Radiative Heat Transfer
Prof. Ankit Bansal
Department of Mechanical and Industrial Engineering
Indian Institute of Technology - Roorkee

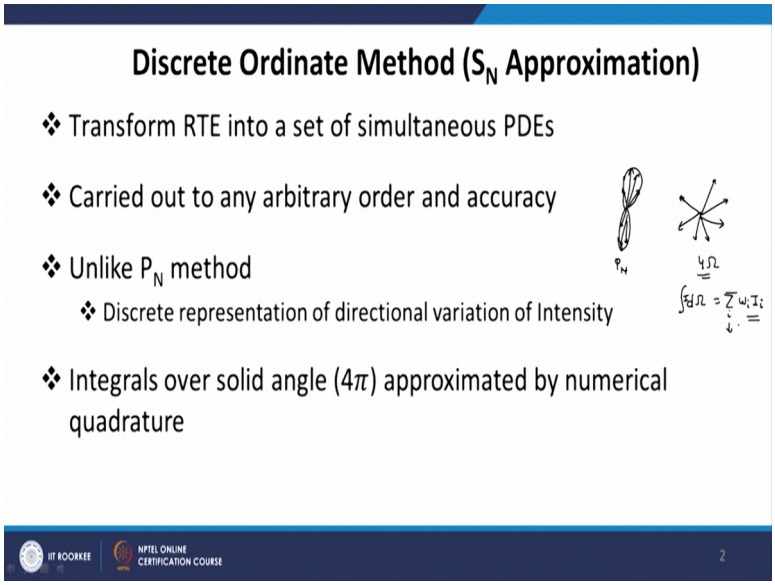
Module - 5
Lecture - 24
Discrete Ordinate Method (DOM)

Hello friends, in the last lecture we discussed the spherical harmonics method P N, for the solution of radiative transfer equation. The method included approximation of the intensity. In today's lecture, we will discuss the method of discrete ordinates, the S N approximation method. The method is very similar to the method of spherical harmonics in the sense that we convert the algebra, the integro differential equation into sort of partial differential equations.

We approximate the intensity in angular direction. And the method has similar accuracy as that of the spherical harmonics method. However, there is a difference between the 2 methods. As opposed to continuous distribution of intensity in spherical harmonics, where we represented the intensity in terms of a series of harmonics. Here we represent intensity as discrete values in different directions.

So, in a sense, the discrete ordinate method finds equivalents in finite difference method of CFD. So, just like we have finite difference method in CFD, this method also relies on discretizing the angular direction using finite angular directions, discrete directions. We also have 1 method which is not we are going to discuss. And the method is finite volume method, where we discretize the angular direction using the finite volume approach. So, this discrete ordinate method is very much similar to what we have the CFD in CFD the finite difference method. So, the objective is same.

(Refer Slide Time: 02:32)



We have to transform our RTE, the integro differential equation into a set of partial differential equations. We can choose as many discrete directions as we want. Of course, the directions have to be decided judicially. And we have to integrate over the solid angles for pi. So, just to explain the difference between the method of spherical harmonics and discrete ordinate method. In spherical harmonics method at any point, we represented the intensity as a continuous function or a series of function.

So, we had a continuous function over theta, in both + direction and – direction. So, that was the P N method. And we can retain as many terms as we want, leading to higher accuracy. So, we have P 1 method, the first order spherical harmonics method. We have P 3 method, the third order spherical harmonics method. However, in the discrete ordinate method, the intensity is not a continuous function of theta.

