Radiative Heat Transfer
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Module - 7
Lecture - 35
k-Distribution Model

Hello friends, in this lecture we will focus on one of the most advanced spectral model in radiative heat transfer, that is the k-distribution model. k-distribution model is a class of models basically, where we map the spectrum into a monotonic function, as we will see. Ultimately, we are representing the entire spectrum. It is basically a global model. And we solve radiative transfer equation using gray gases and weights, just like we did in the weighted-sum-of-gray-gas model.

It has been proved by Modest that weighted-sum-of-gray-gases is nothing but a crude approximation to k-distribution model. Where k-distribution model takes the weighted-sum-of-gray-gasses model to another degree of accuracy. So, let us go into the details. So, the first thing that you should observe is the spectrum.

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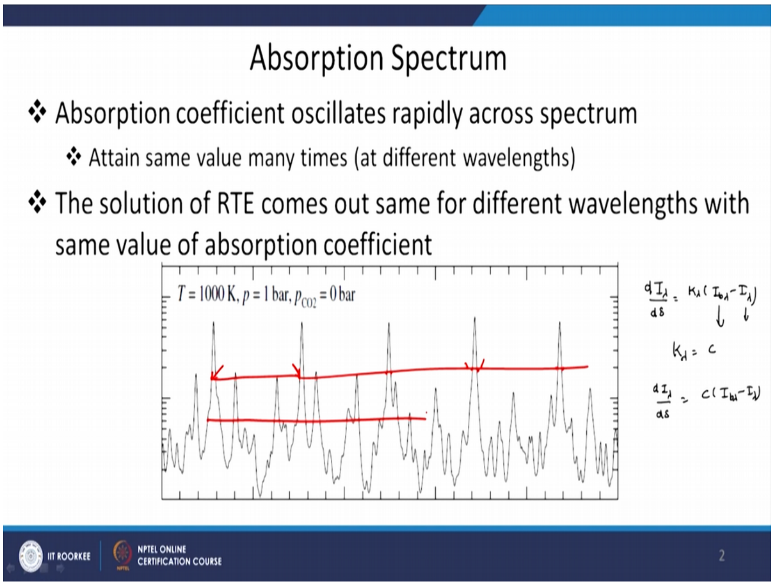

Absorption Spectrum

- Absorption coefficient oscillates rapidly across spectrum
  - Attain same value many times (at different wavelengths)
- The solution of RTE comes out same for different wavelengths with same value of absorption coefficient

$$\frac{dI_\lambda}{ds} = \kappa_\lambda (I_{b\lambda} - I_\lambda)$$

$$\kappa_\lambda = c$$

$$\frac{dI_\lambda}{ds} = c(I_{b\lambda} - I_\lambda)$$

On this slide you see a spectrum of carbon dioxide at atmospheric pressure and 1,000 kelvin typical of combustion applications. And what you see is basically a small range of wavelengths. There will be large number of bands and lines. What you see is a small part of the spectrum, a narrow band rather. And you see that absorption coefficient goes up and down very erratically and continuously.

So, it is called erratic spectrum. What we observe is that in the RTE dI lambda by ds is = kappa lambda I b lambda – I lambda. This is our radiative transfer equation. I b lambda is the black body emissive power which depends on temperature and it does not depend on the absorption coefficient. And I lambda is the unknown variable. So, what we conclude from here is that for all wavelengths where kappa lambda is same, let us say for all wavelengths kappa lambda is C.

If you replace this r in RTE dI lambda by ds is = C I b lambda – I lambda, then we get the same solution. So, for all wavelengths where kappa lambda is constant, we get the same solution. Of course, the effect of Planck function will be felt. Let us say we have 2 wavelengths we pick 2 wavelengths, this 1 and this 1 having the same magnitude of kappa lambda. So, we have picked 2 wavelengths kappa lambda is same.

And over this small wavelength, I b lambda is also not changing very much. So, we get same emission and same absorption. So, we get same intensity. So, for these 2 wavelengths, because absorption coefficient is same and Planck function does not change. So, we get the same answer for the intensity. So, a question we should ask to ourselves: Why to solve radiative transfer equation for these 2 wavelengths separately?

Why not solve only once, because we get the same solution? This point gives the same solution as this point. So, why do we solve for radiative transfer equation again for these points. And we have many number of points. If we increase this line all the way, we have many points where kappa lambda is same. So, for all these wavelengths, provided we can account for change in emission.

Because I b lambda will actually vary over the spectrum. If we can account for variation in I b lambda for all the wavelengths where kappa lambda is same, we can solve the radiative transfer equation only once. So, this is called reordering. So, what we do is, we will not solve the radiative transfer equation in wavelength space, we will solve radiative transfer equation in absorption space.

So, for all values of kappa lambda constant, we will solve only 1 RTE. 1 RTE for these coefficients and 1 RTE may be for these coefficients and so on. So, we will solve varied 1 radiative transfer equation for different levels of absorption coefficient. And this concept is called reordering.

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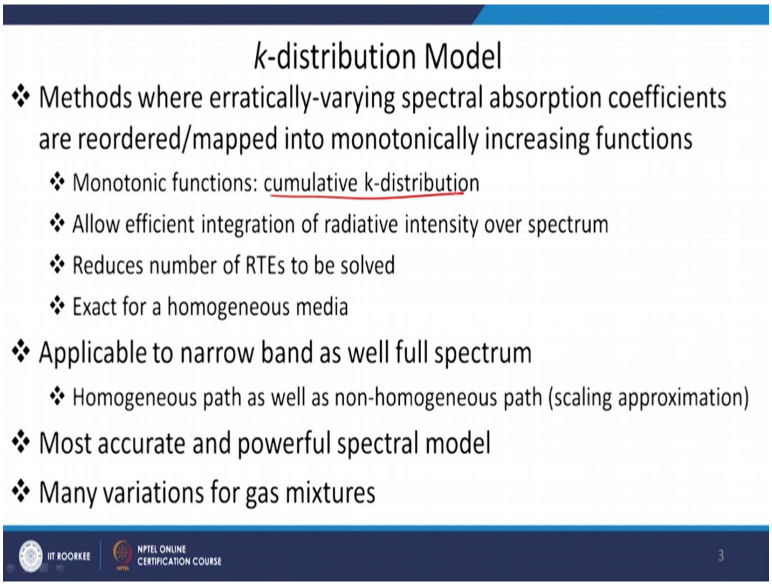

k-distribution Model

- Methods where erratically-varying spectral absorption coefficients are reordered/mapped into monotonically increasing functions
  - Monotonic functions: cumulative k-distribution
  - Allow efficient integration of radiative intensity over spectrum
  - Reduces number of RTEs to be solved
  - Exact for a homogeneous media
- Applicable to narrow band as well full spectrum
  - Homogeneous path as well as non-homogeneous path (scaling approximation)
- Most accurate and powerful spectral model
- Many variations for gas mixtures

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So, they are number of methods. In fact, k-distribution is a class of methods, where erratically varying spectral coefficient is reordered. We will understand what is reordering. But, I tried to explain you in a very crude form that we are solving radiative transfer equation not in wavelength space. Rather, we are solving it in absorption space. 1 RTE we are solving for each value of kappa lambda, constant kappa lambda on all wavelengths.

This is called reordering. And reordering but what basically does is, it converts the function, the erratic function into a cumulative k-distribution, a monotonically increasing smooth function. We will see how this maps the spectrum into a smooth function. And it allows efficient integration of radiative intensity over the spectrum. So, this method is very analogous to weighted-sum-of-gray-gases.

Where is weighted-sum-of-gray-gases we represented the gas mixture, the non-gray gas mixture using sample of gray gases. Here also, we are basically representing it in some in the form of gray gases. Where gray gas represents a level of absorption coefficient. So, each level of absorption coefficient basically represents a gray gas. So, the exact, the method is very efficient, only few RTEs need to be solved, 7 to 8.

