**Radiative Heat Transfer**
**Prof. Ankit Bansal**
**Department of Mechanical and Industrial Engineering**
**Indian Institute of Technology - Roorkee**

**Module - 6**
**Lecture - 27**
# Monte Carlo Method for Thermal Radiation I

Hello friends, in this lecture we will study the Monte Carlo method as applied to solve the radiation problems in complex geometries. The Monte Carlo method is a class of method that basically relies on random sampling to solve mathematical as well as non-mathematical problems. So, I will give you an example. Many news channels conduct pre-poll survey to find out the outcome of an election in our country.

They do this survey to find out the success, expected success of different parties in different regions. Now, for this survey to be successful or effective, the sampling needs to be done. Now, the sample size has to be reasonably good in number and quality. Now, how this agencies do sampling, they go to different villages, they talk to people of different age group, different social background, different economic background, they talk to people in cities, in villages, in different localities of different religion, different caste, different language.

And based on this response, they conclude or make a guess of the expected outcome. Now, this procedure is called Monte Carlo. So, Monte Carlo basically is a method where we do random sampling. So, the sample that we are taking should be purely random. There should not be any biased in the collection of samples. That means, the agency who is conducting the survey should not be biased in taking the samples.

Bias means, they should not take more samples from a particular group and less samples from a, from the other group. If that is done, then the sample would be biased and the expected outcome will not be correct. So, the sample should be purely random. And once we do this random sampling, it is expected that the result will be very close to what we are predicting based on this random sampling.

Now, the same thing we can apply to solve the radiative transfer equation. In numerous examples, we have learnt that the problem of radiative heat transfer is very complex. It involves multiple parameters. The problem may be complex in 3-dimension. The intensity depends on 2 directions. There may be wavelength dependence. And all these parameters makes the problem much more difficult for radiative equations to be solved.

Especially, we get integral equation for radiative transfer between flat surfaces or plane surfaces with vacuum as a participating media, with no participating medium. And we get integro differential equation when we have participating media. So, overall the problem is very complicated. If we try to solve this problem analytically or numerically, we have lot of challenges.

On the other hand, the Monte Carlo method is an effective method that can solve problem of any complexity to arbitrary degree of accuracy without any approximation. All we need to do is random sampling of photons from the surface, from the gas. We have to emit samples randomly. We have to absorb scatter the samples randomly. And we have to trace the history of this photons. And based on this, we can solve the radiative transfer equation.

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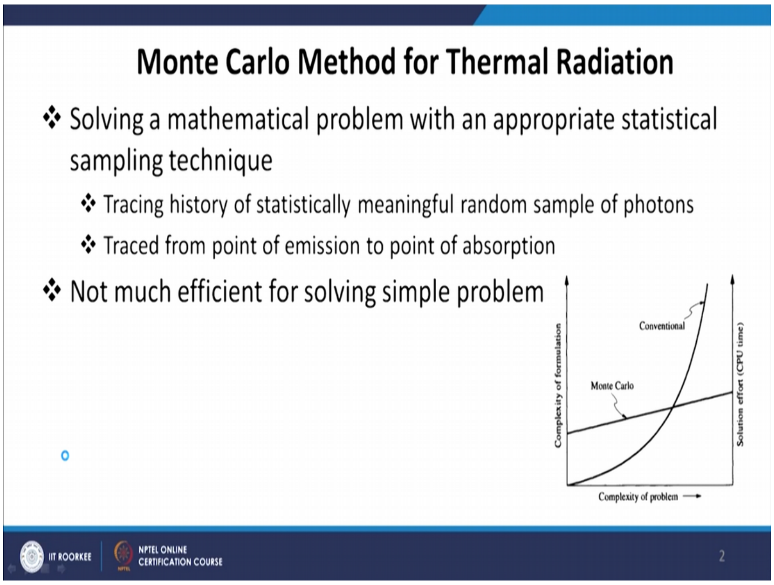


Now, this problem, the Monte Carlo problem is computationally very expensive. Although the problem can be of any complexity, the Monte Carlo method can solve it to arbitrary degree of accuracy. But the problem solving requires significant cost, significant computational cost. We may have to sample millions of photons in the problem before we get good results. So, it is not a good idea to apply the Monte Carlo method to simple problems.

If we look at the time it requires to solve, we see that the Monte Carlo method is less efficient for problems with less complexity. It takes more time than the conventional methods. While, if the problem complexity increases, the Monte Carlo method is more accurate. The effort used by the Monte Carlo method increases linearly, while the effort required to solve problem in standard conventional way increases exponentially.

So, for complex problems, the Monte Carlo method is a powerful tool that can give you accurate results in any complex geometry, without any simplification or approximations. Now, how do we apply this?

