate is  $-L_1 \cos \theta_1 - L_2 \cos(\theta_1 + \theta_2)$ .

So, now, once we have the coordinate, we can differentiate; we will get the velocity. The velocity of the point A, it is the vector representing the velocity is  $L_1 \sin \theta_1$  and  $-L_1 \dot{\theta}_1$ . Similarly, we can get the when we differentiate with respect to  $t$  we get the velocity 2 of velocity of the point B, the vector is here. We differentiate with respect to this, we will get  $L_1 \cos \theta_1 \dot{\theta}_1 + L_2 \cos(\theta_1 + \theta_2) \dot{\theta}_1 + \dot{\theta}_2$ .

So, here we multiply by  $\dot{\theta}_1$  and this is multiplied by  $\dot{\theta}_1 + \dot{\theta}_2$  and then the second one is component is  $L_1 \sin \theta_1 \dot{\theta}_1 + L_2 \sin(\theta_1 + \theta_2) \dot{\theta}_1 + \dot{\theta}_2$ . Here so, different notations are used. Here  $\sin$  of  $\theta_1 + \theta_2$  this expression. So,  $C_{12}$  means  $\cos$  of  $\theta_1 + \theta_2$   $C_1$  means  $\cos$  of  $\theta_1$  etcetera.

So, I think the notations are mixed here  $\sin \theta_1$  here  $\sin(\theta_1 + \theta_2)$   $S_{12}$  means. So, now, if you differentiate with respect to  $t$ , we get velocity of the point A and this second one will give the velocity of the point B.

Now, the modulus of  $V_1$  square will give the velocity of velocity square of the point A that is nothing, but  $L_1^2 \dot{\theta}_1^2$  and the velocity of the point two point B, we can easily see that from here. If you take the square of the first term and plus the square of the second term, we will get  $L_1^2 \dot{\theta}_1^2$ . So, we get like this similarly all other terms can be calculated this is the x component, this is y component. So, if you square this and square this and then add, we get the velocity square of the point B.

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Two degrees of freedom (point mass)

$$K_2 = \frac{1}{2} m_2 V_2^2 = \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1 \dot{\theta}_2) + \frac{1}{2} m_2 2l_1 l_2 C_2 (\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2)$$

So

$$K = \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1 \dot{\theta}_2) + \frac{1}{2} m_2 2l_1 l_2 C_2 (\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2)$$

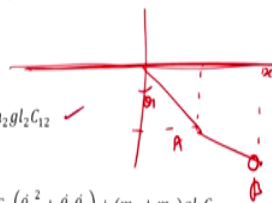
Similarly we can find potential energy as

$$P_1 = -m_1 g l_1 C_1$$

$$P_2 = -m_2 g l_1 C_1 - m_2 g l_2 C_{12}$$

$$P = P_1 + P_2 = -(m_1 + m_2) g l_1 C_1 - m_2 g l_2 C_{12}$$

$$L = K - P$$

$$= \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1 \dot{\theta}_2) + \frac{1}{2} m_2 2l_1 l_2 C_2 (\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2) + (m_1 + m_2) g l_1 C_1 + m_2 g l_2 C_{12}$$


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So, now the kinetic energy of the second link is half  $m_2$  the mass of the second link into the velocity of the point B velocity square of the point B. So, it is given by this expression. So, the total kinetic energy is the kinetic energy of the first link plus kinetic energy of second link that is given by summing the two terms. Similarly the potential energy so, we get to the first two potential energy is given by this expression a  $P_1$  is minus mass into gravity into  $l_1 \cos \theta_1$ . This particular formula the 0 potential line is taken as the x axis itself.

So, the height the point A is here, the height from this particular line is minus  $l_1 \cos \theta_1$  theta is this value. So, minus  $l_1 \cos \theta_1$  is the height. So, the potential energy with respect to this x axis is given by this. Similarly the potential energy of the point B with respect to the x axis is given by  $m_2$  into gravity into the height is given by this. So, now the total potential

energy is calculated, the Lagrangian is calculated and then using the equation for using the equation.

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Two degrees of freedom (point mass)

Derivatives of Lagrangian are:



$$\frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2)l_1^2 \dot{\theta}_1 + m_2 l_2^2 (\dot{\theta}_1 + \dot{\theta}_2) + 2m_2 l_1 l_2 C_2 \dot{\theta}_1 + m_2 l_1 l_2 C_2 \dot{\theta}_2$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_1} = [(m_1 + m_2)l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 C_2] \ddot{\theta}_1 + [m_2 l_2^2 + m_2 l_1 l_2 C_2] \ddot{\theta}_2 - 2m_2 l_1 l_2 S_2 \dot{\theta}_1 \dot{\theta}_2 - m_2 l_1 l_2 S_2 \dot{\theta}_2^2$$

$$\frac{\partial L}{\partial \theta_1} = -(m_1 + m_2)g l_1 S_1 - m_2 g l_2 S_{12}$$

So from Eq. (2)

$$\tau_1 = [(m_1 + m_2)l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 C_2] \ddot{\theta}_1 + [m_2 l_2^2 + m_2 l_1 l_2 C_2] \ddot{\theta}_2 - 2m_2 l_1 l_2 S_2 \dot{\theta}_1 \dot{\theta}_2 - m_2 l_1 l_2 S_2 \dot{\theta}_2^2 + (m_1 + m_2)g l_1 S_1 + m_2 g l_2 S_{12}$$



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We have to calculate the  $\partial L$  by  $\partial \dot{\theta}_1$  and then  $\partial L$  by  $\partial \dot{\theta}_2$  for the second equation. So, first  $\partial L$  by  $\partial \dot{\theta}_1$  directly we can differentiate with respect to  $\dot{\theta}_1$  only. So, this term is to be differentiated, this is to be differentiated.