However, the method is much superior to weighted-sum-of-gray-gases, because it is exact for homogeneous media. We will use absorption coefficient as our variable rather than emissivity data used in the weighted-sum-of-gray-gases model. So, you please recall. In weighted-sum-of-gray-gases model, we used emissivity data to find our parameters, while the k-distribution model is directly based on absorption coefficient itself.

The method can be applied to a small narrow band, as well as to full-spectrum. Narrow band, the advantage is, the Planck function does not vary over the wavelength. So, emission variation is not taking place. While on the full-spectrum basis, the emission coefficient is going to change the Planck function is going to change. So, in the definition of k-distribution, we have to take Planck body Planck function variation into account.

So, we will apply this method in this lecture to the full-spectrum, to understand how this method basically takes into account the variation of Planck function as well. There are many variations of this method applied to gas mixtures such as multi-spectrum, k-distribution model, multi-group k-distribution model. Here we will study only the simplest of these k-distribution model, that is full-spectrum k-distribution model.

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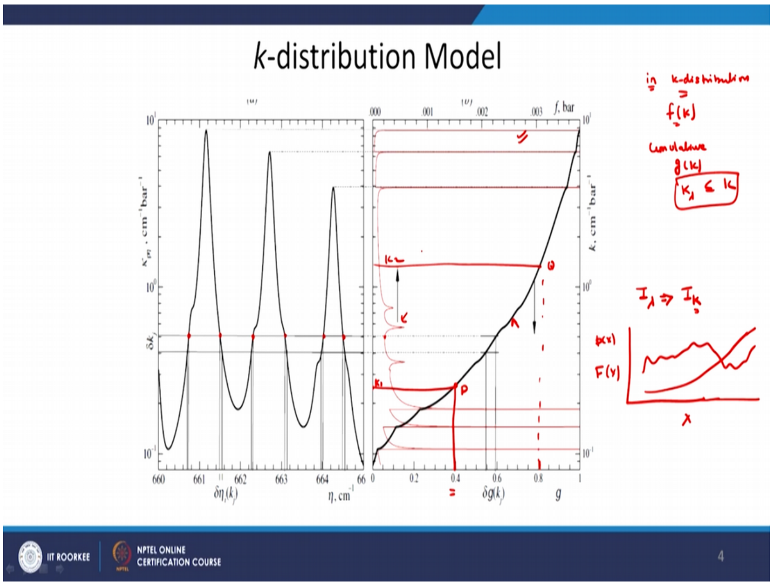

k-distribution Model

This image will explain you all the things you need to know about the k-distribution model. As I said, for each value of kappa lambda, there is a unique solution. So, let us point out these points where the value of absorption coefficient is same. So, all the points that have highlighted, they have same magnitude or absorption coefficient. And they return the same result for intensity.

So, for these intensity, for these points, only for these points that I am highlighting, intensity is not a function of wavelength, rather it is just a function of k, where lambda is spectral coefficient. So, ideally, for each wavelength or wavenumber, the intensity should vary. But because the absorption coefficient is same, we write that intensity is just a function of k. Where k is the magnitude of absorption coefficient at these wavelengths that I have highlighted.

So, what we do is, we represent this, we map it on this red curve. So, this red curve basically represents the mapping of the solution or k-distribution. So, this is called reordering. The red curve here represents the reordered function, where all the unique k points, unique absorption coefficient points have been mapped. So, the red curve is called k-distribution. Now, you see that, although the original spectrum was very erratic, the absorption coefficient was going up and down significantly, but this spectrum looks no better.

Even the k-distribution looks very erratic. At few places, the values are very large. But still there is are variations. So, integrating over this spectrum is also very difficult. So, we will not try to attempt, we will not try to solve radiative transfer equation on this erratically varying k-distribution also. What we will do is, we will go for a cumulative distribution. If we take a cumulative distribution, so remember in probability density function, you have a probability function which may be erratic.

But you also have a cumulative distribution. For example, if I have to plot versus x, the probability distribution may look like this. But the cumulative probability function f x will always vary from 0 to 1 and it will be smooth. So, this point, we have discussed while discussing the Monte Carlo method also. So, the probability distribution or the spectral k-distribution may be erratic.

But, if you take the cumulative distribution, then it will not be erratic. And what is the difference? In k-distribution or in k-distribution, we represent it at a given value f, which is the probability. While in cumulative k-distribution g k we represent < = k. So, that is the difference between the k-distribution and cumulative k-distribution. k-distribution talks about all the values of kappa where kappa is = k.

And cumulative k-distribution talks about all the values of lambda, where kappa lambda is < = k. So, all the values of lambda where kappa lambda is < = k. That is basically the cumulative k-distribution. And you see, this black curve varies very smoothly. So, what this curve represents? At this point, this point basically represents the fraction of spectrum having <.

So, if I just point here; so, what it says is, that 40% of the spectrum has values, k values < this value. Just in normal cumulative distribution function, this particular points, let us call this point p. So, at point p, 40% of the spectrum has k values < this value. Let us call this K 1. Similarly, let us call this point Q. At point q, 80% of the spectrum has absorption coefficient < this value K 2.

So, this is how basically we have mapped the spectrum from erratically varying spectrum to a smoothly varying cumulative k-distribution function. And we will solve the radiative transfer equation on this curve. So, this curve we will solve the RTE on.

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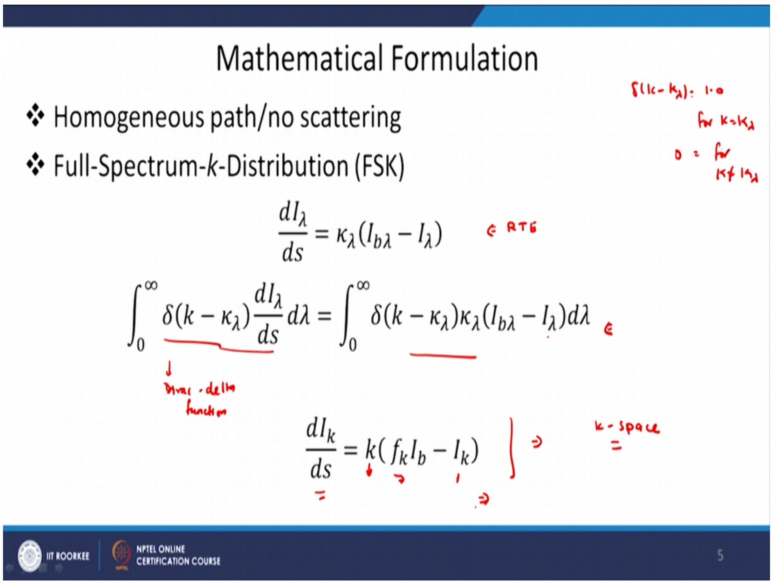

Mathematical Formulation

- Homogeneous path/no scattering
- Full-Spectrum-*k*-Distribution (FSK)

$$\frac{dI_\lambda}{ds} = \kappa_\lambda (I_{b\lambda} - I_\lambda) \quad \leftarrow \text{RTE}$$

$$\int_0^\infty \underbrace{\delta(k - \kappa_\lambda)}_{\text{Dirac-delta function}} \frac{dI_\lambda}{ds} d\lambda = \int_0^\infty \delta(k - \kappa_\lambda)\underline{\kappa_\lambda}(I_{b\lambda} - I_\lambda) d\lambda \leftarrow$$

$$\frac{dI_k}{ds} = k(f_k I_b - I_k) \Big\} \Rightarrow \text{k-space}$$

$\delta(k-\kappa_\lambda) = 1.0$ for $k = \kappa_\lambda$

$0 =$ for $k \neq \kappa_\lambda$

So, let us look at the mathematical formulation of this method. This is our RTE. For a homogeneous path, we will derive. But the method is not limited to homogeneous path. It can be applied to non-homogeneous path also. So, we will apply the, we will develop the mathematics of this method on homogeneous path with no scattering. The method is called full-spectrum, because we will integrate over the entire spectrum.

