Radiative Heat Transfer
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Lecture – 07
Hottel Crossed String Method

Hello friends, we are discussing the view factor within 2 finite area services, in the previous class I discussed the definition of the view factors and applied the area integration technique to find out view factors between two parallel plates. In this lecture, we will do couple of examples and I will introduce you a new method; Hottel-Crossed string method for the evaluation of view factors.

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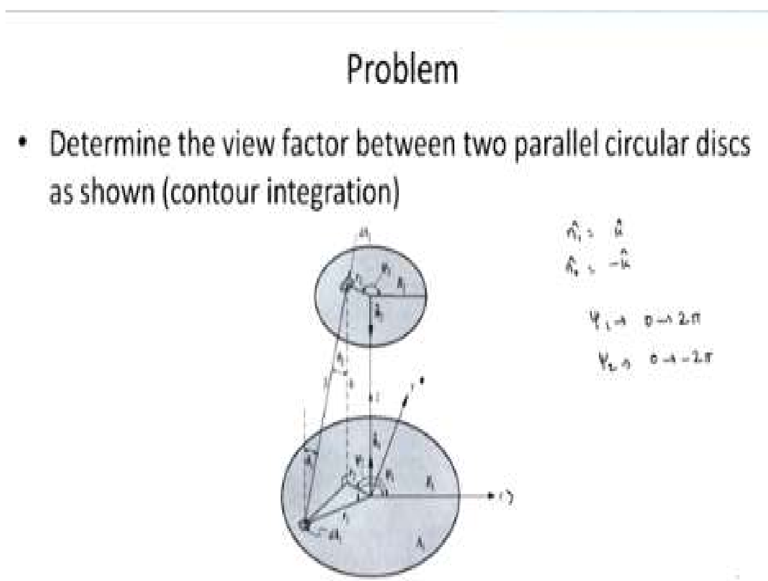

The first problem that we are going to solve in this lecture includes 2 flat circular plates, the plates may have same area or they may have different area, so in this case, we will take that the radius of the 2 plates is different, they are parallel, separated by some distance *h*, we will apply the contour integration technique to find out the view factor between these 2 plates.

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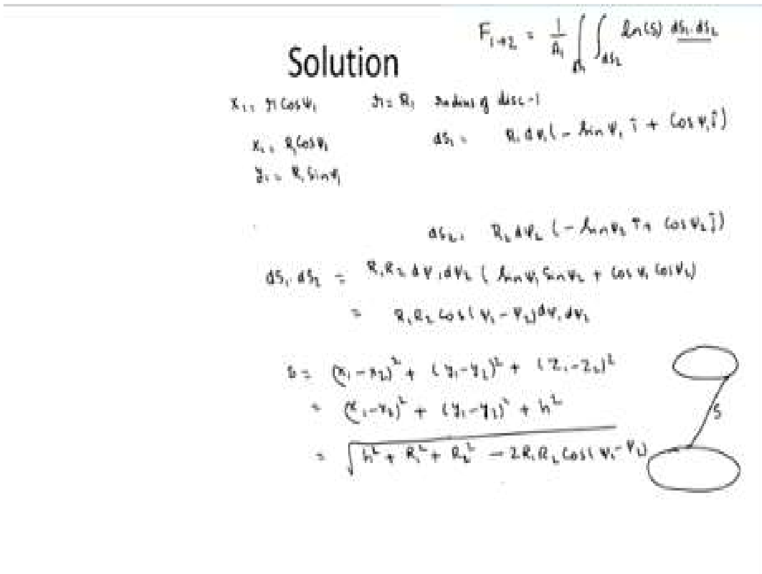

So, as we have already developed the expression for the view factor between surface 1 and 2 is given by as per the Stroke's theorem, we have already done that,

$$F_{1-2} = \frac{1}{A_1}\int_{ds_1}\int_{ds_2} \ln(S) ds_1 ds_2$$

so this is the expression for finding out the view factor between the two configurations. If you just look at the problem, let us understand the coordinate system used.

