

**Radiative Heat Transfer**  
**Prof. Ankit Bansal**  
**Department of Mechanical and Industrial Engineering**  
**Indian Institute of Technology - Roorkee**

**Module - 7**  
**Lecture - 32**  
**Spectral Modelling**

Hello friends, in the previous lectures we discussed the atomic and molecular spectrum. We observed that the spectrum, the absorption coefficient and the emission coefficient of gases vary very erratically, it has very fine resolution, it has subjected to a lot of broadening parameters which makes the lines very thick. So, in a sense, the solution of the radiative transfer equation or any radiation problem needs to be repeated for a large number of wavelengths.

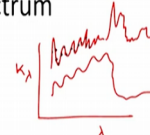
Because the spectrum varies continuously over wavelength and the variation is very erratic, we need to have very fine resolution of the spectrum. It may be few angstroms or it may be less than an angstrom. So, in a sense, depending on the pressure and temperature of the gas that we are considering, we need to resolve very fine spectral resolution. We need to divide the spectrum into a number of points, spectral points.

And we need to solve our radiative transfer equation at each and every wavelength which is going to be very expensive. So, in this lecture we will see some of the things we can do to the spectrum to make it simple and computationally less expensive. As in other branches of engineering, many times we resolve to empirical relations. So, in this lecture, we will also introduce some empirical relations which allow radiative transfer equation, especially the spectral part of the problem to be taken in a very simple and computationally more efficient manner.

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## Spectral Modelling

- ❖ Radiative Transfer equation needs to be solved for each and every wavelength
  - ❖ Thousands of spectral wavelengths
  - ❖ Computationally expensive
  - ❖ Absence of accurate spectral data
- ❖ Need simplified spectral representation of the spectrum
  - ❖ Gray
  - ❖ Mixture of gray gas model
  - ❖ Band Models



$$\frac{dI_\lambda}{ds} = \kappa_\lambda (I_{b\lambda} - I_\lambda) \Leftrightarrow q = \int_0^\infty q_\lambda d\lambda$$



So, radiative transfer equation as we have already developed the relation earlier, depends on wavelength. So, in a 1-dimensional, along path  $s$ , the change in intensity can be written as  $dI_\lambda$  by  $ds$   $\kappa_\lambda I_{b\lambda} - I_\lambda$ . Where  $I_{b\lambda}$  is the black body intensity, the Planck function, the spectral Planck function at a given wavelength.  $I_\lambda$  is the intensity of the light beam traveling in direction  $s$ , at a given wavelength.

And this equation is for a single wavelength. So, when we are solving a radiation problem, we have to solve these type of equations for each and every wavelength. There may be thousands of points. For example, if you have a spectrum of  $\kappa_\lambda$ , that is absorption coefficient varying with wavelength, the variation may be relatively smooth or it may be very erratic. It may be forming bands.

So, we need to represent this is spectrum with number of points. When we do that, we can solve for; so, we need to represent this spectrum into a number of points. And for each point, we need to solve the radiative transfer equation. And then, we have to sum the results. So, for example, if you are interested in heat flux, total heat flux, then we have to first find out the spectral heat flux  $q_\lambda$ .

And then, integrate over all the wavelengths from 0 to infinity. So, this is going to be very very expensive. Because, calculation of single  $q_\lambda$  is going to be expensive. As we have seen in many methods, be it spherical harmonics, be it discrete ordinate, whatever method we choose, the methods are expensive. And if we have to repeat these calculations again and

again for large number of wavelengths, then definitely radiation problem will become very expensive computationally.

In fact, in any combustion problem, if you try to solve radiation, 90% of the time, in computational time, will be consumed by radiation calculation itself. And very little time relatively will be consumed by other calculations like conservation of mass momentum energy and so on. So, definitely we need to do something, otherwise, chunk of time will be consumed by the radiation calculation itself.

So, there are lot of simplifications we can do. Some of the simplifications may not be accurate. But, many times the simplifications are used in practical applications. Even though they may be subjected to inaccuracies, but in engineering our goal is to resolve for an efficient method. We can give practical results in real time rather than going for very very accurate results. So, even if our model is able to give 20 to 30% accuracy in reasonable good amount of computational effort, then we should be feeling happy.


So, there are models based on gray calculations. So, gray calculation is a very conservative approach. We have mixture of gray gas model, where we have more than 1 gray gases. And then, we have band models. So, in this lecture, we will focus our attention on band models, specially the narrowband models that we will focus on.



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### Line-By-Line Method

- ❖ Solve RTE for each wavelength
  - ❖ Expensive
  - ❖ Not always useful as not all wavelengths are important
- ❖ Obtain spectrum of  $\varepsilon_\lambda$  and  $\kappa_\lambda$  for each gas
- ❖ Gas mixture
  - ❖ Absorption coefficients are simply added

$$\underline{\kappa}_\lambda = \sum_{i=1}^{n_g} \kappa_{\lambda i} \quad \checkmark$$



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So, the most accurate method for solving the radiative transfer equation, especially the spectral part that we are talking about. How to solve radiative transfer equation in space and

direction, we have already discussed. Now, we have to integrate over the spectrum. So, we are focusing on the spectrum part here, of the radiative transfer equation. And, line-by-line method means that we have to solve radiative transfer equation without any simplification at each and every wavelength.