We will not limit ourselves to a band, a narrow band or wide band. We will integrate over the entire spectrum. So, what I will do is, multiply the RTE by a function called Dirac delta function. So, this is called Dirac delta function. Now, Dirac delta function is a discrete function, where delta k – kappa lambda is = 1, for k is = kappa lambda. And it is = 0, for k not = kappa lambda.

So, this is called Dirac delta function. And this function, we multiply the RTE with. And then, we integrate over the entire spectrum, both sides, left-hand side and right-hand side. Now, once we multiply this RTE by this function, so we understand that all the wavelengths where kappa lambda is not = k, the RTE simply vanish. So, our left-hand side reduces to dI k by ds. Because, all the wavelengths where kappa lambda is not = k, they will simply with 0.

And all we are left with is dI k by ds. So, the integral over the wavelength space has been converted into a simple I k term. And similarly, on the right-hand side, we have, wherever kappa lambda is = k, kappa lambda will be = k, we just write I lambda as I k and I b lambda multiplied by delta k – kappa lambda integrated over 0 to infinity is written as f k. This is called k-distribution. Let me give you the definition of this f k.

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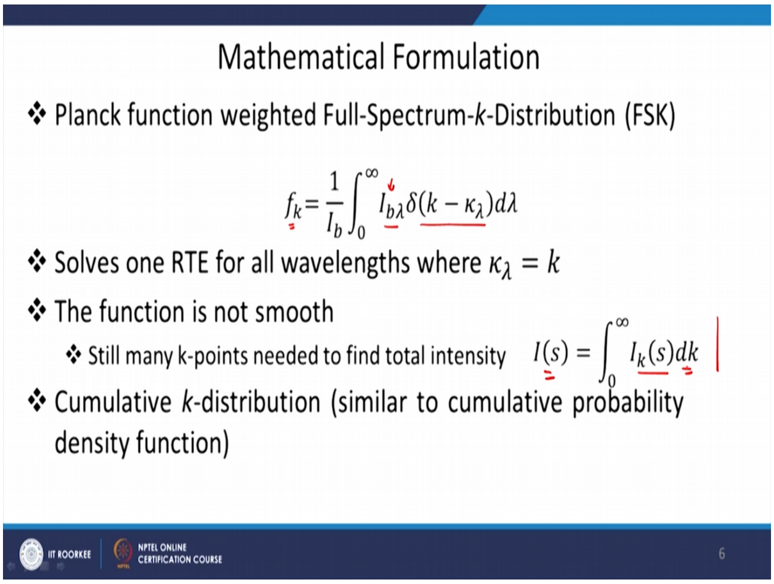

Mathematical Formulation

- Planck function weighted Full-Spectrum-*k*-Distribution (FSK)

$$f_k = \frac{1}{I_b}\int_0^\infty \underline{I_{b\lambda}}\,\underline{\delta(k-\kappa_\lambda)}\,d\lambda$$

- Solves one RTE for all wavelengths where $\kappa_\lambda = k$
- The function is not smooth
  - Still many k-points needed to find total intensity $I(s) = \int_0^\infty \underline{I_k(s)}dk$
- Cumulative *k*-distribution (similar to cumulative probability density function)

So, f k is defined as Planck function weighted full-spectrum-k-distribution. So, Planck function weighted full-spectrum-k-distribution f k is defined as 1 upon I b integral from 0 to infinity I b lambda delta k – kappa lambda d lambda. So, this has been assumed here in this equation and our equation basically simplifies this. Now, this is in k-space. So, we have transformed our equation from lambda space to k-space.

And as I mentioned, the k-space is also very erratic. So, we will not solve this equation in k-space. Rather we will solve in cumulative k-distribution space. But the definition of k-distribution is given by this function, f k is = 1 upon I b 0 to infinity I b lambda delta k – kappa lambda. So, we are solving 1 RTE for all values of lambda where kappa lambda is = k. One, you should also observe here that the Planck function is basically coming in the definition of k-distribution.

So, the variation of k Planck function, the variation of emission is already taken into account. On a narrow band basis, Planck function will be constant. So, we do not have to worry about its variation. But, in the k-distribution, full-spectrum-k-distribution the Planck function appears in the integral. So, the variation of this function is already taken into account. So, we have to integrate now intensity, not in wavelength, but rather in k-space.

So, i k s dk. So, again this function is not very easy to integrate as the k-space function also varies very erratically. So, what we will do is, we will transform it into cumulative k-distribution.

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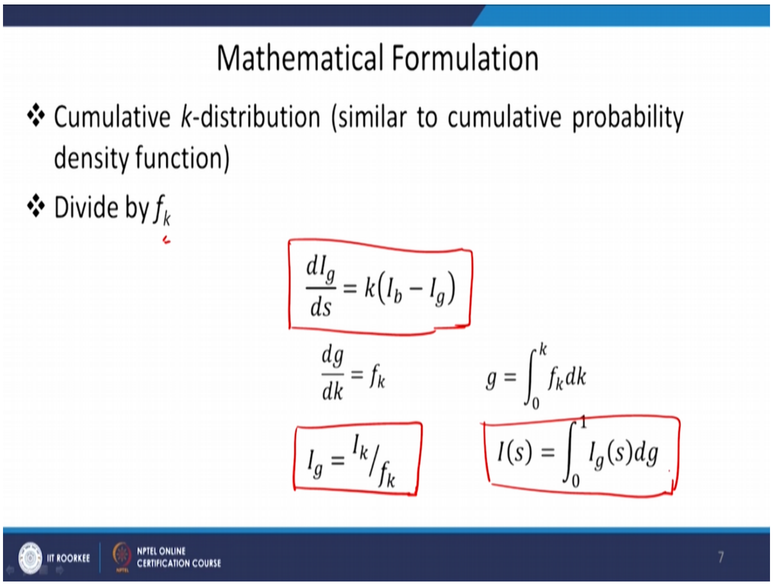

Mathematical Formulation

- Cumulative *k*-distribution (similar to cumulative probability density function)
- Divide by f_k

$$\boxed{\frac{dI_g}{ds} = k\left(I_b - I_g\right)}$$

$$\frac{dg}{dk} = f_k \qquad g = \int_0^k f_k dk$$

$$\boxed{I_g = {}^{I_k}/_{f_k}} \qquad \boxed{I(s) = \int_0^1 I_g(s)dg}$$

So, it is very easy to do it. All we have to do is, just divide this equation. Divide this equation by f k. So, what we get is, dI g by ds is = k times I b – I g. So, we have divided by f k the RTE in k-space, we have divided by f k. And we define I g as I k upon f k. This function is going to be relatively smooth. Because, we have divided by the erratic function f k, this function I g is going to be very very smooth.

And we can use efficient quadrature to integrate this function. So, the RTE, now finally in cumulative k-distribution g is given by dI g by ds. Where k-distribution in g space is going to be very very smooth. We defined dg by dj is = f k. So, how to go from cumulative k-distribution to k-distribution. The derivative dg by dj is = f k or g is = 0 to k f k dk. And cumulative k-distribution, cumulative function always varies from 0 to 1. So, intensity is simply = 0 to 1 I g ds, where we are solving the intensity in g space. And then integrating from 0 to 1.