So, we have *x* direction, *y* direction and *z* direction is pointing upwards, okay so,

$$\widehat{n_1} = \hat{k} \text{ and } \widehat{n_2} = -\hat{k}$$

now with this notation, because the 2 normal vectors are pointing in opposing direction, the azimuthal angle, the way it is measured, the

ψ_1 will vary from 0 to 2π, while ψ_2 will vary from 0 to -2π, it just the book keeping thing, the angle ψ_1 varies from 0 to 2 π and ψ_2 will vary from 0 to -2π.

So, first of all we try to find out dS_1 and dS_2 in terms of the coordinate system, so we can easily write

$$x_1 = r\cos\psi_1$$

and because we are doing line integration, we take just

$$r = R_1$$

where R_1 is the radius disc 1, okay, so x_1 anywhere on path or contour or periphery of the disc will be simply

$$x_1 = R_1 \cos \psi_1$$

and similarly,

$$y_1 = R_1 \sin \psi_1$$

so we can write

$$dS_1 = R_1 d\psi_1(-sin\psi_1\, \hat{\imath} + cos\psi_1 \hat{\jmath})$$

this is all vector.

So, any point on the disc periphery of disc 1 can be represented in the vector along that path as dS_1, where $\hat{\imath}$ and $\hat{\jmath}$ are unit vectors in x and y direction respectively, similarly we can write

$$dS_2 = R_2 d\psi_2(-sin\psi_2\, \hat{\imath} + cos\psi_2 \hat{\jmath})$$

now the dot product between these 2; $dS_1 . dS_2$, so dS_1 and dS_2 are also vector quantities, will simply

$$dS_1 . dS_2 = R_1 R_2 d\psi_1 d\psi_2 (sin\psi_1\, sin\psi_2\ + cos\psi_1 cos\psi_2)$$

And this can be simplified as

$$dS_1 . dS_2 = R_1 R_2 \cos(\psi_1 - \psi_2)\, d\psi_1 d\psi_2$$

this is the dot product of path vector dS_1 and dS_2, now S, the distance between 2 points at peripheries of disc 1 and disc 2 is simply written as

$$S = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

now $z_1 - z_2 = h$

so we can just write this as

$$S = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (h)^2$$

and this can be simplified, if we put the value of x and y, this can be simplified as

$$S = \sqrt{(h)^2 + R_1^2 + R_2^2 - 2R_1 R_2 \cos(\psi_1 - \psi_2)}$$

So, this is going to be and then we have to take under root, so this is going to be the distance between 2 disc; points of two disc, so this is disc 1, any point on this disc and any point on this disc separated by a distance S will have this value,

$$S = \sqrt{(h)^2 + R_1^2 + R_2^2 - 2R_1R_2\cos(\psi_1 - \psi_2)}$$

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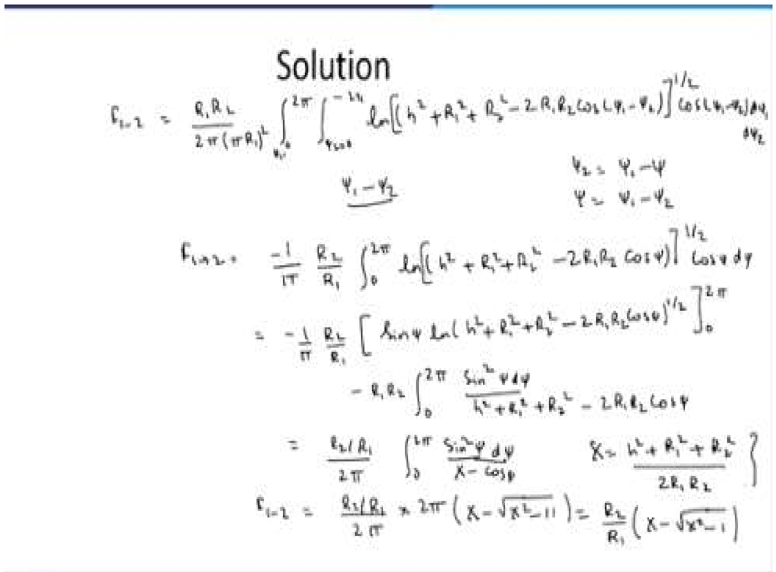

Now, the view factor

$$F_{1-2} = \frac{R_1R_2}{2\pi(\pi R_1^2)^2}\int_0^{2\pi}\int_{\psi_2=0}^{-2\pi} \ln[(h^2 + R_1^2 + R_2^2 - 2R_1R_2\cos(\psi_1 - \psi_2)]^{\frac{1}{2}}\cos(\psi_1 - \psi_2)\, d\psi_1\, d\psi_2$$

so this is going to be the expression of view factor, okay.