So, wherever  $\theta_1$  is appearing we have to differentiate those particular terms and then we have to differentiate it with respect to  $t$  d by dt of this thing. We get this expression and  $\partial L$  by  $\partial \theta_1$ . So, it is independent of  $\theta_1$  wherever only  $\theta_1$  appears. So, here it will be 0 the derivative with respect to  $\theta_1$ , the first term is 0 and the second term is also 0.



So, theta 1 appears in this term cos theta 1. So, we have to differentiate this term and here cos of theta 1 plus theta 2 that can be differentiated. So, we get to the d L by del theta 1 to be like this.

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Two degrees of freedom (point mass)

Similarly,

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 (\dot{\theta}_1 + \dot{\theta}_2) + m_2 l_1 l_2 C_2 \dot{\theta}_1$$



$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 l_2 C_2 \ddot{\theta}_1 - m_2 l_1 l_2 S_2 \dot{\theta}_1 \dot{\theta}_2$$

$$\frac{\partial L}{\partial \theta_2} = -m_2 l_1 l_2 S_2 (\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2) - m_2 g l_2 S_{12}$$

So,

$$\tau_2 = (m_2 l_2^2 + m_2 l_1 l_2 C_2) \ddot{\theta}_1 + m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 S_2 \dot{\theta}_1^2 + m_2 g l_2 S_{12}$$

$\tau_2 = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2}$


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Similarly, for theta 2 del L by del theta dot is this and then its time derivative is given by this and del L by del theta 2 is given. So, we get the formula the torque 2 is d by dt of del L by del theta 2 dot minus del L by del theta 2. So, if you substitute these values, we will get the equation second equation tau 2.

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## Two degrees of freedom (point mass)

Writing equations in matrix form:

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} (m_1 + m_2)l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 C_2 & m_2 l_2^2 + m_2 l_1 l_2 C_2 \\ (m_2 l_2^2 + m_2 l_1 l_2 C_2) & m_2 l_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 & -m_2 l_1 l_2 S_2 \\ m_2 l_1 l_2 S_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} -m_2 l_1 l_2 S_2 & -m_2 g l_1 S_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} (m_1 + m_2)g l_1 S_1 + m_2 g l_2 S_{12} \\ m_2 g l_2 S_{12} \end{bmatrix}$$

So we can write in generalized form as:

$$M(q, \dot{q})\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau$$

where  $M(q, \dot{q})$  is Inertia matrix,  $C(q, \dot{q})$  denotes coriolis and centrifugal forces and  $G(q)$  signifies conservative forces (gravity)

So, we observed that the equation of motion the  $\tau_1$  and  $\tau_2$  can be written in the matrix form like this combining the two equation. Here, the  $\tau_1$  equation is having components of  $\ddot{\theta}_1$ . These are all  $\ddot{\theta}_1$  and this is  $\ddot{\theta}_2$  and remaining are  $\dot{\theta}_1$ .

So, we have a  $\ddot{\theta}_1$ ,  $\ddot{\theta}_2$  terms. Similarly in  $\tau_2$  also we have to  $\ddot{\theta}_1$ ,  $\ddot{\theta}_2$  terms. So, these terms can be collected and returned in the form of a matrix  $\ddot{\theta}$ . So, collecting the second derivative terms as a matrix  $M$  separately the coefficients and the first derivative terms  $\dot{\theta}_1$  and  $\dot{\theta}_2$  etcetera. The second term is called  $C_{qq}$ . Here  $q$  represent a general notation whether it is  $\theta$  or  $d$  because we know that the  $\theta$  represent the revolute angle joint distance. So,  $q$  is in general used for the variables for combining whether it is revolute or prismatic joint.

So, here  $\ddot{q}$  means  $\ddot{\theta}_1$   $\ddot{\theta}_2$ . And the  $C \dot{q} \dot{q}$  means the terms containing  $\theta$  and  $\dot{\theta}$  all the terms whether it is linear or non-linear like this. We can just write collect all the terms having  $\theta_1$  and  $\dot{\theta}_1$  in as a combination.

The third one is wherever the gravity is appearing  $G$  is appearing in this term that is called collect all such terms that is called the gravity term and  $\tau$  is nothing, but this  $\tau_1$  and  $\tau_2$ . So, here general dynamic equation will look like this. It is a two arm manipulator. Therefore, it is a  $M$  is a  $2 \times 2$  matrix  $n$  link manipulator. The matrix  $M$  is the  $n$  by  $n$  matrix;  $M$  is called the inertia matrix and  $C$  is the Coriolis and centrifugal forces says, it contains those forces and then  $G$  contains the gravity term ok.

So, in this manner, we can construct the dynamics equation of the two arm manipulator, but in the coming lectures, we will see how to construct the dynamic equation for general manipulators.

Thank you.