So, of course, this method is the most accurate method, because we are solving the radiative transfer equation for each and every point. There is no simplification as far as mixture of gases is concerned. When we have more than 1 gases in the mixture, the absorption coefficients are simply additive. You can just add absorption coefficient of different gases  $\kappa_{\lambda}$ . And then, your net absorption coefficient  $\kappa_{\lambda}$  will be just a sum of the absorption coefficient of individual gases.

So, at this point also, we do not have any simplification. We will see in later slides that, when we go for simplified models like band models or gray models, it is not simple to add gases. For line-by-line only, we can add gases in an accurate manner. Because we have represented the absorption coefficient of each and every gas at a given wavelength, so total absorption coefficient is just additive and we can add them accurately.

Similarly, we can do for the emission coefficient  $\epsilon_{\lambda}$ . However, line-by-line calculations are very accurate, but they are expensive. On the other hand, the accuracy of these calculations may not be desired all the time. As I said, in engineering applications, even if we are making an error of 20 to 30%, we feel happy, provided the calculations can be done in real time at very less computational effort.

Also, if you look at the spectrum, if you look at the spectrum of  $\kappa_{\lambda}$  versus  $\lambda$ , we see this type of structure. And we observe that there are certain wavelength ranges, where absorption coefficient is small. Now, the magnitude of  $\kappa_{\lambda}$  dictates how much energy is basically transferred. If the value of  $\kappa_{\lambda}$  is small, then very less amount of heat transfer is actually taking place.

So, what we observe is, some of the wavelengths may not be important. Because heat transfer is actually taking place only where the  $\kappa_{\lambda}$  value is large. And very less amount of heat transfer is taking place where  $\kappa_{\lambda}$  value is very small. So, these wavelengths

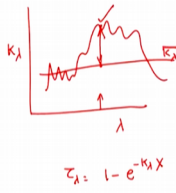
may not be very significant, may not be important for heat transfer point of view. But when we do line-by-line calculation, we have no means to discard these wavelengths.

We end up doing calculations repeatedly for these wavelengths as well. So, in the sense, the line-by-line method may be an overkill which gives you good accuracy but at the same time very high computational cost. And sometimes, this high computational cost may not give you desired accuracy. Because, repeated calculations being done at wavelengths which are of no importance.

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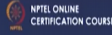
### Mean Absorption Coefficient

- ❖ Planck-mean absorption coefficient
  - ❖ Conserves emission ✓
  - ❖ Results in lower absorption (higher transmissivity)



$\tau_\lambda = 1 - e^{-\kappa_\lambda x}$

$$\kappa_p = \frac{\int_0^\infty I_{b\eta} \kappa_\eta d\eta}{\int_0^\infty I_{b\eta} d\eta} = \frac{\pi}{\sigma T^4} \int_0^\infty I_{b\eta} \kappa_\eta d\eta$$



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The second approach is of course the most extreme approach, which is the mean absorption coefficient. We have a spectrum of absorption coefficient. And what we do is, we try to find a mean value of this absorption coefficient. So, of course, when we take this mean absorption coefficient, then it is not going to have same transmissivity. Because transmissivity tau lambda is given by 1 – e to the power – kappa lambda x.

Now, when we have taken mean, kappa lambda mean; now, there are many ways of taking this mean, but the most preferred mean absorption coefficient is what we call Planck mean absorption coefficient. So, Planck mean absorption coefficient is defined as I b lambda or I b eta. So, as I said earlier also, some researchers prefer wavelength as the subscript, some researchers prefer wavenumber.

Now, most of the combustion people, they prefer wavenumber. So, whatever relations I will show in this lecture will be based on wavenumber. But it does not matter. All these relations

are equally valid on wavelength bases also. So, we define basically Planck mean absorption coefficient as the Planck function multiplied by absorption coefficient integrated over the entire spectrum, divided by the total black body intensity, integrated over the entire spectrum.

So, this is called Planck mean absorption coefficient. The good thing about this method or the mean absorption coefficient is that it captures the emission accurately. If you look at the definition itself of the Planck mean absorption coefficient, it is based on total emission divided by total black body emissive power. So, this method basically captures the emission well. So, it conserves emission.

So, if you have a medium where absorption coefficient is not very large, rather it is optically thin, this method will give you very good results. But for gases which absorb radiation where self-absorption is important, where significant amount of radiation is absorbed, where it is emitted, then this method may be very inaccurate. Why? Because we have reduced the emission or absorption coefficient significantly.