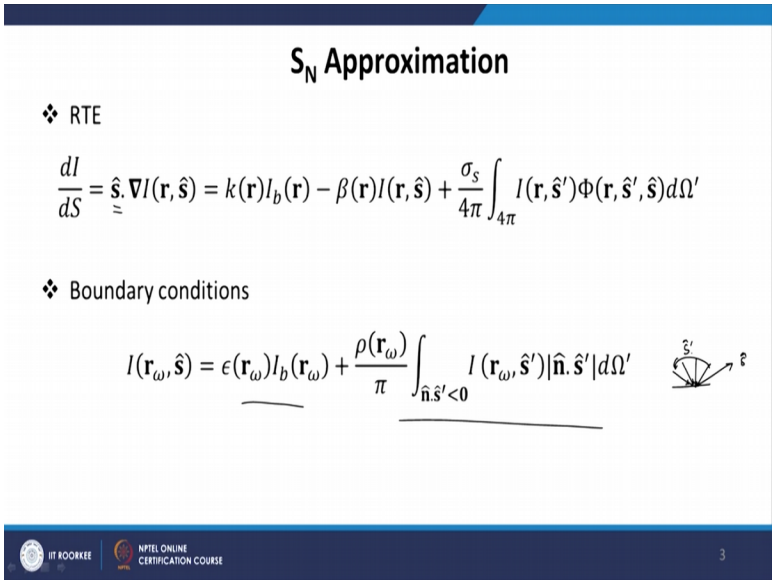
Rather, at any point, we take only discrete directions. So, we choose only discrete directions along which we will solve the RTE. So, we are basically discretizing the angular domain of 4 omega. So, this angular domain, angular solid angle, total solid angle we are discretizing using finite number of discrete directions. And then, we will integrate over all these directions using numerical quadrature.

So, what we will do is basically, we will integrate over the entire solid angle using some numerical scheme, numerical quadrature scheme, where we have w, the weights and intensity. If you are integrating over intensity, then we will have intensity in only those discrete

directions. Where I represents the discrete directions chosen using a suitable scheme. And then, we use numerical quadrature to integrate over the solid angles.

So, that is the difference between the spherical harmonics method and the discrete ordinate method that 1 basically is continuous function of intensity versus theta or azimuthal angle. And here we have a discrete function. And we use numerical quadrature to integrate over solid angle using these discrete values of intensity.

(Refer Slide Time: 05:12)



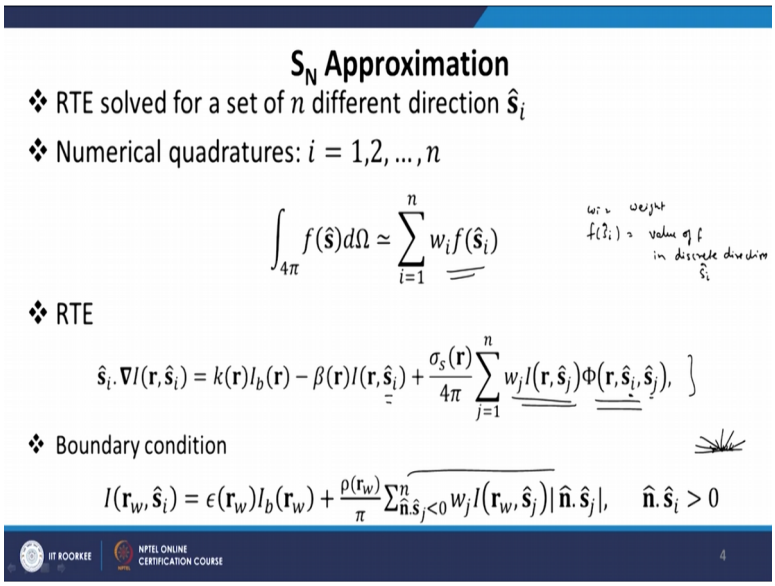
So, we will start the development of this method from the radiative transfer equation. So, for radiative, for absorbing, emitting and scattering medium, we have radiative transfer equation in a given direction s. Radiative transfer equation can be written as the kappa times I b. That is the emission term – extinction by scattering and absorption. And then, in-scattering, that is augmentation by scattering, in-scattering.

And this radiative transfer equation is subjected to boundary condition at the wall, where we have at the wall 2 components, the emission from the wall and reflection from the wall. Reflection will have reflectance rho times irradiation from all the directions. So, we have to integrate over the solid angle for all the intensities coming. So, we are interested in intensity coming from all directions and reflected in a given direction.

And that this direction is s cap. And this, all the directions are basically represented by s cap prime. So, these are the governing equations. We have observed them many times before. Now, how do we approximate in the discrete ordinate method? That is, N S N approximation,

we replace the quadrature or the angular integration over solid angles with appropriate quadrature, numerical quadrature.

(Refer Slide Time: 06:48)



So, if we have a function f, now in this case f will be intensity. So, if we have a function f that we want to integrate over solid angles. Then this integral is replaced by numerical quadrature. So, we replace it by summation i is = 1 to n. w i, where w i is weight. And f s i is value of f in discrete direction, s i cap. So, we have chosen certain directions. And along those directions, we are solving the RTE.