Now, it looks very terrific, the integration looks very terrific, okay those who are expert in mathematics can solve this integration very easily, there are integration tables available, one thing that one should observe is this integration can be simplified very easily because the only thing that appears as a variable is basically, the difference; $\psi_1 - \psi_2$, okay, nowhere we have $\psi_1$and ψ_2 appearing independently in this integrand, okay.

So, we can basically integrate this easily, so we write, just integrating and change of variables, so we write

$$\psi = \psi_1 - \psi_2$$

just a change of variables we have done, so we just reduce this integration from double integral to single integral like this,

$$F_{1-2} = \frac{-R_2}{\pi R_1}\int_0^{2\pi} \ln[(h^2 + R_1^2 + R_2^2 - 2R_1R_2\cos(\psi)]^{\frac{1}{2}} \cos\psi\, d\psi$$

So, by changing this substitution we have just reduce the level of order of integrating by 1 and this will be solved using integration by parts, we will just the write down the steps here;

$$F_{1-2} = \frac{-R_2}{\pi R_1}\left[sin\psi \ln(h^2 + R_1^2 + R_2^2 - 2R_1R_2\cos\psi)^{1/2}\right]_0^{2\pi}$$

$$- R_1R_2\int_0^{2\pi} \frac{sin^2\psi d\psi}{h^2 + R_1^2 + R_2^2 - 2R_1R_2\cos\psi}$$

So we just apply the integration by parts;

$$F_{1-2} = \frac{-R_2}{2\pi R_1}\int_0^{2\pi} \frac{sin^2\psi d\psi}{X - \cos\psi}$$

Here $X = \frac{h^2+R_1^2+R_2^2}{2R_1R_2}$

So, now again we have one more integration to solve but this integration is relatively easy to solve and what we get as a final result for the view factor is

$$F_{1-2} = \frac{R_2}{R_1}\left(X - \sqrt{X^2 - 1}\right)$$

where X is the having this value, so this is how you can apply the contour integration technique to find out complicated view factors between the standard plate configurations, okay.

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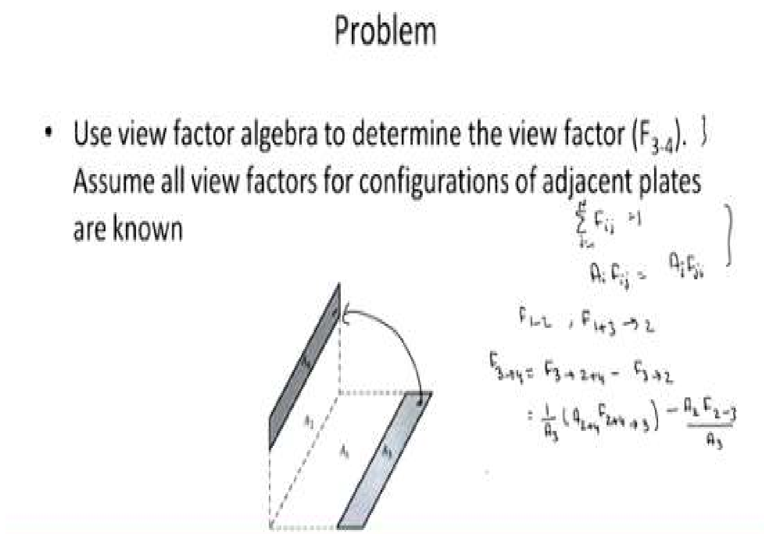

Now, we will do one more problem, okay, this problem basically involves simple algebra; view factor algebra, we already discussed this algebra that means, summation of

$$\sum_{j=1}^{N} F_{ij} = 1$$

and $$A_i F_{ij} = A_j F_{ji}$$

so these are the rules for view factor that we have studied, we apply this rules to find out the view factors between this configuration 3 and 4, okay, so we have to find out the view factors F_{34}.