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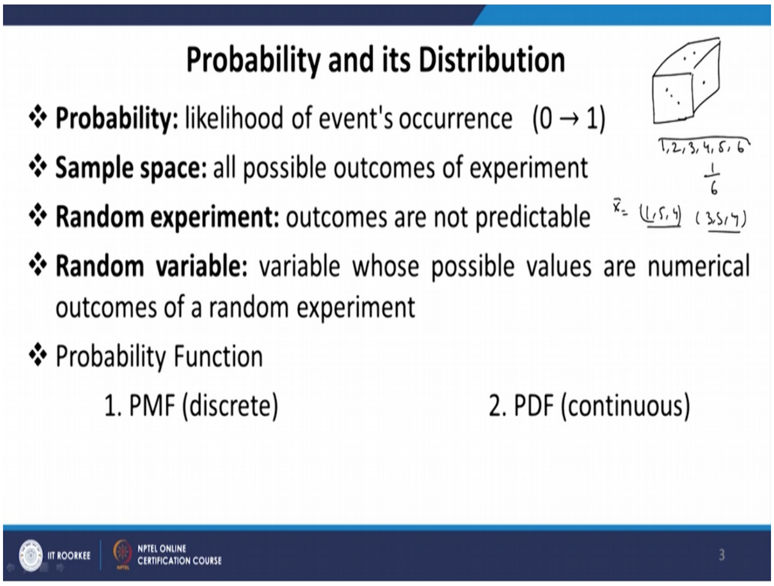


Before we discuss how do we apply the method to solve radiation transfer radiation problem, let us discuss some terminology used in any Monte Carlo procedure. So, we have probability, the first thing. Probability is basically a likelihood of events occurrence. So, this we discussed while discussing the view factor also. So, let us say we have a dice with 6 faces, each numbered from 1 to 6.

Now, when we roll this dice, we can get a number from 1 to 6. So, we can get 1, 2, 3, 4, 5 or 6. And the probability of each of these number is = 1 by 6. We discussed this earlier also. So, the probability of getting 1 is = probability of getting 2 and so on, is = 1 by 6. So, the probability is basically the likelihood of events occurrence. So, what is the likelihood of getting 1; what is the likelihood of getting 2; and so on.

The sample space is basically the all possible outcomes. So, 1 to 6 is the sample space. We can get 1, 2, 3, 4 and so on. So, this is sample space. Random experiment means, we are doing an experiment where we want to roll the dice, let us say 3 times and find out what numbers do we get. So, we can have 1, 5, 4. This may be 1 experiment. We can have 3, 2, 5, 4. And this may be second experiment, and so on.

In each experiment we can have number of samples. And the outcome is going to be purely random, which cannot be predicted. So, random experiment means, outcomes are not predictable. We will get some numbers, but the numbers may be purely arbitrary or random, out of the sample space. Random variable is a variable whose possible values are numerical outcomes of random experiments.

So, this 5, 1, 5, 4 is the outcome of this experiment and they are purely random. We can represent this by a random variable. Let us call x vector is = 1, 5, 4. So, x vector is a random variable which has 3 values 1, 5, 4. And these 3 values are random, purely random, coming from rolling of the dice. Now, we can have discreet values. We can have continuous values. So, this experiment has discrete values 1, 2, 3 and so on.

For one such discrete values, we define what we called probability in mass function. On the other hand, we can have certain numbers which are purely continuous in nature. And for those functions, for those variables, we define what we call probability density function.

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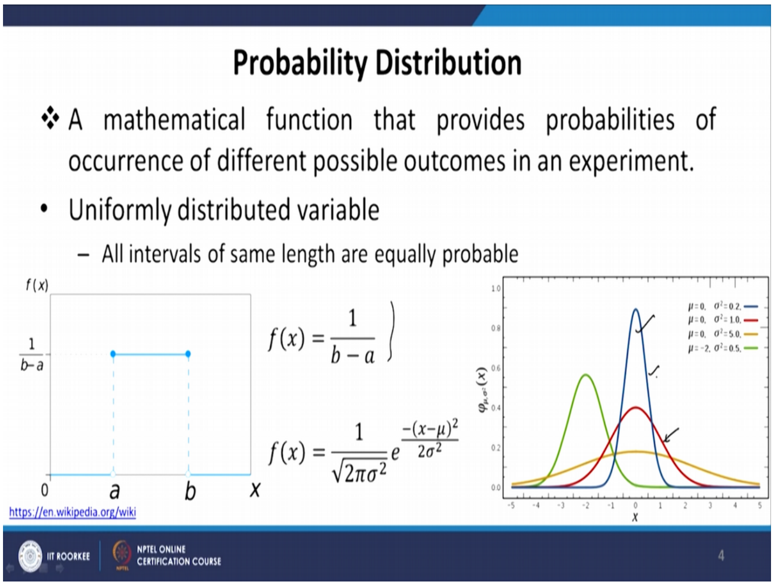

So, probability density function or probability distribution is basically a mathematical function that provides a function to evaluate probabilities of finding a different outcome when we do an experiment. For example, the uniform probability density function is given by f x is = 1 upon b – a. What it says is that, the probability of getting x between a and b is = 1 upon b – a. If a is = 0, b is = 1, then f x is = 1.

That means, the probability of getting a number x; x can have any value 0, 0.001, 0.00002 and so on. It is a continuous function. And the probability of getting any number between 0 and 1; if a is = 0 and b is = 1 is 1. And this probability is independent of x. That means, the probability is same for all values of x. And this is called uniformly distributed variable or uniform distribution.

If a is not = 0, b is not = 1, then the probability will be 1 upon b – a, between a and b and it will be 0 outside this interval. Similarly, we have random variable distributed along what we called normal distribution of Gaussian distribution. It depends on 2 parameters, mu and sigma. Mu is the mean value and sigma is the standard deviation. And we see that the maximum probability in this function is at the mean value.

If you look at it, at the mean value, that is x is = 0, so mean value 0 means, the value of x is 0. So, the probability is maximum. So, the maximum probability in Gaussian distribution or normal distribution is at the mean value. And then, this distribution shows that the probability decreases rapidly. From maximum at the mean, it decreases rapidly in both the direction from the mean value.