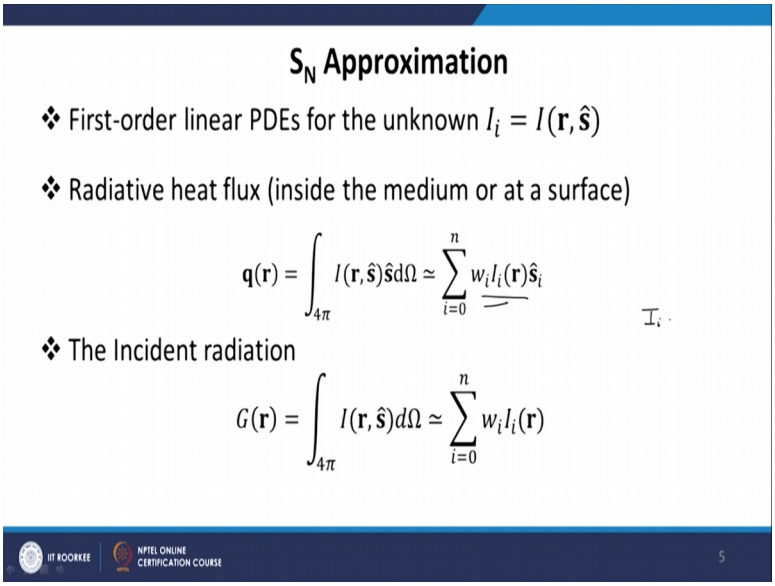
And the integral over solid angles appearing in the RTE is approximated using numerical quadrature. So, now our governing equation, the radiative transfer equation becomes purely differential equation. So, integration does not appear in this equation. We are evaluating intensity in discrete direction s i cap. And their integration has been replaced by numerical quadrature.

So, we have w j intensity coming from all directions s j. The scattering phase function which is now dependent on discrete directions. So, although the scattering phase function is a continuous function, but we have approximated the scattering phase function phi with a discrete function given from 2 directions s i and s j. Similarly, our boundary conditions now is no more an integral equation.

Boundary conditions also simplifies by replacing the integral in the boundary condition with the summation using numerical quadrature. So, the integration in the boundary condition here

for irradiation h is now replaced by a summation for all the incoming directions. So, n cap dot s j cap is < 0 means, we are interested in only incoming directions. So, this has to be, the summation is for all incoming directions over which the reflection has to be accounted for.

(Refer Slide Time: 09:01)

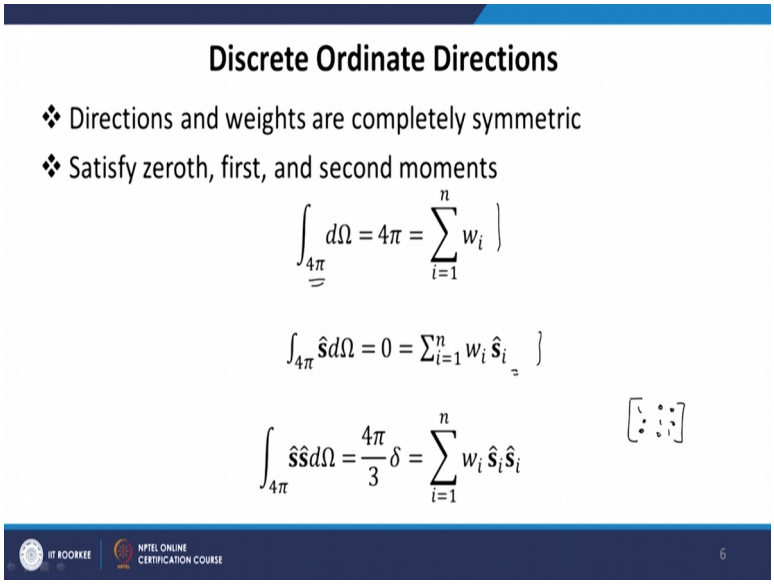


So, using numerical quadrature, we have simplified the governing equations. The integro differential equation has been converted into first order linear partial differential equation in unknown intensity I subscript i. So, there will be number of discrete directions. So, we have to solve a number of equations. Number of equations depend on how many directions, discrete directions we have chosen in our analysis.