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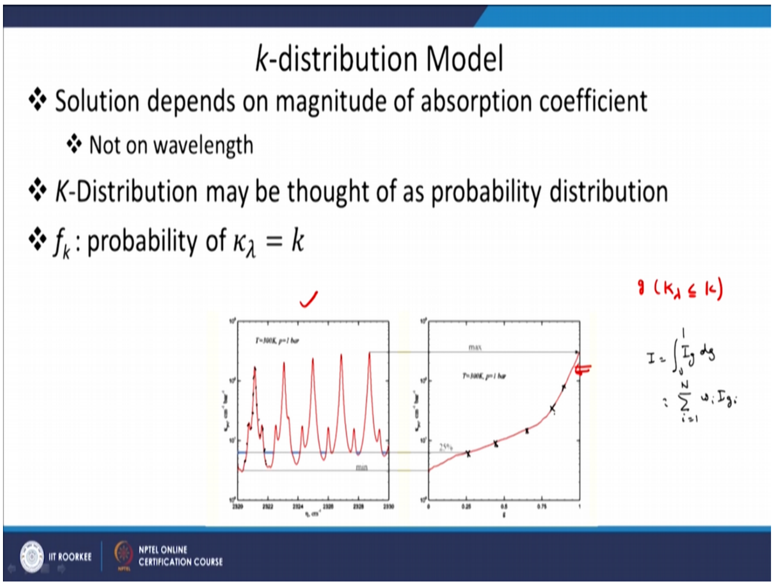

k-distribution Model

- Solution depends on magnitude of absorption coefficient
  - Not on wavelength
- K-Distribution may be thought of as probability distribution
- f_k : probability of $\kappa_\lambda = k$

$g\ (\kappa_\lambda \le k)$

$$I = \int_0^1 I_g\, dg \approx \sum_{i=1}^{N} \omega_i I_{g_i}$$

So, again, to summarize this we have, we started with a spectrum. So, this is the spectrum which varies erratically over wavelength space. We have mapped this spectrum in k-distribution, which is cumulative function. Okay. Each point on this spectrum basically gives you the probability of kappa lambda < = k. So, f k is probability function. While g is the cumulative probability function.

So, g gives you probability of kappa lambda < = k. That is the cumulative probability or cumulative function. We will solve this RTE for some points on this curve. So, we will solve our RTE on few points only on this curve. And then, use quadrature to integrate our intensity. So, we do not have to integrate along this wavelength space by representing this wavelength by millions of thousands of points over wavelength.

All we have to do is map the spectrum into a smooth function and then solve radiative transfer equation over few points on this RTE. And then integrate the intensity as simply Ig dg from 0 to 1 or simply using quadrature, we can write as i is = 1 to N w i I gi, where w i and I gi are weights and intensity at a selected quadrature points in this curve.

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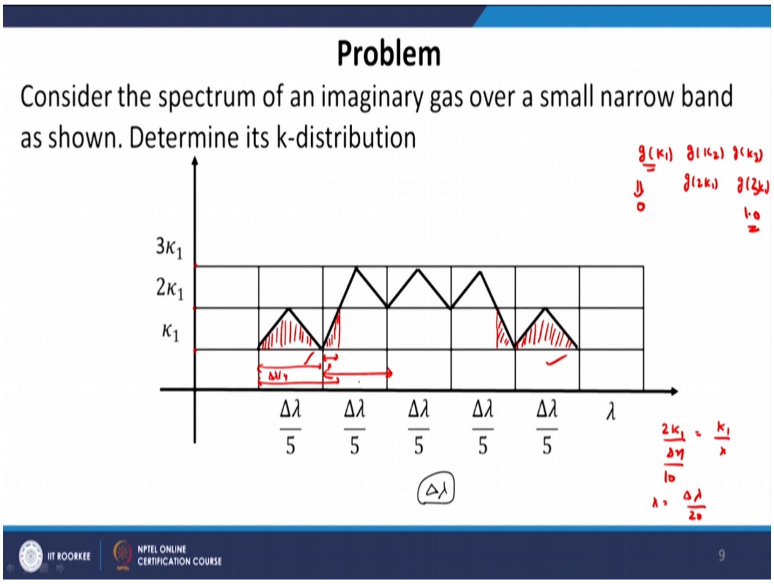

Problem

Consider the spectrum of an imaginary gas over a small narrow band as shown. Determine its k-distribution

Let us solve 1 problem and understand how the k-distributions are generated. Basically, this allows you to, this is a simple mathematical problem where we have taken a simplified spectrum. The spectrum looks like this. We have a small narrow band. The total width of this narrow band is delta lambda. And the absorption coefficient varies from kappa 1 at the lowest level to 3 kappa 1 at the highest level.

So, small narrow band, the spectrum, the absorption spectrum is given here, a simplified absorption spectrum. This not realistic. This is just a fictitious imaginary spectrum. We have to generate k-distribution for this or we have to map this spectrum from wavelength space to cumulative function. So, let us see how we can do it.

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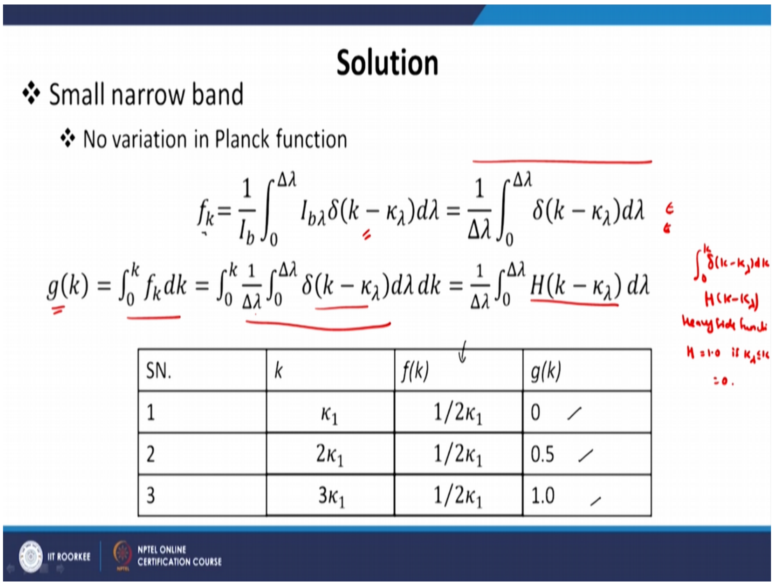

Solution

- Small narrow band
  - No variation in Planck function

$$f_k = \frac{1}{I_b}\int_0^{\Delta\lambda} I_{b\lambda}\,\delta(k-\kappa_\lambda)\,d\lambda = \frac{1}{\Delta\lambda}\int_0^{\Delta\lambda} \delta(k-\kappa_\lambda)\,d\lambda$$

$$g(k) = \int_0^k f_k\,dk = \int_0^k \frac{1}{\Delta\lambda}\int_0^{\Delta\lambda} \delta(k-\kappa_\lambda)\,d\lambda\,dk = \frac{1}{\Delta\lambda}\int_0^{\Delta\lambda} H(k-\kappa_\lambda)\,d\lambda$$

$\int_0^k \delta(k-\kappa_\lambda)\,dk$

$H(k-\kappa_\lambda)$

heavy side functi

$H = 1.0$ if $\kappa_\lambda \le k$

$= 0.$

SN.	k	$f(k)$	$g(k)$
1	κ_1	$1/2\kappa_1$	0 /
2	$2\kappa_1$	$1/2\kappa_1$	0.5 /
3	$3\kappa_1$	$1/2\kappa_1$	1.0 /

We start with the definition of k-distribution f k. So, f k as I have already given you the formula is given by 1 upon I b 0 to delta lambda I b lambda delta derived delta function k – kappa lambda d lambda. So, this is the definition of the k-distribution. Now, we are dealing with only a narrow band. And over this narrow band, the Planck function is not going to vary much. So, we can take it out of the integral.

And our k-distribution basically reduces to this formula, 1 upon delta lambda 0 to delta lambda del k k – kappa lambda d lambda. So, this is the simple integral. Now, we will not, we are not interested in calculating f k. Although I will give you the values of f k. But f k is not easy to evaluate. And the RTE is very difficult to solve in k-space. As I said, it is erratic. We will find cumulative function.

So, cumulative function is defined as 0 to k f k. So, all the values of kappa < = k. So, we have to integrate the function f over 0 to k. So, we substituted the value of f k here. Change the integration. Take 0 to k inside and 0 to delta lambda outside. So, 1 upon delta lambda is taken out. 0 to delta lambda is taken out. And we get 0 to k integral 0 to k over this Dirac delta function. And we get what we call another function.