Now, when we solve this problem, we assume that the view factors adjacent plates are known that means, we know F_{12}, this is known, we know that $F_{1+3\to 2}$ is known, okay, so these are adjacent plates, they may have same area or they may not have same area, so by assuming that the view factors for the adjacent plates are known, we can find out the view factors between known adjacent plates 3 and 4 using this view factor algebra.

So, we write $$F_{3\to 4} = F_{3\to 2+4} - F_{3\to 2}$$

this is simply the summation rule,

$$F_{3\to 4} = \frac{1}{A_3}(A_{2+4}F_{2+4\to 3}) - \frac{A_2 F_{23}}{A_3}$$

So, we have used both the summation rule and the reciprocity theorem to find out $F_{3\to 4}$, now, $F_{2+4\to 3}$ is also not known, so let us again look at these quantities.

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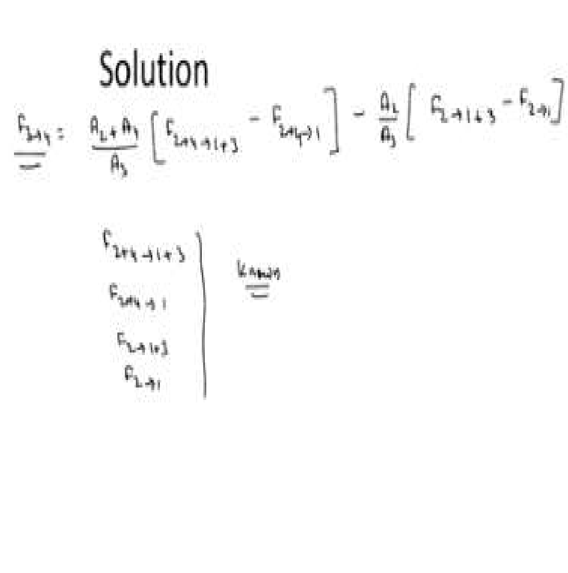

$$F_{3\to4} = \frac{A_2 + A_4}{A_3}\left[F_{2+4\to1+3} - F_{2+4\to1}\right] - \frac{A_2}{A_3}\left[F_{2\to1+3} - F_{2\to1}\right]$$

So, all these quantities as we started with are known okay, these values are available at the back of the book, you can easily read this values, okay and based on this view factor algebra, we can; we could find the view factor $F_{3\to4}$ for the plates which are not adjacent to each other, okay.

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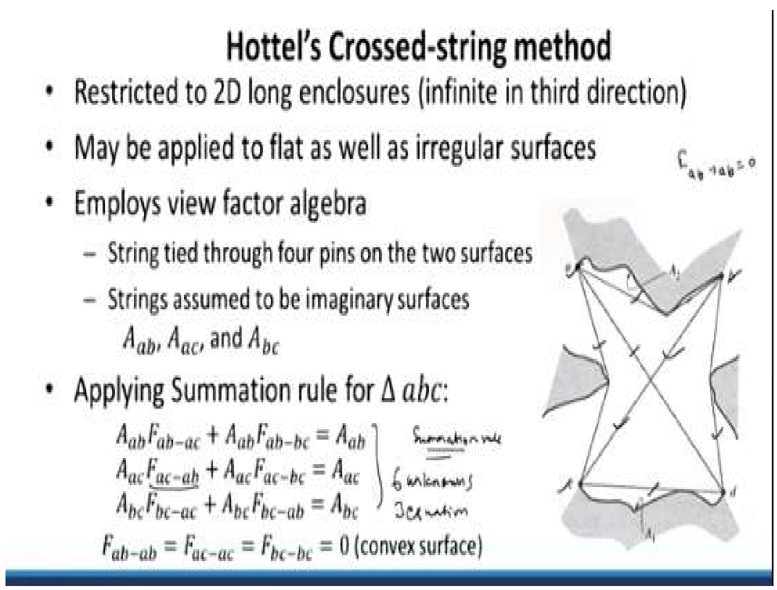

Now, we will discuss the other method; the Hottel-Crossed string method, this method is very powerful and very simple in calculating the view factors, the only condition is that it is restricted to 2 dimensional enclosures that means the surface is for which the view factor is being calculated have to be infinitely long in the third dimensions, okay, for this type of surfaces, we can calculate the view factor using Hottel Crossed-string method, okay.