Now, once the intensity is known, we can calculate the radiative heat flux. So, radiative heat flux is intensity times the direction vector s cap integrated over all solid angles. Now, we can replace I with I i and we integrate over the solid angle using the quadrature. So, our flux becomes summation i is = 0 to n, w i I subscript i s i. Similarly, the incident radiation is integration of intensity over the solid angle.

And this can be written as simply summation w i I subscript i. So, once the discrete intensity or the intensity in discrete direction I subscribe i is known, the quantities of our interest heat flux and irradiation are easily calculated using this summation in the quadrature.

(Refer Slide Time: 10:24)

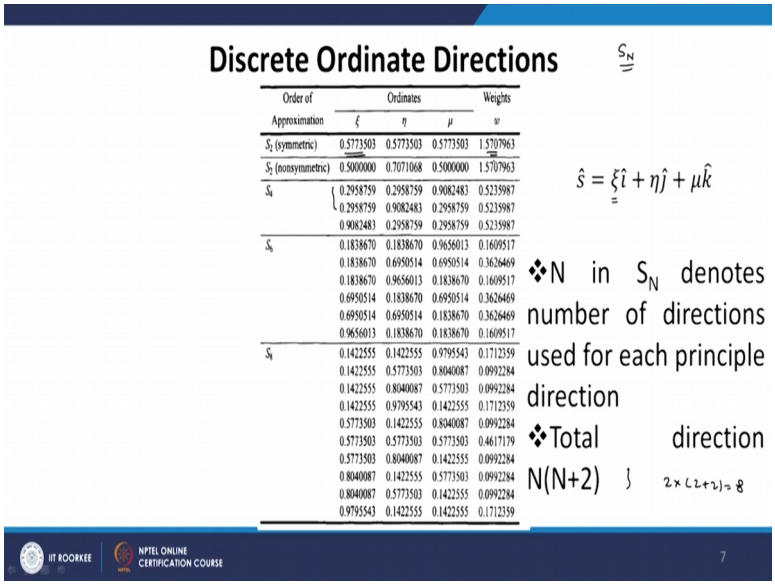


Now, how these directions are chosen? So, we have to choose a certain number of directions. But how do we choose this directions? It has to satisfy certain conditions. The condition basically includes zeroth, first and second moments over the solid angle. So, for example, the total solid angle over 4 pi gives, should give you total intensity. So, the summation of the weights all the weights i is = 1 to n should be = 4 pi.

Similarly, the first moment over the solid angle s d omega should be = 0. That is, the flux from all the directions should add up to 0. So, it becomes in summation, that summation i is = 1 to n, w i s i should be = 0. And similarly, the second moment, s cap times s cap d omega should be = 4 by 3 delta, where delta is basically the matrix, identity matrix. And this should be = summation i is = 1 to n w i s i cap times s i cap.

So, this value should be = the identity matrix times 4 by 4 pi by 3. So, when we choose directions, our directions, chosen directions should satisfy these rules. And based on this, the directions can be chosen.

(Refer Slide Time: 12:01)

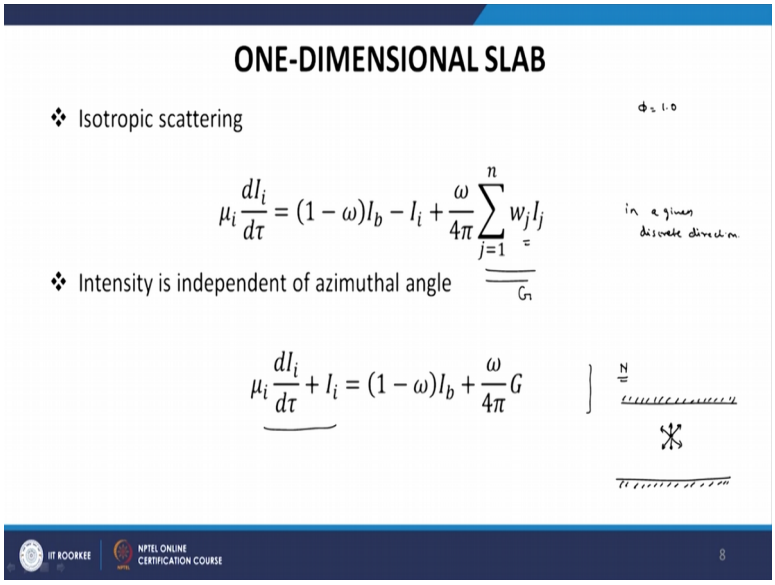


Now, for different values of approximations. So, we are talking about S N approximation, where subscript N represents the order of the method. If N is = 2, we have second order discrete ordinate method. If N is = 4, we have fourth order discrete ordinate method and so on. Now, this N represents number of directions used for each principal directions. So, suppose we have taken 2 values of zeta, now any direction can be represented by its direction cosines.