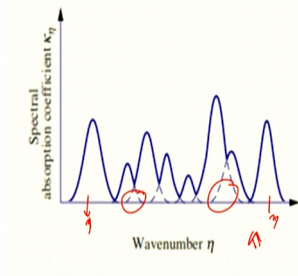
If a photon is traveling at this wavelength, then it will see very less amount of absorption coefficient and it will not be absorbed. But in actual practice, the absorption coefficient is large and it should be absorbed. So, what basically we are doing is, a photon which should have been absorbed because of large amount of absorption coefficient is allowed to transmit by taking a small value of absorption coefficient, a small value of mean absorption coefficient.

So, this basically results in lower absorption and higher transmissivity. And definitely, this method is not accurate for gases where absorption is important. For optically thin case, this method may be accurate.

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## Narrow Band Model

- ❖ Averaging over few lines or wavenumber/wavelength
  - ❖ Exact description of line position and magnitude not required
- ❖ Statistical description
  - ❖ Random placement of lines and strength from probability distribution
  - ❖ Simplifies mathematics involved
  - ❖ Saves computational time
- ❖ Parameters fitted with experimental data for good accuracy (15-30 %)



The first simplification towards an accurate method and we are talking about accuracy of 15 to 30% here. We are not talking about accuracy of 100% or 90% here. So, this narrowband model that we are going to talk will have accuracy of the order of 15 to 30%. So, what we do in narrowband model is, we take a few lines, a few wavenumbers and we take an average over that spectral range.

So, unlike the gray method, we have taken the average over the entire spectrum from 0 to infinity. We are now going to take average only over a small range. Okay. We have taken few lines as is shown here. These lines will have different magnitudes. They, these lines will have overlap as is seen clearly observed here. So, lines have different strength. Lines have different overlap.

And we are taking an average over these small wavelength range, where we have only few lines. Now, the point of argument here; why this method is successful? why this method is able to work? The argument is that, in rotational lines, especially when we have molecular spectrum, the lines, there are large number of lines. And these lines are having different magnitudes. And that is seen in this is spectrum also.

We have lines which are of different magnitude, different type of overlap is there. So, the argument is that, we do not need to have exact knowledge, which line is where. We are not interested in where, what wavelength, wavenumber this line is located, what wavenumber this line is located. So, the exact knowledge of where the spectral line is located, the exact location or the magnitude of these lines is not important.

It can be randomly placed. So, that is what basically the approach of this narrowband distribution, narrowband model is. We assume that the lines are randomly located. They have random strength, they have random location. And we apply the theory of random numbers, the probability distribution and based on that theory, we are able to simplify significantly the mathematics involved.

And this gives you good results within the accuracy range of 15 to 30%. Lot of the computational time is saved. And what you get is basically a reasonably accurate model without resolving to complex calculations. Of course, because there is no quantum mechanical, there is no logic in these calculations, so we have to fit the parameters involved with experimental data.

So, the method basically resolves on experimental fitting. So, we develop a correlation. This correlation has no sound physical background. But, all we do is, we fit the parameters involved in this correlations from experimental data. And this experimental fitting basically dictates the accuracy of the model.

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### Narrow Band Models

- ❖ Black surfaces with no scattering
- ❖ Spectral integrals

$$\int_0^{\infty} \kappa_{\eta} I_{b\eta} d\eta$$

$$\int_0^{\infty} I_{b\eta} \left[ 1 - \exp\left(-\int_0^X \kappa_{\eta} dX\right) \right] d\eta$$

$X = \text{Optical path length}$

- ❖ Dividing the spectrum into narrow bands

$$\int_0^{\infty} I_{b\eta} \left\{ \frac{1}{\Delta\eta} \int_{\eta-\Delta\eta/2}^{\eta+\Delta\eta/2} \kappa_{\eta} d\eta' \right\} d\eta$$

$\Delta\eta \Rightarrow I_b \text{ does not change}$

So, what we will do is, we will do averaging in, let us go into the details of this narrowband model. The method is only applicable to black surfaces with no scattering. And it will be clear why it is restricted to black surfaces with no scattering. All we are interested in is averaging. We want to average these 2 integrals which appear in radiative transfer equation. If you look at the radiative transfer equation, the integrals, when you integrate the radiative transfer equation on spectral bases we get 2 integrals.



1 is the integral kappa eta I b eta and another is the 1 – exponential over the path length of kappa eta dx which is basically the transmissivity. So, these 2 integrals appear in radiative transfer equation when it is integrated over wavenumber. Now, if you look at this expression, we want to simplify this. We want to take an average. So, what we do is, we divide the spectrum into number of bands.

So, we have spectrum going up to 0 to infinity in wavelength space. So, what we do is, we divide this spectrum into small small bands. These are our small bands in which we have divided the spectrum. And let us say 1 band that we take has width of delta eta. The width of 1 small band is delta eta. So, we want to integrate our result over this small band. And over this small band, we assume that I b lambda, the black body function does not change.