Now, the surfaces may be flat or they may have an irregular, geometry, for example we want to calculate the view factor as shown in this figure, the surface A_1 is also irregular, and surface A_2 is also irregular. So, how do we find, what we do in this method is; we put 4 pins, now this pins are called *ABCD*, oaky, so these pins; 2 pins on the surface A_2 and 2 pins on the surface A_1, okay.

And we put wire or a thread, okay, now we call it string, because the method is called crossed string method, so we have this one string *AC*, okay, another string is tied between *A* and *D*, similarly *A* and *B*, *C* and *D* and *C* and *B*, okay, so we have put 6 strings, okay, total 6 strings are put, okay and these strings are assume to be tied to each other; tied to the pins, they are tight and they are convex in shape, okay.

So, once we have this, we assume that the strings are representing basically an imaginary surface, so for example, string *AC*, you can think of it as in the surface that is extending in the third direction, so you just the assume it to be a surface, so just by measuring the length of this strings, the method can calculate the view factors okay, so the method does not involve any complex mathematics.

It basically need some measurements of the string lengths which for standard configuration is pretty easy and it needs some view factor algebra, so what we do is; we have the summation rule; the summation rule is applied to the various view factors, so for example the triangle *abc*; triangle *a, b* and *c*, we have this surface *ab*, so

$F_{ab\text{->}ac} + F_{ab\text{->}bc}\ F_{ab\text{->}bc}$; *bc* is this length and $F_{ab\text{->}ac} = 1$ or *ab*, we have multiplied by area *ab* throughout.

$$F_{ab\text{->}ab}=0$$

because the surfaces are convex , so

$$F_{ab\text{->}ab} = 0$$

so we have applied the summation rule here, similarly we apply this summation rule to surface *ac* and surface *bc*, okay, so we get 3 equations by applying the summation rule to the 3 sides of the triangle *abc*.

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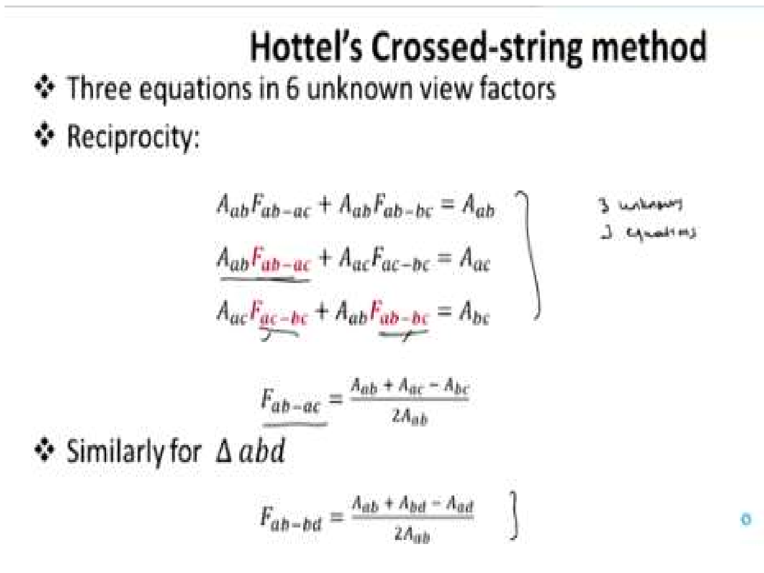

Similarly, what we do is; we apply the method to the triangle *abd*, okay, we apply the method; similar method to the triangle *abd*, okay, now we; so, in this 3 equations, we have 6 unknowns and we have only 3 equations, okay, so we need some more relations and these relations are basically found using the reciprocity theorem okay, so reciprocity theorem basically, we replace these view factors in red, okay.

So, for example in the equation, we have $F_{ac\text{->}ab}$, so we have replace it with $F_{ab\text{->}ac}$, okay, similarly we have replaced these 3 view factors using the reciprocity theorem, now these 3 equations, we have 3 unknowns and 3 equations, we use a simple algebra to solve this and what we get is the view factor

$$F_{ab-ac} = \frac{A_{ab} + A_{ac} - A_{bc}}{2A_{ab}}$$

and same thing we do in the triangle *abd*.