So, S direction can be represented as zeta i cap + eta j cap + mu k cap, where eta, zeta and mu are constants. So, we have taken 2 values. If we have taken 2 values of zeta, then it will be called second order discrete ordinate method. Now, we will have total 8 directions in this case. So, total directions will be 2 times 2 + 2, that is total 8 directions. So, for a 2 values of zeta, we will have total 8 directions.

But we will just call it second order discrete ordinate method. Similarly, if we choose 3 values of zeta, we will have 3 + 2, 5 times 3, 15 directions in all. So, as opposed to spherical harmonics method where we have first order and third order; even order terms were not there. Here we have even order for the method, that is S 2, S 4, S 6, S 8 and so on. S 3 and S 5 method is not basically done here. The weights of these ordinates is also given in this table.

(Refer Slide Time: 13:46)



Now, let us apply this method of discrete ordinate to 1-dimensional slab with isotropic scattering. So, once we have isotropic scattering, the value of phi will be = 1. So, our directional radiative transfer equation in direction mu i will be simply = mu i d I subscript i d tau is = 1 – omega I b – I i + omega upon 4 pi sum of the over all the quadrature weights, j is = 1 to n, w j i j. So, this is in a given direction, in a given discrete direction, the radiative transfer equation.

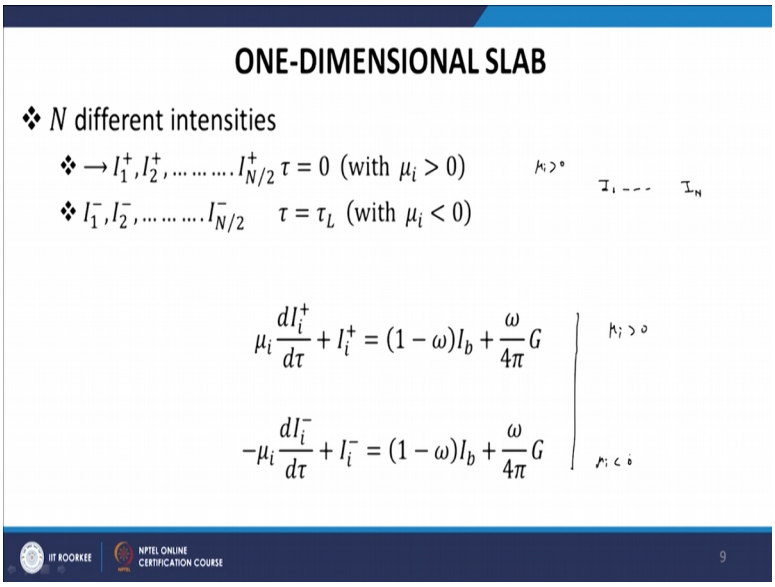
Now, for 1-dimensional slab, the intensity is independent of azimuthal angle. It depends on only on mu. And we observe that the method basically can also be written as first term remaining the same, the last term omega upon 4 pi j is = 1 to n w j i j. Now this summation is where nothing but incident radiation G. So, we replace it by G. So, what we get is the last term in the equation as omega G by 4 pi. Where omega is single scattering albedo.

Now, we have this governing equation. Total number of direction is N. This is in a given direction. So, total number of directions is N. So, in plane parallel slab, N by 2; if you are interested in finding intensity at a given point. So, let us say we have certain number of directions. So, N by 2 directions will be in the upward direction and N by 2 directions will be in the downward direction.