So, 0 to k delta k – kappa lambda d dk. This becomes what we called heavy side function H, k – kappa lambda. This we call heavy side function. So, H is basically = 1. If kappa lambda is < = k. And it is 0 otherwise. So, if kappa lambda is < = k, then it will be = 1. It is called heavy side step function. Okay. And it will be 0 otherwise. So, we will use this equation to solve our problem.

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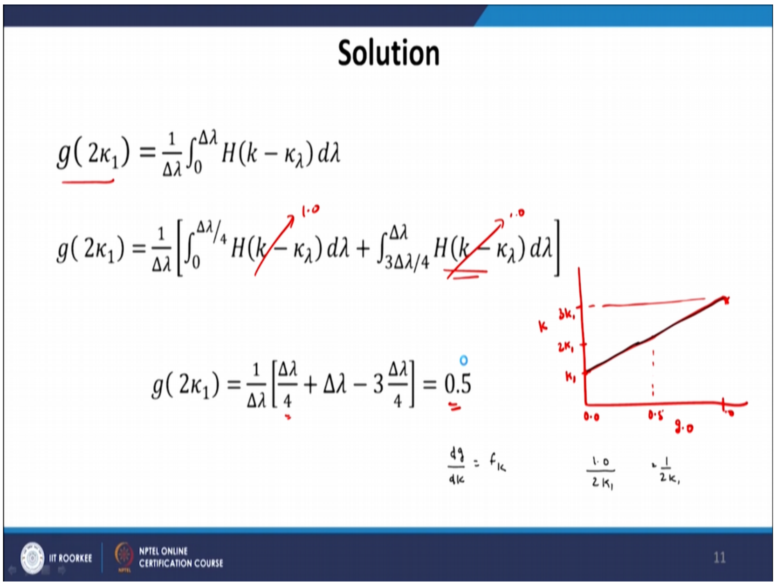

Solution

$$g(2\kappa_1) = \frac{1}{\Delta\lambda}\int_0^{\Delta\lambda} H(k - \kappa_\lambda)\, d\lambda$$

$$g(2\kappa_1) = \frac{1}{\Delta\lambda}\left[\int_0^{\Delta\lambda/4} H(k - \kappa_\lambda)\, d\lambda + \int_{3\Delta\lambda/4}^{\Delta\lambda} H(k - \kappa_\lambda)\, d\lambda\right]$$

1.0

1.0

$$g(2\kappa_1) = \frac{1}{\Delta\lambda}\left[\frac{\Delta\lambda}{4} + \Delta\lambda - 3\frac{\Delta\lambda}{4}\right] = 0.5$$

$$\frac{dg}{dk} = f_k \qquad \frac{1.0}{2K_1} = \frac{1}{2K_1}$$

Now, 1 thing we should observe is, when we plot the cumulative k-distribution; so, we will have 3 points. We will plot the k-distribution for 3 points. One is K 1, 2 K 1 and 3 K 1. So these, for 3, these 3 points we will plot the cumulative k-distribution. You should understand that, what these represents. So, g kappa 1, g kappa 2, g kappa 3. So, g kappa 2 is nothing but g 2 times kappa 1. g kappa 3 is g times 2 3 kappa 1.

So, what does g kappa 1 represents? It represents the fraction of spectrum below kappa 1. So, g kappa 1 represents the fraction of spectrum below the value of kappa 1. And it is 0. There is no spectrum below this value. All the spectrum is above kappa 1. Similarly, g kappa 3 represents the fraction of spectrum below kappa 3. That is below 3 kappa 1. So entire spectrum is below this kappa 3. So, this will be = 1. So, g kappa 1 is 0. g kappa 3 is 1. All we need to find out is g kappa 2. So, this is the only unknown here we need to find.

So, g kappa 2 or g 2 times kappa 1 is = 1 upon lambda 0 to delta lambda heavy side function k – kappa lambda d lambda. This function, heavy side function, we have to integrate over the entire spectrum that has been given. So, we will observe here, the value of; so, we are basically interested in integrating this function over this spectral range. The one I am highlighting. So, < = 2 kappa 1. And then, this one.

So, we are interested in only the shaded part of the spectrum we need to integrate. Although, I can use symmetry, because area under curve, this one and this one are same. So, I can use symmetry, but I will split the integral into 2 parts. Now, just by simple trigonometry. So, total width is delta lambda by 5. This width, we can just find out using trigonometry. Let me just show you the trigonometry here.

2 times kappa 1 upon delta lambda over 10 is =, let us call this x. So, it will be kappa 1 by x. So, this gives you x is = delta lambda by 20. So, this x is delta lambda by 20. So, we split our integral here. 1 upon delta lambda 0 to delta lambda by 4. Where delta lambda by 4, let me show you what is delta lambda by 4. This distance is delta lambda by 4. How? Because, delta lambda + delta lambda upon 5, the width of this region, delta lambda upon 5 + delta lambda by 20, total is delta lambda by 4.

So, 0 to delta lambda by 4 H k – kappa lambda d lambda +. Similarly, 3 delta lambda by 4 to delta lambda, the second part of the spectrum d lambda. Now, under this range, H is always = 1. For all the values of kappa lambda < = k, heavy side function is = 1. So, you can just substitute is as 1, as 1. And we can easily integrate this function. This integral comes out to be. 1 upon delta lambda, delta lambda by 4 + delta lambda – 3 delta lambda by 4 is = 0.5.

So, we have obtained the k-distribution. We can plot the k-distribution here. This is the value of g on the x-axis. And the value of k; so, k is kappa 1, 2 kappa 1, and 3 kappa 1. So, at 3 kappa 1, g value goes from 0 to 1. And this is 1. At 3 kappa 1 g is 1. At kappa 1 g is 0. And at 2 kappa 1, g is 0.5. This is 0.5. So, however basically is linear. So, we get a linear function or linear distribution.

This is the k-distribution we have got. So, that is what I have put in the table also. So, g value is 0 at the first point, 0.5 at second point and 1 at the third point. Now, f is easy to evaluate. How to evaluate? So, we know dg by dk is = f k. So, the f function is simply dg by dk. Now, dg by dk is the slope of this line. So, line slope is basically given by 1.0 upon 2 kappa 1. Okay. So, 1 upon 2 kappa 1 is the slope.

So, the f k function is same for all the values, all the points. So, f k is 1 upon 2 kappa 1, 1 upon 2 kappa 1 and so on. But this is a, something we are not interested in. We are always interested in cumulative distribution. So, cumulative distribution is linear. So, we can solve our radiative equation on this linear curve.