And how fast it decays away from the mean depends on the standard deviation. If your standard deviation is small, then it decays much faster. If your standard deviation is large, then it decays much slow. So, the red curve here has standard deviation, which is basically root 1 or 1. And it decays much slowly from the mean value. While this curve blue, has a standard deviation of root 0.2. And it decays much faster than the red distribution.

So, this is normal distribution. Now, which distribution shall we use in solving a problem depends on the type of problem. So, in a election we may assume that all people carry equal opinion. We may use a uniform distribution. But if we come to know that certain group of people is more likely to have a particular opinion than another group of people, then we may not use a uniform distribution. And we may have to go for some other distribution. So, we must understand the problem before we actually apply what kind of probability density function we should use in our problem.

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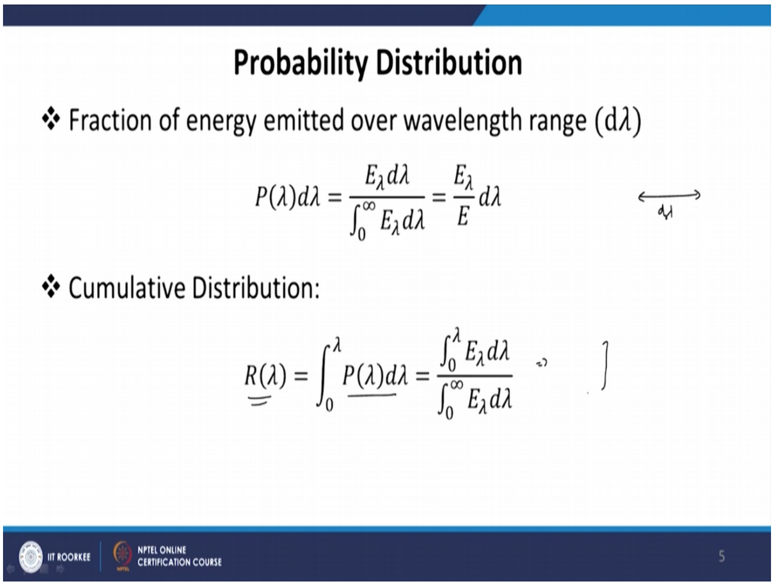


In Monte Carlo methods applied to radiative transfer problems, we define different type of probability distributions. For example, we can define for wavelength selection. So, everything is sampled randomly, the location, the emission point, direction, even the wavelength is sampled randomly in Monte Carlo method for radiative transfer. And for wavelength selection, we define the probability density function, which is basically = the emitted energy in a small wavelength interval.

So, we have a small wavelength interval d lambda. How much energy is emitted in that interval, divided by total energy emitted gives you a probability distribution. From this probability distribution, we need to find out random photons carrying energy in this wavelength interval d lambda. So, we will see how this probability distribution will be used in the Monte Carlo method for radiative problems.

In probability, we also talk about cumulative distribution function. So, probability distribution function gives you the probability of finding the given value at a given value of x, but probability distribution gives you total probability of getting a variable for all values of x < or = that value. So, for example, the cumulative distribution R lambda in this space is defined as sum or integrated probability over all the wavelengths, < the value lambda.

So, R lambda is the cumulative probability from 0 to lambda. And this can be defined as 0 to lambda E lambda divided by total emission. So, we take total amount of energy emitted from all the wavelengths < lambda and divided by total amount of energy emitted over all wavelengths. So, this is called cumulative distribution. So, normally we apply this cumulative distribution rather than probability density function for sampling in our problems.

So, mostly we will apply this cumulative distribution function, because it is easy to use cumulative distribution function for sampling. And we will use this for the solution of the problems. So, general procedure for photon Monte Carlo basically involves large statistical samples.

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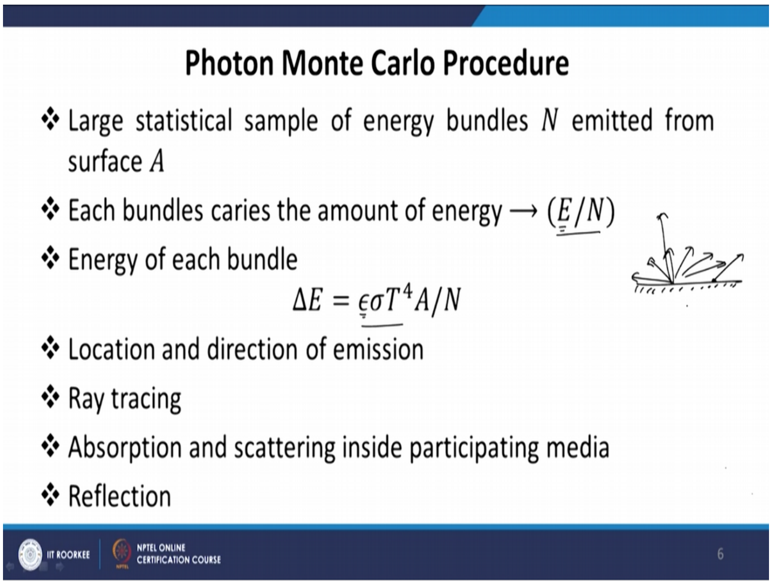


We have to divide the problem into number of photons. So, if you have a surface, let us say. And we want to find out heat transfer from the surface. What we will do is, we will randomly emit photons, let us say total N number of bundles, energy bundles, randomly in all direction. Okay. So, we have to divide the problem into random or statistical samples, each sample carrying equal amount of energy.