Because the spectrum delta eta is very small. Only few wavenumbers are present. So, we assume that I b lambda does not change. And we can take it out of this integral. So, let us divide the spectrum into narrow bands. And we take an averaging. So, what we do is, the same kappa eta d eta we average or we integrate from range eta delta eta by 2. We integrate over the small narrowband. And then, what will do is, I b eta is taken out of the integral. So, I b eta now is out of this small integral. Earlier, it was multiplied by kappa eta, but now we have taken it out of the integral.

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### Narrow Band Models

$$\int_0^\infty I_{b\eta} \left\{ \frac{1}{\Delta\eta} \int_{\eta-\Delta\eta/2}^{\eta+\Delta\eta/2} \left[ 1 - \exp\left(-\int_0^X \kappa_\eta dX\right) \right] d\eta' \right\} d\eta$$

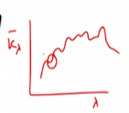
❖ Averaged spectral coefficient



$$\bar{\kappa}_\eta(\eta) = \frac{1}{\Delta\eta} \int_{\eta-\Delta\eta/2}^{\eta+\Delta\eta/2} \kappa_\eta d\eta'$$

Spectral average absorption coeff.

$$\bar{\epsilon}_\eta(\eta) = \frac{1}{\Delta\eta} \int_{\eta-\Delta\eta/2}^{\eta+\Delta\eta/2} \left[ 1 - \exp\left(-\int_0^X \kappa_\eta dX\right) \right] d\eta'$$

Spectral average emissivity.





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And same thing we do for the second integral, which is the integral of emissivity, 1 – exponential over the path integral. So, we also take it average over the small narrowband. And we define 2 quantities, average spectral absorption coefficient and average spectral

emissivity. So, this is a spectral average absorption coefficient. And this is spectral average emissivity. Now, we are using the both spectral and average.

So, what does this mean? What, why we are using spectral an average, both the terms simultaneously? So, it is average over small narrowband. But it varies from narrowband to narrowband. That is why we are using the term spectral. If you look at the variation now of absorption coefficient, it will not be very very abrupt. It will be rather very smooth. If you look at the narrowband absorption coefficient, it will be rather smooth, because we have averaged over a small narrowband.

Now, it is not very erratic as was in the earlier case. Now, we have taken an average over small wavenumber range. So, it becomes relatively smooth and the integration becomes relatively easier. So, this is the first part of the narrowband model. The second part is of course, that we do not know the, we do not want to put the information of exact lines, exact wavelength and exact magnitude. So, we have everything in random. So, in this category, we have 2 models.

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### Narrow Band Models

- Elsasser Model:
  - Equally spaced lines
  - Equal intensity
  - Less overlap (over predicts transmissivity)
- Statistical Model:
  - Random spacing and intensity
  - More overlap
  - Error difference 20 % between two models

The top graph shows the Elsasser model with equally spaced, equal-intensity lines. The bottom graph shows the Statistical model with random spacing and intensity, resulting in more overlap between lines.

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The Elsasser model which assumes that the lines are equally spaced. They have equal intensity given by  $S$ . So, the intensity of the line is given by  $S$  at the center. The distance between lines is same as is given here  $d$ . So,  $d$  is a separation between 2 lines. So, Elsasser model is a very simple model which assumes that all the lines in a band are of same magnitude of same separation distance  $d$ .

Of course, there is small overlap here. The overlap is very minimal here and it over predicts transmissivity. So, if you do a calculation based on this model, you will get results higher heat flux as compared to actual value. Because it leads to higher transmissivity. On the other hand, there is a another model, statistical model, which assumes that the lines are randomly located. They have random position given by a probability distribution.

It leads to higher amount of overlap between the lines, because lines now are not equally spaced. They are randomly located. So, lines may significantly overlap. And this result basically gives you less amount of transmissivity, more overlap, less transmissivity. So, it tends to under-predict the heat flux. So, if you use Elsasser model, the heat flux will be over predicted. If you use statistical model, the heat flux will be under predicted.

But the difference will not be more than 20%. So, the model, both the models, they give very good results. While the error between the 2 models may not be very large, compared to the other models that we are going to discuss. So, the 2 models will have accuracy of the order 20% or so. So, first we will discuss the Elsasser model. In Elsasser model, what we assume is there are infinitely in many lines all separated by a distance  $d$ .