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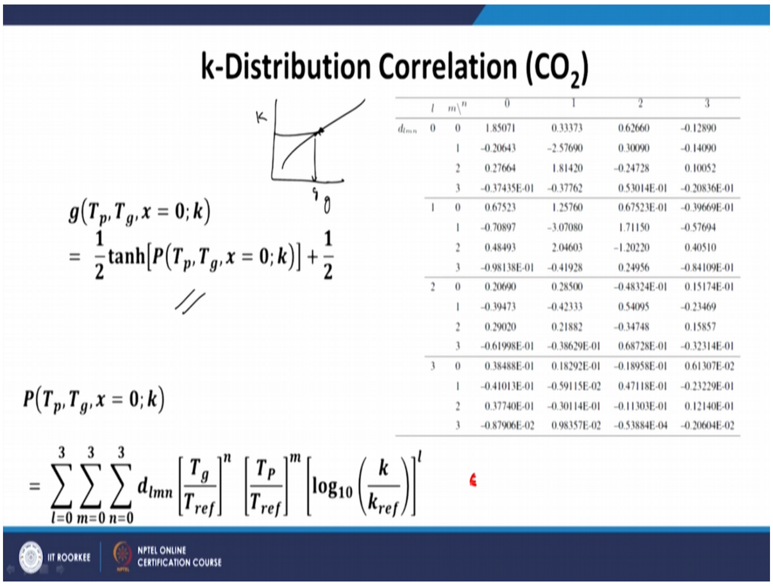

k-Distribution Correlation (CO_2)

$$g(T_p, T_g, x = 0; k)$$
$$= \frac{1}{2}\tanh\left[P(T_p, T_g, x = 0; k)\right] + \frac{1}{2}$$

//

$$P(T_p, T_g, x = 0; k)$$
$$= \sum_{l=0}^{3}\sum_{m=0}^{3}\sum_{n=0}^{3} d_{lmn}\left[\frac{T_g}{T_{ref}}\right]^n \left[\frac{T_P}{T_{ref}}\right]^m \left[\log_{10}\left(\frac{k}{k_{ref}}\right)\right]^l$$

	l	$m\backslash n$	0	1	2	3
d_{lmn}	0	0	1.85071	0.33373	0.62660	−0.12890
		1	−0.20643	−2.57690	0.30090	−0.14090
		2	0.27664	1.81420	−0.24728	0.10052
		3	−0.37435E-01	−0.37762	0.53014E-01	−0.20836E-01
	1	0	0.67523	1.25760	0.67523E-01	−0.39669E-01
		1	−0.70897	−3.07080	1.71150	−0.57694
		2	0.48493	2.04603	−1.20220	0.40510
		3	−0.98138E-01	−0.41928	0.24956	−0.84109E-01
	2	0	0.20690	0.28500	−0.48324E-01	0.15174E-01
		1	−0.39473	−0.42333	0.54095	−0.23469
		2	0.29020	0.21882	−0.34748	0.15857
		3	−0.61998E-01	−0.38629E-01	0.68728E-01	−0.32314E-01
	3	0	0.38488E-01	0.18292E-01	−0.18958E-01	0.61307E-02
		1	−0.41013E-01	−0.59115E-02	0.47118E-01	−0.23229E-01
		2	0.37740E-01	−0.30114E-01	−0.11303E-01	0.12140E-01
		3	−0.87906E-02	0.98357E-02	−0.53884E-04	−0.20604E-02

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So, this was for a fictitious spectrum. Now, how do we generate k-distribution for actual spectrum? For example, carbon dioxide, which is typical for combustion problems. How do we generate the spectrum? How do we generate spectrum for H 2 O, water vapor? Now, 1 way is, you can use codes that basically calculates the k-distribution. First generate k absorption spectrum line-by-line absorption spectrum you generate for any gas C O 2 or H 2 O.

You can mix line-by-line absorption coefficient for the gas mixture. And then generate the k-distribution from first principle. It is going to be expensive. You can have to, you will have to apply the mathematics that I have developed here to generate k-distribution from raw absorption spectrum. On the other hand, you can use correlations to generate k-distribution. The k-distribution correlations have been developed.

For example, for C O 2, you can find out the value of g for any value of k. So, we may have k versus g curve. And this may look like this. For this, for any value of k, you can find the value of g using this relation. This is the correlation that has been given for gases like C O 2 and H 2 O. So, for the C O 2, you can use this correlation to find out the value of x, value of k and g. The only thing you should remember is that this is available for x is = 0.

So, x is = 0 means, we have only 1 gas present, there is no other gas present in this. So, only for the pure mixture, basically we finding this curve. Now, P function is basically fitted on 2 parameters, T g and T p. T p is basically the temperature for on which the Planck function is evaluated. So, there are 2 temperatures 2 unknown temperatures in this correlation T g and T

p. T g is the actual temperature of the gas, while T p is the temperature at which the spectrum black body intensity is calculated.

So, this point, I have not emphasized because we have just discussed a single homogeneous layer. But in many applications, the temperature of emission may be different than temperature of absorption. And this relation basically takes into account this variation.

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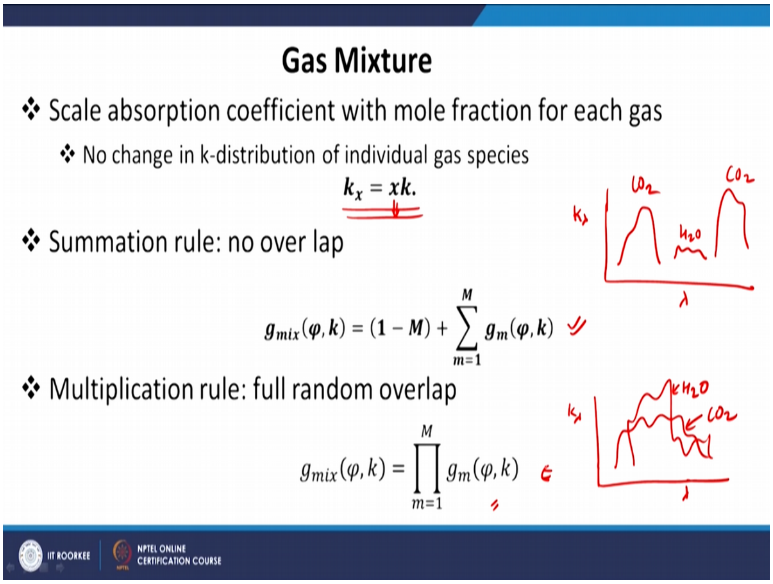

Gas Mixture

- Scale absorption coefficient with mole fraction for each gas
  - No change in k-distribution of individual gas species

$$\underline{\underline{k_x = xk.}}$$

- Summation rule: no over lap

$$g_{mix}(\varphi,k) = (1-M) + \sum_{m=1}^{M} g_m(\varphi,k)$$

- Multiplication rule: full random overlap

$$g_{mix}(\varphi,k) = \prod_{m=1}^{M} g_m(\varphi,k)$$

Now, for the gas mixture, we can generate k-distributions very easily. All you have to do, you have to use the first this correlation for individual gases. Suppose we have a mixture of carbon dioxide and water vapor, all you have to do, you apply this correlation to individual gases. To apply, you apply this correlation to a C O 2, similarly you apply this correlation to H 2 O separately.

Now, when we have a mixture, the mole fraction will not be = 1. That means, the mole fraction of individual gases will be different. There is no change in k-distribution if you have mixture of gases. All we have to observe is that absorption coefficient will scale by mole fraction. If you have single gas, you will get 1 value of k. You generate the k-distribution. If you have mixture, the absorption coefficient will scale by mole fraction.

So, k x is the actual k value when we have a gas mixture. And then, you generate the k-distribution for 1 species, you generate the distribution for another species. And then you can mix them. It is called mixing, mixing the k-distribution. You can mix the k-distribution of individual species by the summation rule. So, the mixture k-distribution is given by this mixing rule, summation rule.

It is, it gives you very good accuracy when the gases do not overlap. For example, you have a spectrum, lambda versus kappa lambda. You have 1 band of C O 2; and then another band of H 2 O; and then again may be a C O 2 band. So, whenever you have this kind of spectrum which do not overlap, there are non-overlapping regions between the 2 gases. The summation rule gives you very good results.

You can use multiplication rule. You can multiply individual k-distributions and find out the mixture k-distribution. This gives you good accuracy when there is full overlap. That means, you have 1 gas. And then, there is another gas. 1 is C O 2 and another is let us say H 2 O. And there is full overlap between absorption coefficient. Then, this method will give you better results. So, depending on the applications you can either use submission rule or multiplication rule to find out k-distribution for the gas mixture.