So, total energy, let us say is E emitted from the surface. Then each bundle, each energy bundle will carry an energy = E by N. And if the emittance of the surface is let us say epsilon, the total amount of energy emitted is epsilon c tigma T 4 by A. And energy of each bundle will be this total energy divided by the total number of bundles N. Once we know how many bundles to emit, we have to find out the properties of this bundle.

What will be the location of a particular bundle; what will be the direction in which it will be emitted; whether this bundle will travel without any attenuation; or will it be subjected to absorption and scattering if there is a medium adjacent to the plate. We have to trace this bundle from its origin to its final destination. And wherever it reaches, whether the other surface or the gas, we have to find out whether it is subjected to scattering.

If scattered, in which direction the scattered photon will go; if reflected, which direction the reflected photon will go. So, all this things, we have to basically incorporate in solving the radiation problem using the photon Monte Carlo method.

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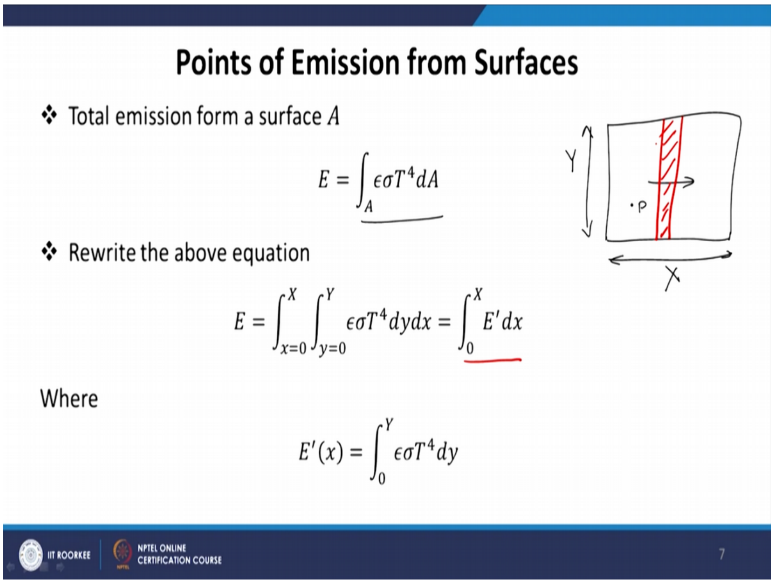


So, the first thing that we will discuss in the photon Monte Carlo or Monte Carlo method as applied to radiation problem is, the emission from a surface. Now, we let us say we have a surface. The dimension of the surface is x and this is y. Okay. This 2-dimensional plate. And we have to find out at what location a photon is emitted. Let us call this point p. So normally, you will expect that that the, there is a uniform probability, the photon is equally likely to be emitted from any point on this plate, while this is true for a plate having normal isotropic properties.

That means, if the plate has uniform temperature over its entire area, if the plate has uniform emittance over its entire area, then definitely the probability will be uniform. We can use uniform distribution to find out the point of emission on this plate. But what will happen if the temperature is varying? If let us say 1 end is at lower temperature and other end is at higher temperature, which direction you think the photons will be emitted more?

So obviously, where the plate is hot, more photons will be emitted, where the plate is less, having less temperature, less number of photons will be emitted from that location. So, we will give a general relation. And then, we will simplify the relation for a special case of isothermal isotropic plate. So, we define total amount of energy emitted from this plate as epsilon sigma T 4 dA.

And this can be splitted into double integral over this plate, x is = 0 to x. y is = 0 to y sigma T 4 epsilon dydx. Now, we define another quantity E prime, where E prime is basically the amount of energy emitted at all ys at a given x. So, E prime x is amount of energy emitted, at all ys at a given x. So, this is E prime x. Total amount of energy emitted at a given x for all values of y.

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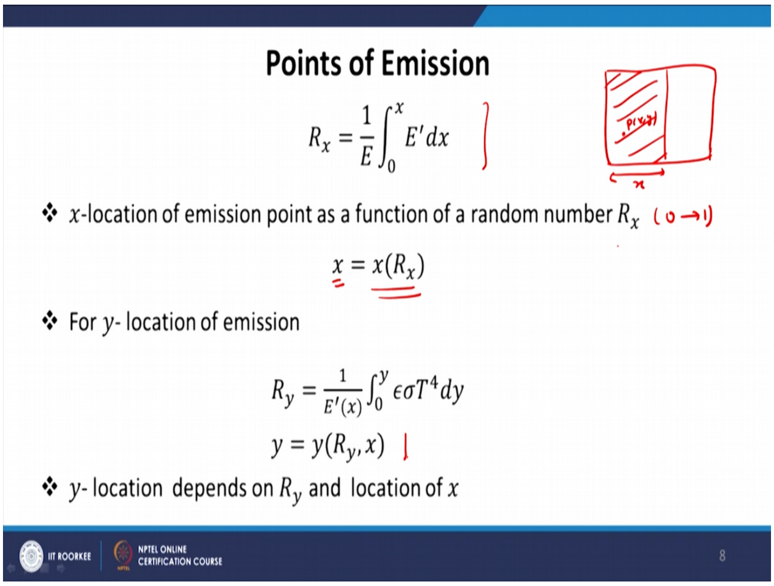


Now, we define a cumulative distribution R x. So, cumulative probability distribution R x is basically the amount of energy emitted 0 to x E prime dx upon 1 upon E. That is, amount of energy. So, this is the plate. Amount of energy emitted from this plate, divided by total energy, gives you the cumulative probability of emission from this plate for all values of y. Now, this relation needs to be inverted.