So, those directions which are going in upward direction are basically originated from the bottom plate. And intensity direction going in the downward direction are basically originated from the top late. So, we can assume that, out of N directions, N by 2 will be in the upward

moving direction and N by 2 will be in the downward moving direction. So, based on that we can write down the intensities

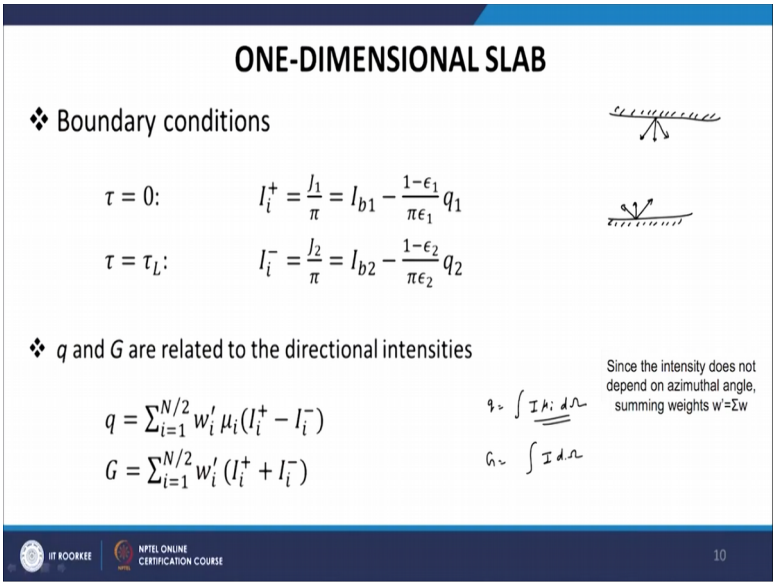
(Refer Slide Time: 16:02)



So, unknown intensities I 1 +, I 2 + and so on up to I N by 2 +. Here, + represents the intensity in the upward direction. That is, mu i > 0. And I 1 –, I 2 – and so on up to I N by 2 – represents intensity in the downward direction at the plate, tau is = tau L with mu i >, < 0. Okay. So, we have total N unknowns. So, unknowns are total N, I 1 up to I N. Out of these unknowns N by 2 intensities are in the upward direction and rest N by 2 are in the downward direction.

So, the radiative transfer equation for this 1-dimensional case can be written for mu i > 0 as, mu i d I + d tau + I i + is = the source term. And similarly, for mu I < 0, that is downward movement, – mu i d I – d tau + I i – and the source term.

(Refer Slide Time: 17:09)



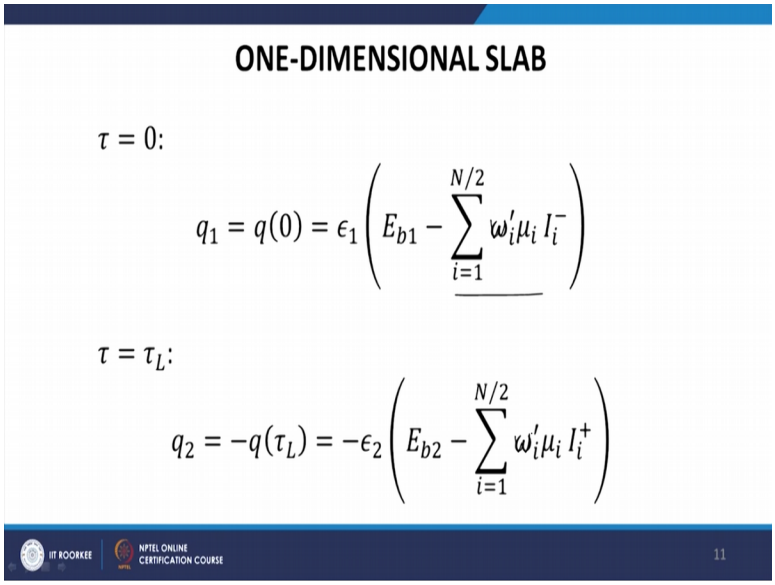
The boundary conditions can be written at the bottom wall. At the bottom wall we have tau is = 0 and intensity is moving in the upward direction. So, we write it as I i +, certain discrete directions. And this intensity will be written, is = J 1 upon pi is = I b 1 – 1 – epsilon 1 upon pi i epsilon 1 q 1. So, we can write down the boundary condition either in terms of radiosity J 1 upon pi or we can write down in terms of emission and reflection.

So, first term is reflection, I b 1 emission I b 1 and second term is basically the reflection part. Similarly, at the top boundary, the boundary condition can be written. At top boundary, tau is = tau L. We have certain discrete directions. And in these discrete directions, the intensity will be = J 2 upon pi. And this value should be = I b 2 – 1 – epsilon 2 upon pi epsilon 2 q 2. Now, q and G, as we have done many times for the approximate methods and P 1 method.