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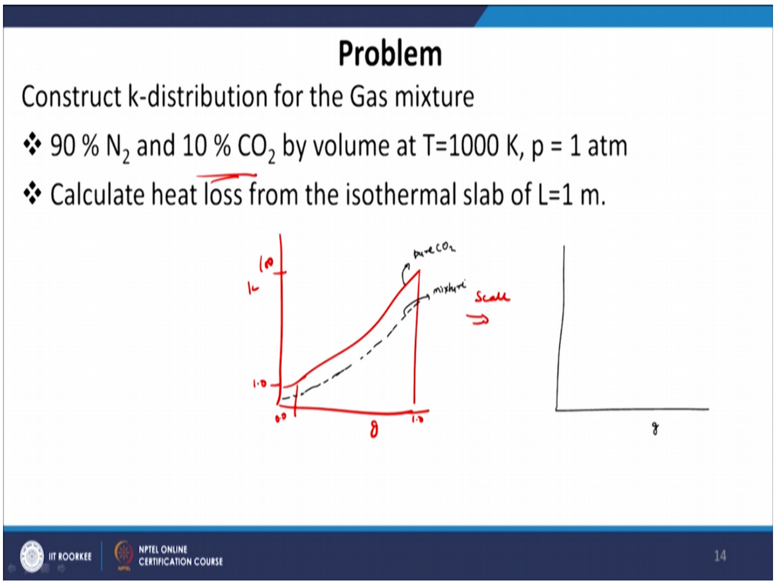

Problem

Construct k-distribution for the Gas mixture

- 90 % N_2 and 10 % CO_2 by volume at T=1000 K, p = 1 atm
- Calculate heat loss from the isothermal slab of L=1 m.

Let us basically, we can use the correlation that we have developed. I will not try to solve this problem actually in the lecture. I will just give you the outline, how we can solve the k-distribution for the gas mixture having the composition of 90% nitrogen and 10% C O 2 by volume. So, we do not have a single gas, although there is only 1 radiating gas. So, nitrogen we know will not radiate energy.

It will not have its own spectrum. But, the presence of nitrogen results in small mole fraction of C O 2. So, nitrogen do not emit radiation. It does not interfere as such. Although it affects broadening, but there is not much effect of nitrogen on C O 2 spectrum. But the mole fraction of C O 2 has reduced. So, how do we solve this problem? So, we will use this correlation first to generate the k-distribution for C O 2.

So, this relation is used first to generate the k-distribution for C O 2. Now, because C O 2 is only 10%; so, first we generate the k-distribution for this. Let us call let us say this is the k-distribution for C O 2. Okay. The magnitude of k varies from let us say 1 to 100. This is all cooked up data. It does not have the actual values. I am just giving you an idea how to solve this problem.

So, we generate k-distribution from the correlation at 1,000 kelvin 1 atmospheric pressure. And the value of k goes from 1 to 100. Okay. From this range 1 and 0. Now, we know that there is only 10% C O 2. So, k values will not vary from 1 to 100. So, we scale. So, how do we scale this? The g value will remain same. The only thing is, the curve will shift. Let me just draw it here itself. So, the curve will shift little down.

So, this will be the k-distribution, actual k-distribution when there is only 10% C O 2. So, it will shift, it will scale by a factor x. Okay. So, this is for pure C O 2 and this is for mixture. The k-distribution will shift up and down depending on the concentration of the gas. Now, once we have this concentration;

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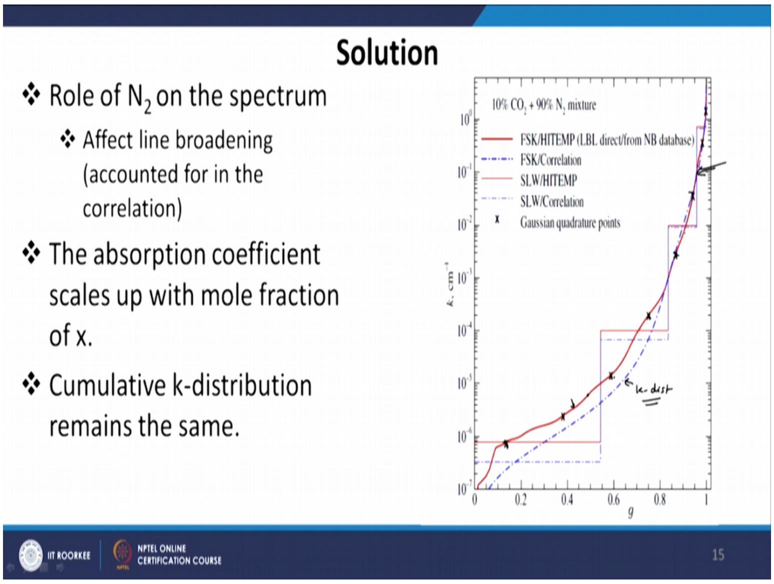

Solution

- Role of N_2 on the spectrum
  - Affect line broadening (accounted for in the correlation)
- The absorption coefficient scales up with mole fraction of x.
- Cumulative k-distribution remains the same.

I have shown you the k-distribution calculated from this; the blue curve basically shows the k-distribution evaluated from the correlation. And this is the actual k-distribution. We see that, for very low values of k, the agreement is not good. But for large values of k, the agreement is pretty good. So, 1 k-distribution is evaluated from correlations and another k-distribution is evaluated directly from the absorption coefficient data from high temp, line-by-line data.

So, this is actual k-distribution, this is fitted data from correlations. And this matches well. And we have obtained the k-distribution as I outline first by correlations and then scaling the absorption coefficient by the concentration of C O 2. Now, we want to solve for the heat flux. So, for heat flux, we need to integrate. So, we choose some points marked by cross. So, we choose certain points for integration.

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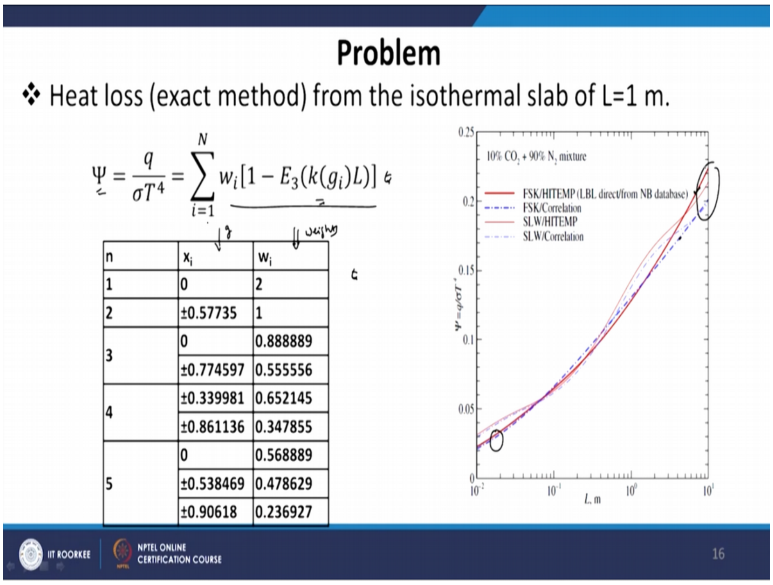

Problem

- Heat loss (exact method) from the isothermal slab of L=1 m.

$$\underset{-}{\Psi} = \frac{q}{\sigma T^4} = \sum_{i=1}^{N} \underbrace{w_i[1 - E_3(k(g_i)L)]}_{=}$$

↓g ⇓weights

n	x_i	w_i
1	0	2
2	±0.57735	1
3	0	0.888889
	±0.774597	0.555556
4	±0.339981	0.652145
	±0.861136	0.347855
5	0	0.568889
	±0.538469	0.478629
	±0.90618	0.236927

And the heat flux, non-dimensional heat flux, we evaluated this in the previous lecture also for the weighted-sum-of-gray-gasses for same formula, exact method for isothermal slab, the heat flux non-dimensional heat flux can be written as weighted-sum-of-gray-gasses. So, i is = 1 to N, w i is the weight, exponential integral of third order k values, that is the gray absorption coefficient values times l.

Now, this we, k value is basically are picked from these cross. The k values comes from there. These cross basically gives you the value of k and corresponding weights are given in this table. So, this is the Gaussian quadrature that we are using for integration. For Gaussian quadrature, we can pick the values of x and w. This is the value of g and this is the value of weights.