So, what we have is, random number on the left-hand side. And unknown x on the other side. So, we are interested in finding the location p. Okay. So, this p needs to be, this location p which has coordinate x and y needs to be determined. Now, x and y are unknown, which are on the right-hand side. While random number which you can easily generate from any computer language using random number generator is on the left-hand side.

So, this relation cannot be inverted easily. You have to apply some strategy to invert this and find out the value of x. And for this type of problems, we will see that this may be little challenging. But the point is, we have to find out the value of x by inverting the relation. Given the value of R x, find out the value of x. Okay. So, where R x is a random number between 0 and 1.

So, R x is a random number between 0 and 1, because probability distribution may be uniformly distributed or it may be distributed using Gaussian. But cumulative distribution will always be between 0 and 1. So, we will use the cumulative distribution. And this R x, basically we will take 1 value from 0 to 1 and we will invert this relation to find out the value of x. Similarly, for y, we define R y, the random number R y is basically 1 upon E prime x.

That is energy emitted at all ys at a given x divided by 0 to y epsilon sigma t 4 d y. And we can invert this relation. So, y will be a function of x and R y now. Because the temperature of the plate is not uniform, so the random location will depend on x. Okay. So, in general, this relation is very difficult to invert, but we will see that for certain simplification we can invert it.

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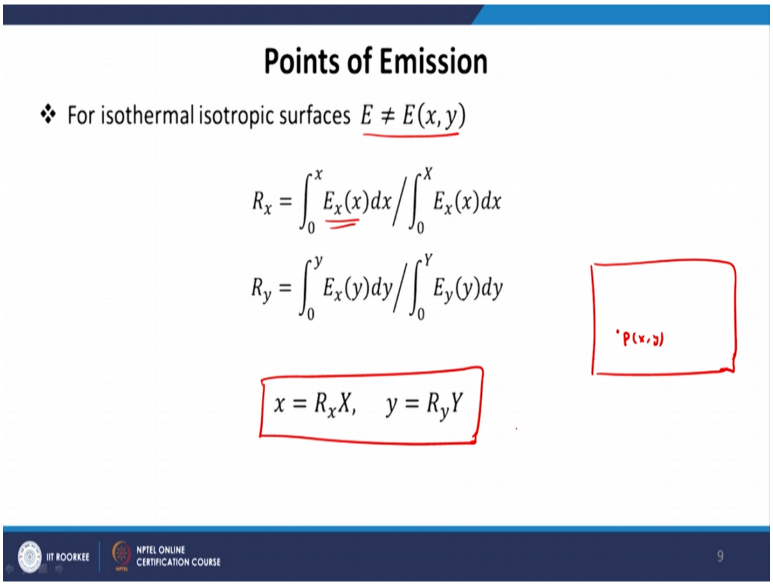


So, mostly we deal with isothermal surfaces. That means, the function E, the energy emitted from the surface is not a function of x and y. And from, for that case, the E x will basically be taken out of the integral. And what we get is simply this relation, x is = R x times capital X; y is = R y times capital Y. So, we have easily evaluated with, we do not need to invert this relation anymore.

So, for isothermal surfaces, the relation is simple algebraic relation. And we can easily calculate the value of x and y at any point. So, by this relation, we have to pick 2 random numbers R x and R y. And the location of point p x and y can be easily calculated based on this relation.

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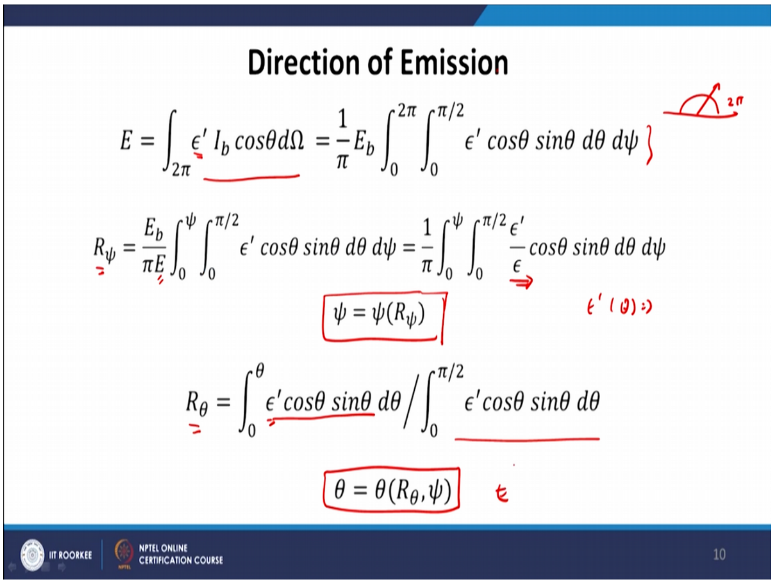


The second is the direction of emission. Now, direction of emission we will start from general relation where the emittance is a function of direction. If it is a function of direction, then the relation is going to be complicated, be very difficult to invert, we will see. But if the direction, if the emittance is isotropic, that means the value of emittance does not depend on direction. Then, the relation will be simplified.