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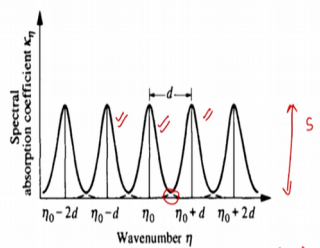
### Elsasser Model

- ❖ Infinitely many lines
  - ❖ Separated by constant distance  $d$
  - ❖ Lorentz line broadening

$$\kappa_{\eta} = \sum_{i=-\infty}^{\infty} \frac{S}{\pi} \frac{b_L}{(\eta - \eta_0 - id)^2 + b_L^2}$$



- ❖ Arbitrary band center  $\eta_0$
- ❖ Series solution

$$\kappa_{\eta} = \frac{S}{d} \frac{\sinh 2\beta}{\cosh 2\beta - \cos(z - z_0)}$$



$$\beta \equiv \frac{\pi b_L}{d}$$

$$z \equiv \frac{2\pi\eta}{d}$$



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Now, these lines have strength  $S$ . That means, at the center, the strength is  $S$ . The distance between the 2 lines is  $d$ . They have width  $b_L$ . So, they follow Lorentzian line broadening. These are the assumptions in this model. There are many many lines, infinitely many in fact, separated by distance  $d$ , all have same shape given by Lorentz profile. Lorentz profile, we

discussed in the previous lecture on line radiation is given by this relation,  $bL$  where  $bL$  is Lorentz width;  $\eta - \eta_0 - \eta_0^2 + bL$  square.

If  $\eta$  basically represents the line number. If  $\eta = 0$ , we get the center line, this one. If  $\eta$  is 1, then we get line on the left. If  $\eta$  is  $-1$ , we get the line on the right. So, we have infinitely many lines equally spaced. Now, this makes an infinite series. We can sum the series to calculate our absorption coefficient  $\kappa_\eta$ . But to make it easy, this series has been evaluated by number of researchers.

And this series solution is given by this formula,  $\kappa_\eta = S \frac{\sinh 2\beta \cosh 2\beta - \cos z - z}{2\beta}$ , where  $\beta$  is line overlap parameter  $\pi bL/d$ . So,  $\beta$  is basically line overlap parameter which basically dictates how much overlap exists between lines. Of course, this is going to be very less for this model, but still there is some overlap in the line wings. And  $\beta$  is basically the magnitude or it basically gives you an idea how much overlap exist between the 2 models.

$\sinh$  represents the hyperbolic sine function.  $\cosh$  represents the hyperbolic cosine function. And this formula in closed form can calculate the absorption coefficient  $\kappa_\eta$ . But we are not interested in absorption coefficient. We are interested in mean absorption coefficient. So, we have to take an average over the few lines.

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### The Elsasser Models

- ❖ Average spectral coefficients Lines are periodic  
-π → π

$$\bar{\kappa}_\eta = \frac{S}{d}$$

- ❖ The spectrally averaged emissivity

$$\bar{\epsilon}_\eta = 1 - \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(-\frac{2\beta x \sinh 2\beta}{\cosh 2\beta - \cos z}\right) dz$$

$\beta$  = line overlap parameter  $\frac{\pi bL}{d}$

$x$  = line strength parameter

$$x = \frac{SX}{2\pi bL}$$

$$\tau = \frac{S}{d} X = 2\beta x$$

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And in fact, these lines are periodic in nature. If you have observed, the Elsasser model has lines equally spaced. They are periodic in nature with period from  $-\pi$  to  $\pi$ . So, what we can

do is, we can integrate and find out an average absorption coefficient. So, average absorption coefficient is given by  $S/d$  where  $S$  is line strength parameter,  $d$  is separation distance. Spectral emissivity, average emissivity is given by this relation.

Of course, to evaluate this integral is not easy. And I will not attempt in this lecture how to evaluate this integral. This can be evaluated numerically. As we have seen in this course, many integrals, they become very complicated. And we do not have any method to solve them analytically. So, this integral gives you the average emissivity of the narrowband. And this has to be evaluated using some numerical scheme.

Beta is line overlap parameter, we have already discussed. Beta is given by  $\pi b L/d$ .  $b L$  is Lorentzian width,  $d$  is separation distance,  $X$  is line strength parameter where  $S$  is line strength  $x$  is path length  $2/b L$  and  $\tau$  is optical thickness which is defined as  $s/s/d$  into  $X$  or  $2\beta x$ . So, this is the Elsasser model. The second model we have is the statistical model which assumes that the lines are randomly placed.

They are not equally spaced, they are randomly placed. So, it is purely probabilistic in nature. But it turns out to be accurate, because ultimately the parameters of this model will be fitted from experimental data. So, we have 3 types of model in this category. And the first one assume that all line have equal strength. It is called uniform distribution or uniform statistical model. So, all lines have same strength.

So, there is no randomness as such. All the lines are of same strength. The second model is Goody model which gives that there is a probability distribution from which lines can be selected.