So, for corresponding values of g; now you can use 1 point also, you can use 2 point quadrature, 3 point quadrature and so on. For number of quadrature points, you just find out the value of g and correspondingly you find out the value of k from the curve. And then multiply by the weight and solve for the radiative heat flux. And the result is given in this image. And you see that the radiative heat flux, this blue curve and this red curve which is exact, they good, they give you very good agreement in the optically thin region.

And there is some inaccuracy in the optically thick region where length is large. And the same observation you observed in the weighted-sum-of-gray-gasses also. In weighted-sum-of-gray-gas model also, we observe that agreement was very good when l was small and agreement was relatively poor when the l was large. Same observation is observed in k-distribution also. But the accuracy of k-distribution is superior to weighted-sum-of-gray-gases by many factors.

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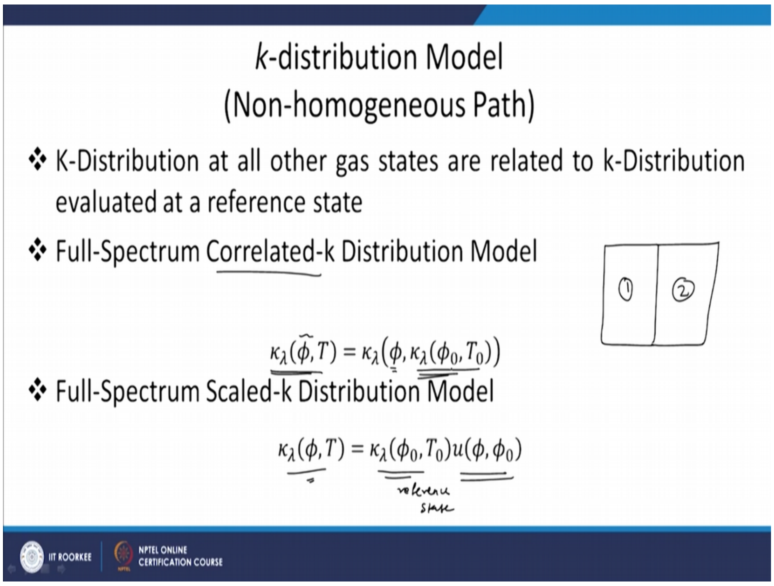

k-distribution Model
(Non-homogeneous Path)

- K-Distribution at all other gas states are related to k-Distribution evaluated at a reference state
- Full-Spectrum Correlated-k Distribution Model

$$\underline{\kappa_\lambda(\tilde{\phi},T)} = \kappa_\lambda\left(\underline{\phi},\underline{\underline{\kappa_\lambda(\phi_0,T_0)}}\right)$$

- Full-Spectrum Scaled-k Distribution Model

$$\underline{\kappa_\lambda(\phi,T)} = \underbrace{\underline{\underline{\kappa_\lambda(\phi_0,T_0)}}}_{\text{reference state}}\underline{\underline{u(\phi,\phi_0)}}$$

Now, how do we apply to non-homogeneous path? So, so far, we have understood the k-distribution for homogeneous path. We have only 1 gas layer. Now, we may have more than 1 gas layer. So, let us say this is 1 gas layer. We may have another gas layer. And there may be many more. So, how do we apply this method? So, all the methods that we have discussed so far, they suffer in the treatment of non-homogeneous path.

There is no direct way of dealing with non-homogeneous path. Whenever there is a variation in gas concentration or temperature, the methods basically, they have no exact way of dealing with this. So, in k-distribution, there are 2 ways. We discussed scaling approximation in weighted-sum-of-gray-gases model also, where we said that absorption coefficient at any gas condition phi is basically = absorption coefficient at reference state.

This is the reference state. And times a scaling function u. So, scaling function, we discussed in weighted-sum-of-gray-gas model also. So, scaling function applies to k-distribution also. We can use scaling approximation in k-distribution also. But in k-distribution, there is one more method of dealing with non-homogeneous path and that is called correlated-k. In correlated-k, what we say is that absorption coefficient at any wavelength lambda at any gas condition phi is basically depends on the gas condition phi and absorption coefficient evaluated at gas condition phi nought.

This is called correlated-k. It will be more clear when I show you the image of correlated-k method. There is slight difference in the 2 approximations. The correlated-k and scaled k. But, correlated-k is much superior as we will be looking at this in the next slide.

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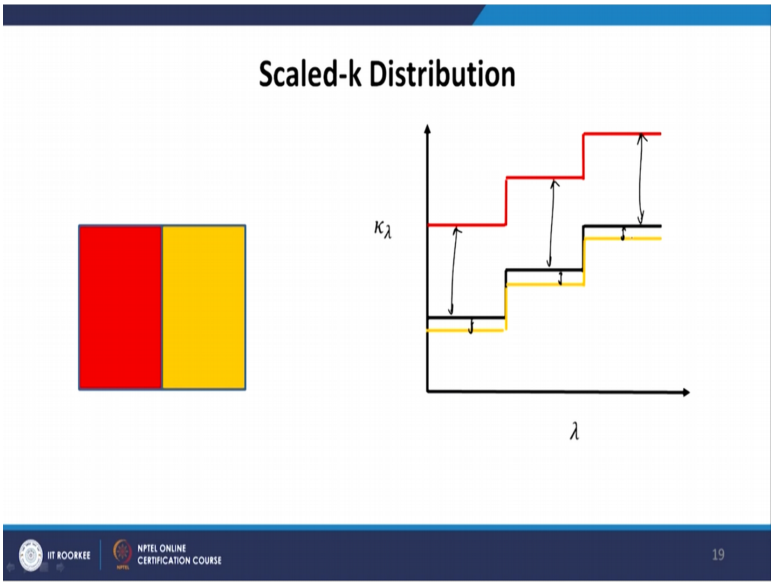

Scaled-k Distribution

So, what is correlated-k-distribution? So, let us say we have 2 gases, gas 1 and gas 2. One of the gas is hot. Another gas may be cold. 1 gas chamber may, 1 cell may contain high concentration of C O 2. Another cell may contain low concentration of C O 2. So, overall, the path is non-homogeneous. So, what we do is, we generate reference k-distribution or reference kappa. I have taken it here constant value. But, it can be continuously varying.

So, let us say this is the reference absorption spectrum. Now, what is meant by correlated-k is that, for all the values of lambda where kappa lambda has a unique value, those wavelengths will see a scale up. And the scale up may be different for different wavelengths. So, for example, the cell 1, see this scale up. But this scale up is different for different wavelengths. Here it is small, here it is intermediate.

So, correlated-k basically means that absorption coefficients are correlated. They depend on 2 values. 1, they depend on gas state phi. But they also depend on the value of absorption coefficient at the reference state, kappa lambda phi nought. So, it depends on 2 values. Here kappa lambda phi nought value was different and the scale up was large. Here there was different absorption coefficient value. You saw a different scale up.

Here the absorption coefficient was different and you see a different scale up. This is called absorption, correlated absorption coefficient or correlated-k. Similarly, here we also see at gas state 2, there is a different scale up, different scale up and different scale up. So, at different magnitudes of kappa lambda phi nought, the scale up is different. While in scale k it is same. Independent of kappa lambda magnitude, the scale up is same.

This is called scaling approximation. Everywhere it is same. This is same, this is same, this is same. This is called scaling approximation. So, the scaling function depends only on gas concentration. It does not depend on the magnitude of absorption coefficient. So, scaling approximation is little more restrictive. That means, scaling is applied uniformly over the spectrum, while the correlated k-distribution is more reasonable, more better approximation, because it takes into account the different part of the spectrum that may see different scale up.

So, we end this lecture on k-distribution here. And this basically ends the topic of spectral modelling. In the next lecture, we will discuss the radiative properties of scattering particles. So, thank you.