So, again we write the relation. Total amount of energy emitted by the surface is directional emittance multiplied by I b cos theta and integrated over the solid angle. So, amount of energy emitted in a given direction. And then, we have to integrate over the entire solid angle of 2 pi. This will give the total amount of energy. And we can write down d omega, the solid angle as sin theta d theta d psi.

Now, we define the cumulative distribution. So, cumulative distribution is amount of energy emitted from 0 to psi. That means, a range of azimuthal angle divided by total amount of energy E. So, we have R psi is = E b upon pi epsilon prime cos theta sin theta d theta d psi. Where we have to integrate for all polar angle 0 to pi by 2, but only a range of azimuthal angle 0 to psi. Now, we substitute the value of E from this relation into this expression for R psi and we get 1 upon pi 0 to psi 0 to pi by 2 epsilon prime upon epsilon.

Where epsilon is now average emittance over the entire hemisphere. Okay. So, epsilon is the average value of the emittance over the entire hemisphere. We discuss this while driving the relation for flat surface and we see that psi is basically a function of R psi. Okay. Now, if we do not know the variation of epsilon prime, if we do not know how epsilon prime varies with theta, then this relation is going to be difficult to invert.

And we may have to do some numerical calculation for finding the direction of emittance. Similarly, for theta angle, we write R theta is = 0 to theta epsilon prime cos theta sin theta d theta. And then, over the entire solid angle or entire range of theta angles. And we get theta as a function of R theta and psi. And this relation is going to be difficult to invert for theta. Now, we will take a simplified case where emittance does not depend on wavelength.

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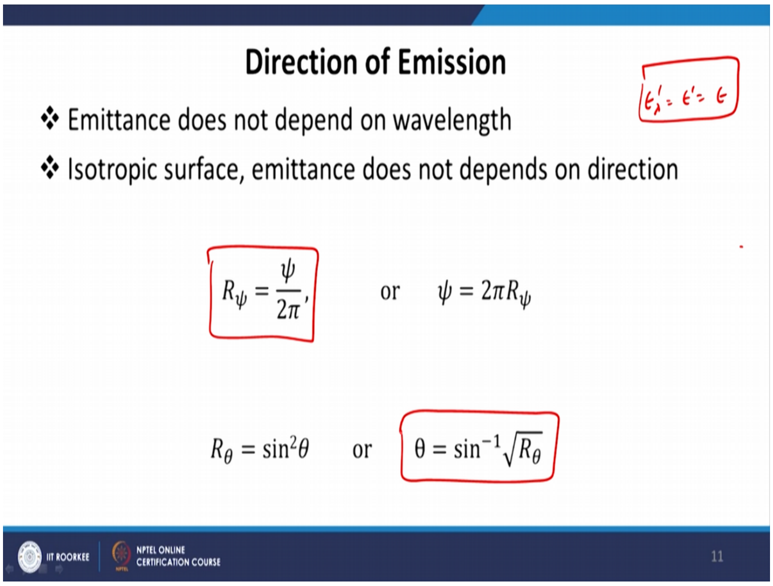


And it is isotropic and it does not depend on direction. So, we assume that epsilon lambda prime is = epsilon prime is = epsilon. That means, it is independent of wavelength and direction. Now, what we can do is, we can take it out. This epsilon prime can be taken out of the integral from this expression or it will just cancel out. And the relation is simplified as R psi is = psi upon 2 pi.

So, we can take or choose azimuthal angle directly. All we have to do is, pick a random number between 0 and 1, R psi multiplied by 2 pi. And that will give us the azimuthal angle. Similarly, the, this equation is simplified as R theta is = sin square theta or theta is = sin inverse R theta root R theta. We pick a random number, take it inverse, take it square root and take it sin inverse and we get the direction theta. So, in this way, we can find out the direction of emission theta and psi for different type of surfaces, isotropic or non-isotropic.

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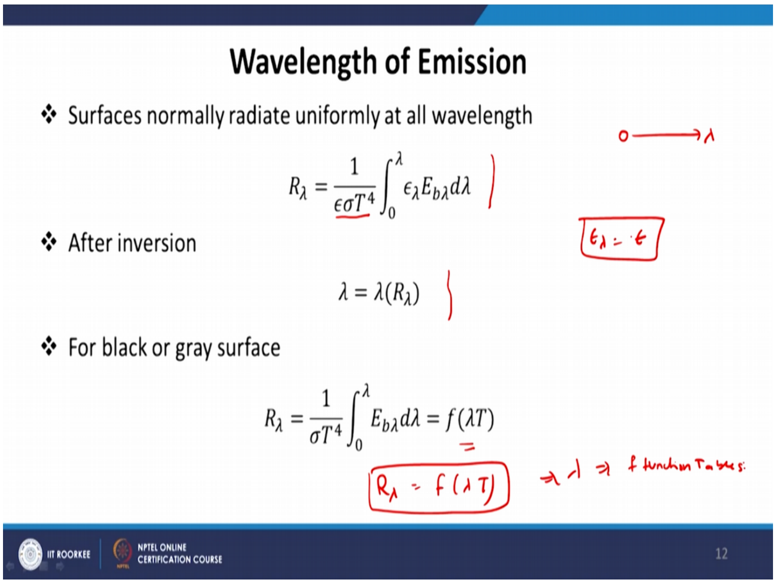


The third thing is wavelength. How to select the wavelength of the photon? So, we have emitted N number of photon bundles. Each bundle has total energy as same, but the wavelength of the photon bundles may be different. So, we defined a probability, cumulative probability d function as total amount of energy emitted from the surface in the denominator and total amount of energy emitted from the surface in a given wavelength range 0 to lambda.