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## Statistical Models

- ❖ **Uniform statistical model:** All lines have equal strength

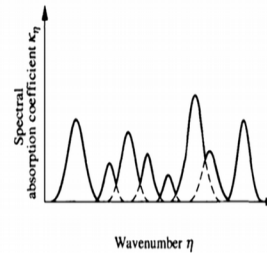
$$S = \bar{S} = \text{const.}$$

- ❖ **Goody model:**

$$p(S) = \frac{1}{\bar{S}} \exp\left(-\frac{S}{\bar{S}}\right) \quad 0 \leq S < \infty$$

- ❖ **Malkmus model:**

$$p(S) = \frac{1}{S} \exp\left(-\frac{S}{\bar{S}}\right) \quad 0 \leq S < \infty$$



And this probability distribution is given by  $1/S$  upon  $S$  prime, where  $S$  prime is average line strength, exponential  $S$  over  $S$  prime. So, this is the Goody model. And the third model is Malkmus model which basically replaces the denominator  $S$  bar, the mean line strength with  $S$ . That is, the particular value of  $S$ . So, there is a small difference between Goody model and the Malkmus model.

Now, Goody model and Malkmus model has enjoyed very popularity in combustion as well as in atmospheric sciences. So, these 2 models we will discuss. And the uniform statistical model we will not discuss. I will not go into the derivation of the results. I will just give you the final outcome of this results. Now, statistical models, again we are interested in finding the value of a mean emissivity.

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## Statistical Models

- ❖ All models give identical result for spectral average emissivity

$$\bar{\epsilon}_\eta = 1 - \exp\left(-\frac{\bar{W}}{d}\right)$$

- ❖  $d$  = average line spacing

$$d = \frac{\Delta\eta}{N}$$

- ❖ Average equivalent line width

$$\bar{W} = \frac{1}{N} \sum_{i=1}^N W_i$$



So, mean emissivity, it turns out to be same for all the models. So, that is a good thing. The expression, the final expression for mean emissivity turns out to be same for all 3 models. It is given by  $1 - \exp(-\bar{W}/d)$ . Where  $\bar{W}$  is average equivalent line width and  $d$  is average line spacing given by  $\Delta \eta / N$ . Now,  $\bar{W}$  will be different of course for different models.

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### The Statistical Models

❖ Goody model:

$$\frac{\bar{W}}{d} = \frac{\bar{S}}{d} X \left/ \left( 1 + \frac{\bar{S}X}{\pi \bar{b}_L} \right)^{1/2} \right. = \tau / (1 + \tau/\beta)^{1/2} \quad \checkmark$$

❖ Malkmus model:

$$\frac{\bar{W}}{d} = \frac{\pi \bar{b}_L}{2d} \left[ \left( 1 + \frac{4\bar{S}X}{\pi \bar{b}_L} \right)^{1/2} - 1 \right] = \frac{\beta}{2} [(1 + 4\tau/\beta)^{1/2} - 1] \quad \checkmark$$

❖ Parameters  $\bar{b}_L/d$  and  $\bar{S}/d$  found by fitting experimental data

So, the expression for  $\bar{W}/d$  is given here for the Goody model. It will be best understood when we take 1 example. So, Goody model will have  $\bar{W}/d$  given by this relation. And Malkmus model has  $\bar{W}/d$  given by this relation. Okay. So, I will not going to explain how this relations are derived. There is a complicated theory behind these 2 models, how the models are, how the results are derived.

But it turns out to be that the emissivity formula is same for both the models. And the parameter  $\bar{W}/d$  can be calculated using the formulas given on this slide. Now, why this 2 models that we have discussed, the statistical and the Elsasser model are able to produce results with such good accuracy is basically the parameters. So, 2 parameters that are important here are  $\bar{b}_L/d$ ; that means, average line width divided by line separation  $d$  and average line strength divided by line separation  $d$ .

These 2 parameters that are required to solve the problem are found by fitting experimental data. So, this is where basically the accuracy of these models basically rely. So, these are kind of semi-empirical models where some parameters are basically based on experimental data. And rest of the things are based on simple theory, simple observation like the lines are

randomly located, given by a certain probabilities distribution. So, we will use these models to solve 1 problem in a while. Now, what we have done here is, so far.

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### Gas Mixtures

❖ Assumptions:

- ❖ Absorption coefficients of different gases are statistically independent

$$\bar{\tau}_{\eta, \text{mix}} = 1 - \bar{\epsilon}_{\eta, \text{mix}} = \prod_{m=1}^M \bar{\tau}_{\eta, m}$$

$P(A \cdot B) = P(A) \cdot P(B)$

no-scatter  
single  
gas  
uniform  
homogeneous  
 $k_{\lambda} \neq k_{\lambda}(\beta)$

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We have so far assumed that we have a gas layer. There is a single gas in this layer. Okay. This gas emits or radiates over a spectrum. We have also assumed that the black, walls are black. These are black walls. And we also assume that there is no scattering. So, these models are basically derived for number of simplifications. We have a single gas. We have uniform absorption coefficient, uniform or homogeneous.

Homogeneous means that it does not vary with space. So,  $k_{\lambda}$  is not a function of space. Okay. So, it is constant. Throughout the cell, the absorption coefficient is constant. The only parameter it changes with is wavelength. So, under this simplification, we have derived this data. Now, suppose we have again homogeneous case, but we have mixture of gases. In many combustion applications, you will find that carbon dioxide and water vapor, they coexist.