So, we get view factor

$$F_{ab-bd} = \frac{A_{ab} + A_{bd} - A_{ad}}{2A_{ab}}$$

so just by applying the summation rule and reciprocity theorem, the 2 triangles, we have calculated the view factor; *ab-ac* and *ab-bd*, okay, now again we applied the summation rule to the triangle, okay, so we have *abcd* quadrilateral, we apply the summation rule.

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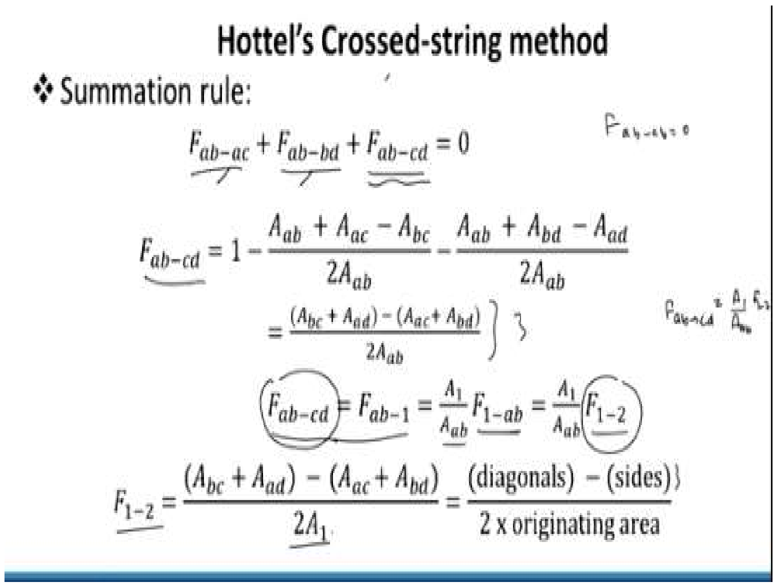

And we get another relation;

$$F_{ab-a} + F_{ab-b} + F_{ab-cd} = 0$$

so we again apply the summation rule okay, now this time not to the triangle but to the quadrilateral and the expression for

F_{ab-bd} is already found, F_{ab-ac} is already found, so these 2 expression we have already found okay, now we basically find; solve for *abcd*, okay, this is the unknown here and when we put the values of F_{ab-ac} and F_{ab-bd} in this relation, we can simplify this, F_{ab-cd}.

So, all the view factors are now known okay, so between F_{ab-cd}, okay what is; let us look at it what is *ab-cd*, so, *ab* is this surface and *cd* is this surface, okay, so this does not represent the actual surface, this represent the string surface over this real surface, okay so we have found the view factor F_{ab-cd} represented by imaginary stringed surface okay over the real surface okay, now we have convert this view factor to the real surface.

Now, the argument here is that whatever radiation leaving *ab*, all the radiation leaving *ab* that falls on surface *cd* will actually go to surface A_1, right, so we have radiation emitted from surface *ab* falling on surface *cd*, the string surface and the string surface is tied in such a way that all the radiation that falls on *cd* will actually fall on A_1, okay, so based on this argument,

$$F_{ab-cd} = F_{ab-1}$$

And then applying the reciprocity theorem, we write

$$F_{ab-cd} = \frac{A_1}{A_{ab}} F_{1-ab}$$

now similar arguments hold true for these surface, all the radiation leaving surface A_1 and travelling to surface *ab* will actually fall on A_2, so we can also write

$$F_{1-ab} = F_{1-2}$$

So by using this reciprocity theorem, we have related $F_{ab\text{-}cd}$ to F_{12} and the relation is basically

$$F_{ab-cd} = \frac{A_1}{A_{ab}} F_{1-2}$$

So, finally we get F_{1-2} =; substituting the value of this expression in this relation, so we put the value of F_{ab-cd} in this expression, we get F_{1-2}, the view factor as

$$F_{1-2} = \frac{(A_{bc} + A_{ad}) - (A_{ac} + A_{bd})}{2A_1} = \frac{(\text{diagonals}) - (\text{sides})}{2 \text{ x originating area}}$$

which are the basically, diagonals, so some of diagonal lengths, so you have to measure or you should know the length of the string in the diagonal direction – A_{ac} + A_{bd} which is the sides of the quadrilateral, okay.