So, q and G are related to these intensities. So, we can write down q. So, q is basically = summation of intensity mu i d omega. So, we can simply write down this as summation i is = 1 to N by 2 w I i mu i I i + – I i –. And G is =, similarly, G is defined as integration over I d omega. So, we can write down this as summation i is = 1 to N by 2 w i – w i prime I i + + I i –.

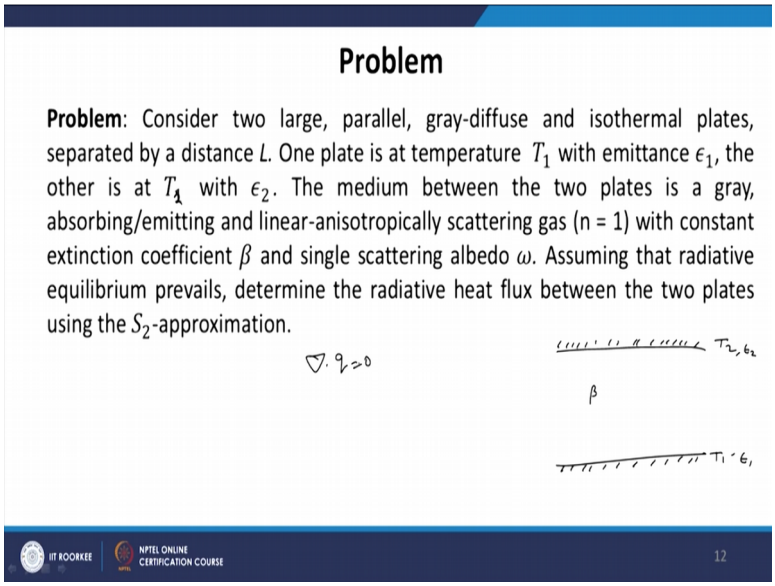
So, we can write down our radiative heat flux and incident radiation in terms of the variables that we are solving the radiative transfer equation, that is I + and I –. So, in terms of I + and I –, the radiative heat flux q and G can be written using or approximating this integral using summation as has been done here.

(Refer Slide Time: 19:29)



So, once we do that finally, our boundary condition can be written at q is = tau is = 0. That is, q 1 is = epsilon 1 E b 1 – the reflected part of the radiation i is = 1 to N by 2 w i; this is w; w i prime mu i I i –. And similarly, at the top boundary q 2 is = – q tau L. And this will be = – epsilon 2 E b 2. And then, sum over N by 2 weight mu i I i +. So, this is the expression for the heat flux at the boundaries. Now, let us apply this method to solve 1 problem.

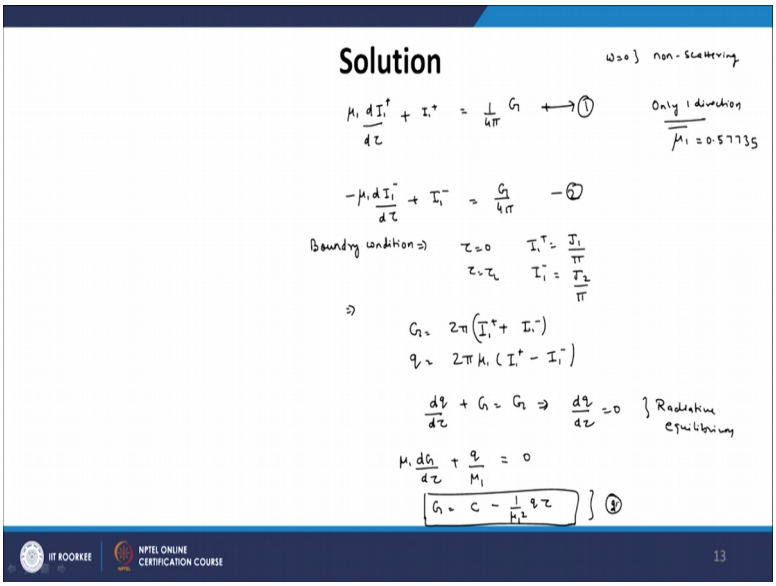
(Refer Slide Time: 20:15)



So, we have 2 large parallel gray-diffuse plates. The bottom plate, the first plate has temperature T 1 and emittance epsilon 1. The other plate is at temperature T 2 emittance epsilon 2. The medium between 2 plates is gray, absorbing and emitting. And linearly isotropically scattering gas; so, we take a extinction coefficient beta for the medium. And assuming radiative equilibrium prevails, that is, del dot q is = 0, we have to find out radiative heat flux between the 2 plates using the S 2 approximation method.