And they radiate similarly. In atmospheric sciences we have carbon dioxide and water vapor. So, mostly we have at least mixture of 2 gases, maybe more. But, we have at least 2 gases. So, how do we solve a problem with mixture of gases. Now, there is again a very simple analysis. So, what we define is, that the mixture transmissivity is basically a product of individual transmissivity.



So, this is a very simple relation based on probability theory. So, if you have a probability of 2 events and both the events are independent, then you know that probability of A into B is simply probability of A and probability of B. So, probability of A times B is simply = probability of A times probability of B. Based on probability theory, now we have assumed here random location of lines.

So, it turns out to be that we can apply this relation for mixture of gases also. So, transmissivity tau eta mean transmissivity of the mixture is = mean transmissivity of individual gases multiplied together. So, we can solve for mixture of gas mixture very easily using this assumption.

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### Nonhomogeneous Gases

- ❖ Curtis-Godson Approximation
- ❖ Local  $\tau$  and  $\beta$  replaced by path-averaged values
  - ❖  $\bar{\tau}$  and  $\bar{\beta}$

$$\left(\frac{\bar{W}}{d}\right)^2 = \frac{4/\pi}{C_b} \bar{\tau} \bar{\beta} = \frac{4}{d^2} \int_0^x \bar{S}(X) b_L(X) dX$$

$$\bar{\beta} = \frac{1}{\bar{\tau}} \int_0^x \frac{\bar{S}}{d} \beta dX$$

The second thing is homogeneous non-homogeneous path. Suppose we have now 2 layers or 3 layers or more, let us say we have number of layers. Radiation enters from this side and leaves from here. Here the absorption coefficient kappa lambda, let us call it 1. Here we have absorption coefficient kappa lambda 2. And here we have absorption coefficient kappa lambda 3. This situation may arise in combustion application, in furnace for example.

Where furnace has continuous temperature distribution. Gases are in different concentration, at different heights of furnace. So, absorption coefficient is continuously going to vary. It will not be uniform because absorption coefficient depends on temperature and concentration of the gas. So, kappa lambda is a absorption coefficient in gas 1 or cell 1. You can call it cell or zone 1, cell 2 and cell 3.

This is a non-homogeneous problem, non-homogeneous in space. Because within this space, the absorption coefficient is changing. Now, how to solve for this type of problem? Because ultimately, what we have derived is a formula for mean absorption coefficient in cell 1 and mean absorption, mean emissivity in cell 1. These formulas we have derived. But how do we apply this formulas to the total path?

Again, to do that, we have again 1 approximation, very effective approximation given by Curtis-Godson. The Curtis-Godson approximation is based on the same formula. But now, we have to find out our tau tilde and beta tilde, which are now based on path averaged value. So, the transmissivity tau and overlap parameter beta is now replaced by path averaged value. Okay. So, transmissivity, we have taken an average over the entire path.

And beta which is line overlap parameter, we have taken over the entire path. And these quantities are basically derived using this relations. Okay. Again, we have integral to solve. And this integral varies from 0 to x, where x is now total path length. Okay. So, over the path, the beta will, the kappa will change. And we have to take an average or we have to integrate over the entire path.

So, this is going to be very very complicated. So, we will not going to touch upon this. We will just do 1 problem for a simplified case of one cell, homogeneous case. And for this case, we have following data given for a pure gas at 300 kelvin and 0.75 atmospheric pressure.

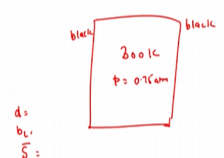
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### Problem

Following data is given for a narrow band of a pure gas at 300 K and 0.75 atm pressure

- ❖ Mean line spacing,  $d = 0.6 \text{ cm}^{-1}$
- ❖ Mean line half width,  $b_l = 0.03 \text{ cm}^{-1}$
- ❖ Mean line strength,  $S = 0.08 \text{ cm}^{-2} \text{ atm}^{-1}$

Determine mean spectral emissivity of the band for geometric path length  $X=1 \text{ cm}$ , using the Elsasser model and Malkmus model



$\bar{\epsilon}_\eta \quad X = 1 \text{ cm}$

So, we have a single cell. We have a gas here and it is bounded by black walls; both sides are black. The temperature of the gas is 300 kelvin. The pressure of the gas is 0.75 atm. And we have been given certain parameters.  $d$  is given, mean line spacing. You can relate this either with Elsasser model or with statistical model. Both will have mean line spacing  $d$  as 0.6 centimeter inverse.

We are working in units of wavenumber. We also have mean line half width  $b_L$ , which is used in Lorentzian profile. And we also have mean line strength  $S$  or  $\bar{S}$  as 0.08 centimeter  $^{-2}$  atmosphere inverse. We have to find out determine mean spectral emissivity. So, we have to find out mean spectral emissivity of the band, narrowband for geometrical path length  $x$  is = 1 centimeter, using the Elsasser model and Malkmus model.