So, you should know the sides, okay either by measurement or by standard configuration and then 2 divided by A_1; A_1 is the originating area, so this is the hottel's crossed string method as you see that this is a very simple method just by using the view factor algebra, summation rule and the reciprocity theorem, we have calculated the view factor between 2 surfaces which are not represented by any mathematical religion, which are complex in geometry, okay.

Now, to demonstrate the power of this method people, we will solve the same problem that we solved using the area integration method and we found that area integration method involves complicated integration and algebra okay, so we will solve this problem using this method.

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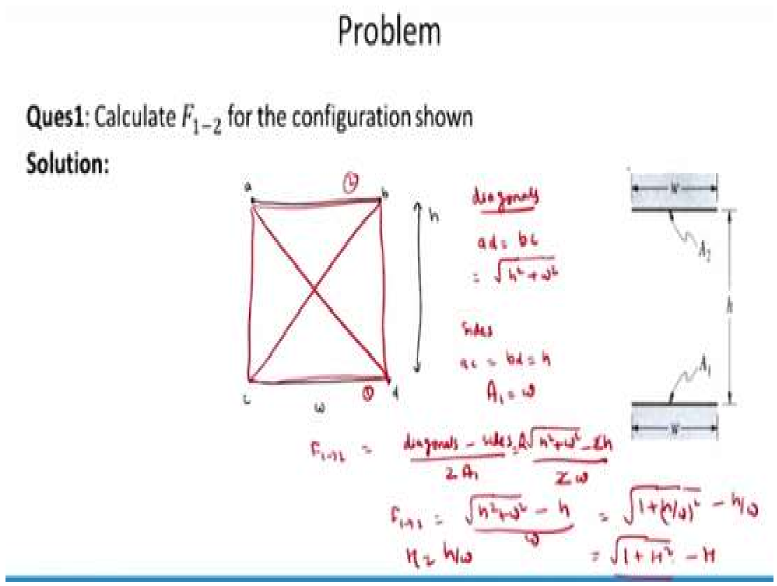

So, we have again as shown 2 plates, the width of each plate is w and the separation distance is basically the h, this is the simple configuration for this method but complicated problems can be solved easily. So, what we do is; we put 4 pins; pin a, pin b, pin c and pin d, okay, so we have put 4 pins, then we tied 6 strings, one string, another string, okay let me just put it using different colour.

So, this is one string, this is another string, this is one string, this is another string and 2 string will be just parallel to the plates, so this six 6 we have put, so the diagonals have ad and bc have same length, okay and the length is basically given by root, this distance is h and width w,
Diagonal = $\sqrt{h^2 + w^2}$, this is the length of the diagonals and the sides $ac = bd$ is simply = h, okay.

Now, as per the formula F_{12}, this is surface 1, this is surface 2,

$$F_{1\to2} = \frac{diagonal - sides}{2A_1}$$

where A_1 is simply = w, assuming the depth of the plates in the plane of the board is 1, so

$$F_{1\to2} = \frac{2\sqrt{h^2 + w^2} - 2h}{2w}$$

so 2 will cancel out from the numerator and denominator, so we get

$$F_{1\to2} = \frac{\sqrt{h^2 + w^2} - h}{w}$$

And this becomes

$$F_{1\to 2} = \sqrt{1 + (h/w)^2} - h/w$$

$$F_{1\to 2} = \sqrt{1 + (H)^2} - H$$

so we have already found this expression using the area integration method that involves complicated double integration and the expression that we found for the view factor, if you can recall was similar,

$\sqrt{1 + (H)^2} - H$ where H is the ratio of the separation distance divided by width of the plates and this method can very easily give you the view factor for complicated geometries as well.

So, the method is very powerful with the only limitation that the plates have to be infinite in depth, if you have finite length plates, the method may not be applied. So, thank you very much, this basically, concludes one important topic on view factor evaluation, we still have a number of methods for the evaluation of view factor namely the inside sphere method and the Monte Carlo method that we will take in the subsequent chapters, thank you.