So, 0 to lambda is amount of energy emitted by the plate in this wavelength range cumulative wavelength range. And this ratio gives you the cumulative distribution function for the wavelength selection. Now, again we have to invert this relation. And inverting this relation may be easy if the plate is assumed to be black or gray. That means the epsilon lambda is = epsilon. If we assume this, then the relation can be easily inverted.

And we have identified this basically nothing but f function. That is, the fractional black body emissive power for which the tables are easily available in the book. So, you have to pick 1 random number R lambda. And you have to look at the table of the f function in the book and find out the lambda. So, lambda will come from f function tables. So, this will have to look at the tables to find out the lambda. So, this is how you will find out the wavelength of the emitted photon from the surface.

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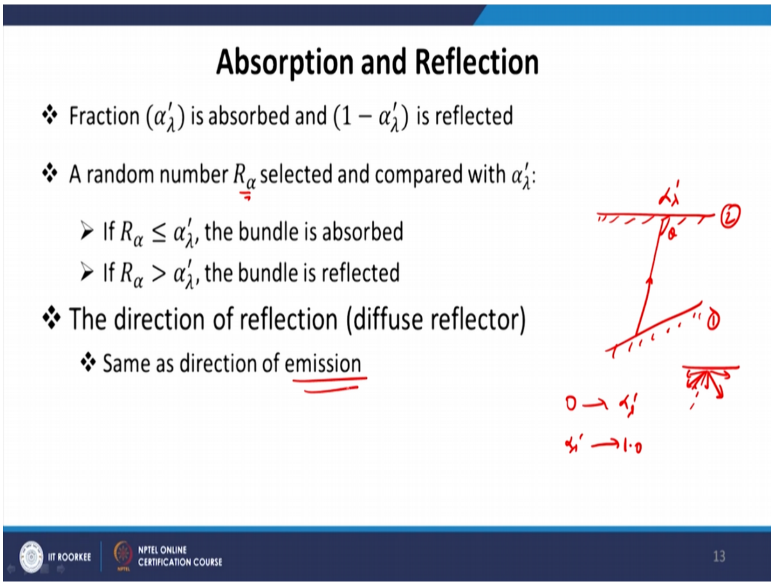


Now, let us say we have 2 surfaces. 1 photon is emitted from the surface and we want to see whether this will be reflected or absorbed by the other surface. We have surface 1, we have surface 2. So, let us say surface 2 has absorptance alpha lambda prime. Okay. Which depends on wavelength and which depends on direction. Now, if this photon strikes this plate at certain angle theta, then the absorptance is alpha lambda prime and 1 – alpha lambda prime is the reflectance.

Now, we will relate this reflectance with or absorptance with the random number. So, what we say is, we pick a random number R alpha. And if R alpha is < the value of absorptance, the bundle will be absorbed. And if the value of R alpha is > this absorptance value, the bundle will be reflected. So, we have related this absorptance or reflectance with random number. So, 1 to 0 to certain fraction of alpha lambda prime, the chances are for absorptance and alpha lambda prime to 1, the chances are for reflectance.

So, in this way, we can find out whether the photon will be absorbed or it will be reflected from the radiation. Now, if it is reflected let us say, then we have 2 possibilities. The reflected radiation may be specular. Then we have to find out the direction of the new photon after reflection. This direction may be specular. We have to apply the relation of Snell's law to find out the direction.

Or it may be diffuse. That means, it has equal probability in going in any direction after reflection. So, if the surface is diffuse, the photon after reflection may go in any direction.

And it is basically same as emission. So, you have to apply the same relation for emission that to this reflection, for diffuse reflection. So, let us solve 1 problem.

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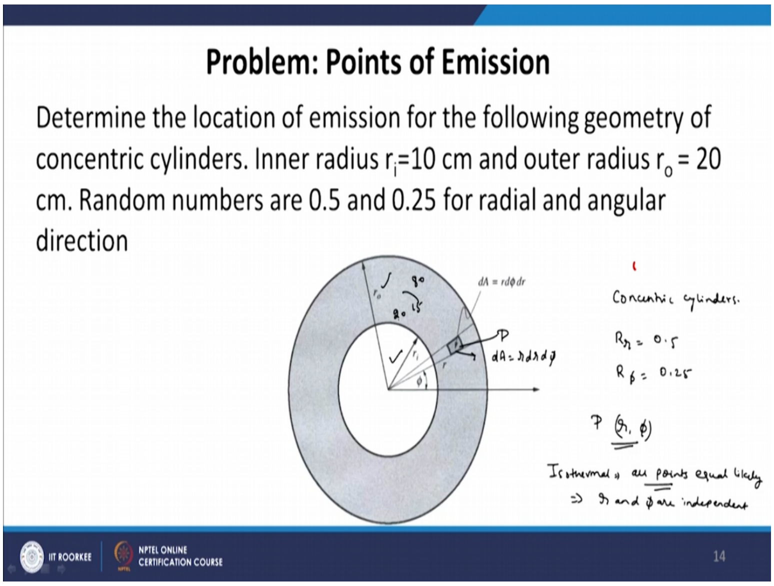


And we will see how to find out the emission location for the geometry given in this problem. So, we have basically a concentric cylinders. The inner radius, so concentric cylinders; the inner radius is r i and the outer radius is r nought. We have to find out the point of emission on this annulus. The value of r i is given as 10 cm, the outer radius is 20 cm. Now, we have to repeat this process in a Monte Carlo procedure for large number of randomly selected photons.