So, the first thing that you should observe is, that the units of line strength or absorption coefficient, mean absorption coefficient as  $\tau$  in terms of pressure. So, we have to take path length based on pressure. So, remember in the very first few lectures, we discussed that absorption coefficient can be defined either based on simply path or it can be defined based on pressure based or can be density based. So, in this problem we have pressure based absorption coefficient. So, we have to take the path length accordingly in this problem.

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### Solution

$s = 1 \text{ cm}$   
 $p = 0.75 \text{ atm}$

$x = p \cdot s$   
 $x = 0.75 \text{ cm atm}$

$$x = \frac{SX}{2\pi b_L} = \frac{0.08 \text{ cm}^{-2} \text{ atm}^{-1} \cdot 0.75 \text{ cm atm}}{2\pi \times 0.03 \text{ cm}^{-1}}$$

$$x = \frac{1}{\pi}$$



$$\beta = \frac{\pi b_L}{d} = \frac{\pi \times 0.03}{0.6} = \frac{\pi}{20}$$

$$\tau = 2\beta x = 0.1$$

$\bar{\epsilon}_\eta = 0.0867$       Elsasser Model

Statistical Model      Malkmus model

$$\bar{\epsilon}_\eta = 1 - \exp\left\{-\frac{\pi}{2} \frac{\pi}{20} \left[1 + \frac{4 \times 0.1}{\pi/20}\right]^{1/2} - 1\right\} = 0.0670$$



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So, we defined, first of all path length based on pressure. So, we multiply by pressure and take the path length. Now,  $s$  is given as 1 cm and pressure is given as 0.75 atm. So,  $x$  now becomes = 0.75 centimeter atm. So, this is the path length for the problem that we have

found. Next, so we are going to, let me just show you the quickly what relations we are going to use. So, for Elsasser model, we are going to calculate  $X$ .

We are going to calculate mean emissivity from this relation. Of course, solving this integral is not easy. So, I will just give you the final result and will not try to solve this integral in this lecture. So, we first find out small  $x$  line strength parameter as  $= S \text{ times } X \text{ upon } 2 \pi b L$ . Okay. And this will be  $=$ , now  $s$  0.08 centimeter  $- 2$  atmosphere inverse.  $x$  we have already calculated 0.75 centimeter atm upon  $2 \pi$ .

$b L$  is given as 0.03 centimeter inverse, 0.03 centimeter inverse. So, this value of  $x$  comes out to be  $1 \text{ upon } \pi$ . Okay. The next parameter we need is beta, line overlap parameter. So, beta is basically  $= \pi \text{ times } b L \text{ upon } d$ , line overlap parameter. This will be  $\pi \cdot b L$  is given again, 0.03. And  $d$  is given as 0.6 from here.  $d$  is 0.6. So, this will become  $= \pi \text{ by } 20$ . Okay. So, from here, now tau is  $= 2 \text{ beta } x$ . So, this will be  $=$  simply 0.1. Okay.

Now, we have calculated all the parameters required. Now, the mean emissivity can easily be calculated. So, we can use this formula. Let me just show you this formula again. So, we have to put the value of beta and  $x$ . And integrate here from  $-\pi$  to  $\pi$  to find out the mean transmissivity. Okay. So, again I will not try to solve this here in this lecture. So, this value, I am just directly writing. Will be  $= 0.0867$ .

This is the mean emissivity of a gas layer using the Elsasser model. So, this is the mean amount, mean emissivity of the cell or the path, given path filled with its particular gas using the Elsasser model. Now, let us do using the statistical model. So, we use the Malkmus model here. So, let me just show you what formula we will be using for the Malkmus model here. Okay. So, we have a homogeneous path.

So, this is the formula we are going to use.  $1 - \text{exponential} - W \text{ by } d$ . Where  $W \text{ by } d$  is given by the this relation,  $\text{beta by } 2 \cdot 1 + 4 \text{ tau upon beta half} - 1$ . So, for the Malkmus model, we can calculate emissivity. I will just directly write the results. It will come out to be  $= 1 - \text{exponential} - 1 \text{ by } 2 \pi \text{ by } 20$ . Just writing the values in the expression without actually showing you the calculation here.  $1 + 4 \text{ into } 0.1 \pi \text{ by } 20 \text{ power half} - 1$ . And this value comes out to be 0.0670. Okay.

So, this is how we can calculate the emissivity of a gas layer, a homogeneous gas layer using the 2 narrowband models. That is the Elsasser model which is uniformly spaced lines and statistical model, statistical Malkmus model, random placement of lines. So, thank you. In the next lecture, we will discuss wide band models, which further improve the efficiency of radiative calculation in combustion and with less, very small sacrifice in accuracy of the calculations. We can speed up the calculations by averaging over the entire Ro-vibrational band. Thank you.