But in this example, we will just do it for a single photon. And the value of random numbers in 2 direction, that is for radial direction is 0.5, and in angular direction R phi is = 0.25. So, with these 2 values of random number already given, we have to find out the location r and phi for point of emission. So, let us say we have point of emission. Let us call this point p. So, we have to find out the location of p as a function of r and phi.

Now, the plate is assumed to be isotopic, isothermal. That means, each point on this plate is equal likely. All the points on this plate, annulus plate are equal likely, because the plate is isothermal. So, we have to find out. So, plate is isothermal. So, all points equal likely. Okay. Now, all points are equal likely, because of the plate being isothermal. And 1 thing we have to note here is that r and phi are independent.

So, we noted in the case of a flat plate of, a flat rectangular plate, that if the plate is not isothermal then the y will depend on x. But in this case, because the plate is isothermal, then the r location will be independent of phi.

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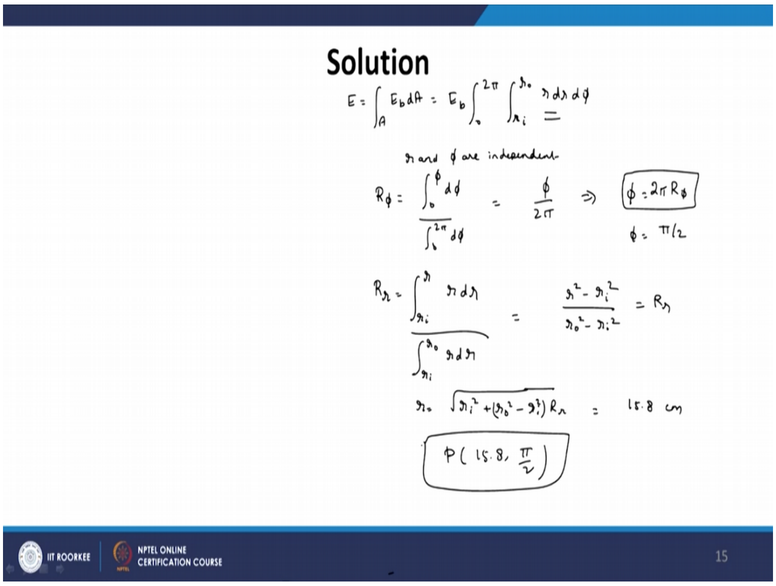


So, how do we solve this problem? So, we write the total amount of energy E emitted from this plate as integrated over the entire area. Now, let us say the plate is black. So, E b times dA. And this will be = E b 0 to pi r i to r nought rdrd phi. Where rdrd phi is the area of this small element rdrd phi. So, this is the total amount of energy emitted from this surface. And we have to integrate, double integrate from value of r from r i to r nought, inner radius to outer radius.

And we have to integrate over the entire angle phi 0 to 2 pi. Now, because r and phi are independent, because of the plate being isothermal, we redefine a cumulative distribution R phi as = 0 to phi d phi. Note that there is no dependence on phi. The integrant is R and it does not depend on phi. So, we get 0 to 2 pi d phi. And this will be = phi upon 2 pi. And it means, the angular direction is nothing but R phi times 2 pi.

And this relation basically we found earlier also. So, all angular directions are equal likely. So, we get a linear relation. You have to pick a random number from 0 to 1 and all directions are equal likely. So, phi is = 2 pi R phi. Now, the random number R r, cumulative distribution in the radial direction is = integration from r i to r nought rdr divided by; so this will be = let us, this will be, we have to take r.

And this will be r i to r nought rdr. Okay. That means total area in the denominator divided by up to a certain radius r. And this will be = r square – r i square upon r nought square – r i square. So, we get, and this will be simply = the random number r. So, all you have to pick is basically the random number R r. And based on that, you can find out the radius of emission as r i square + r nought square – r i square R r.

So, we have given, been given 2 random numbers. R phi is 0.25 and R r is 0.5. So, from this, we get phi is = pi by 2. And from this relation we get r is = 15.8 centimeter. Okay. So, this is the solution. The point of emission p is nothing but 15.8 pi by 2. This is the solution. 1 thing you note is that, the result r gives you under root r i square + r nought square – r i square times R r. So, in this relation, one should observe that; let us say we pick 100 photons randomly.

Then, out of 100 photons; so, we have inner radius as 10 centimeter, outer radius is 20 centimeter. 10 centimeter is inner radius, outer radius is 20 centimeter. And let us say, the mean radius of this annulus is 15 cm. So, mean radius is 15 centimeter. So, if we pick 100 photons, you will find that, more than 50 will be on the outer periphery. So, let us say we have 100 photons taken. 80 may be here, 20 may be inside.

That means, for radius < 15 centimeter. So, why is it so? Why we have more number of photons with radius more than the mean radius? The reason is because of the area. The area of the annulus plate is not uniformly located. If you look at the area between 15 cm and 20 centimeter, you have more area. If you look at area between 10 centimeter and 15 centimeter you have less area.

So, naturally the number of photons will be more for radius > the mean radius 15 centimeter. So, we I will end the first part of this Monte Carlo method here. In the next lecture we will discuss the Monte Carlo method for the participating media, where the plates are basically bounded by the participating gas absorbing and emitting gas. Thank you.