So, S 2 approximation method as we have discussed, is based on single direction. So, let us apply this method to solve this problem. So, we have to take the governing equation first. Our governing equation is given by these 2 equations, mu i; so, these are our governing equation. So, let us write down these equations for the case.

(Refer Slide Time: 21:28)



So, we have mu 1 d I 1 + by d tau + I 1 + is = 1 upon 4 pi G. And this will have a 1 more term for scattering. So, for the time being, we will just solve a simplified problem assuming omega is = 0. So, let us just solve this problem for a non-scattering case that will give us a simple solution. So, assume let us say there is no scattering. So, omega is = 0. Our equation now becomes simply this mu 1 upon d I 1 + d tau + I 1 + is = 4 pi 1 upon 4 pi G. Okay.

So, there will be only 1 direction. Okay. So, only 1 direction will be there for the S 2 method given by mu 1. Similarly, the second equation can be written as – mu 1 d I 1 – by d tau + I 1 – is = G upon 4 pi. This is our second equation. Okay. The boundary conditions, the boundary condition can be written as; at tau is = 0 I 1 + is = J 1 upon pi and at tau is = tau L, I 1 – is = J 2 upon pi. So, these are the boundary condition. Okay.

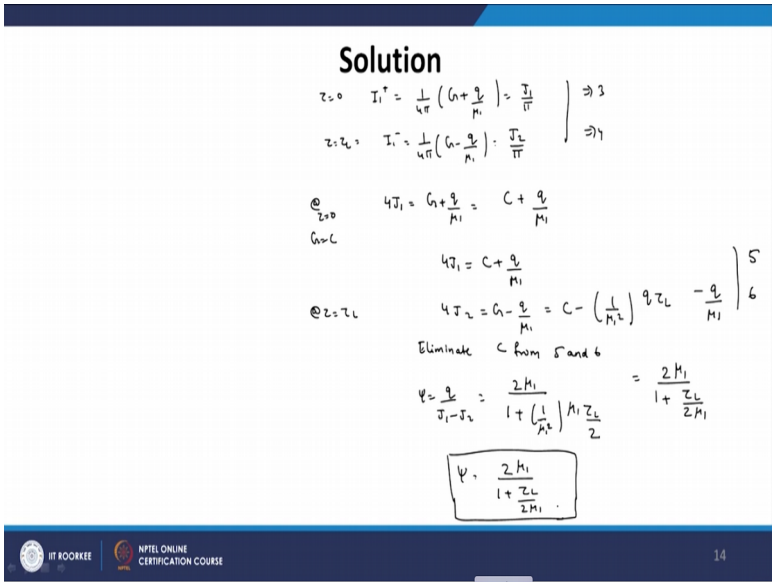
Now we have only 1 direction mu 1. Mu 1 is given by 0.57735. So, remember mu 1 is nothing but cos theta. Okay. So, this is a given direction. And the value of mu 1 is 0.57735. Okay. As is given in this table also. So, this value we are taking S 2 symmetric; So, there are 2 versions of second order discrete ordinate method, symmetric and non-symmetric. So, we are taking this symmetric method.

And the direction is 0.5773503. And w, the weight is 1.5707963. Okay. So, once we do that, now, let us solve this equation 1 and 2. So, we write the expression for G we derived here. Okay. So, G is = summation i is = 1 to N by 2 w i prime I i i + + I –. And similarly, for q; so, we write G is = 2 pi I 1 + I 1 –. And q is = 2 pi mu 1 I 1 + – I 1 –. Okay. So, what we are trying to do is, we are trying to convert our equation 1 and 2 from I 1 to in terms of G and q. Okay.

So, once we do that, our equation is transformed as dq by d tau + G is = G. Or simply, d by d tau is = 0. So, that is radiative equilibrium. So, this is a redundant equation. Because it reiterates the statement of the problem, that is radiative equilibrium. Okay. So, this equation of no use. The second equation can be written as, mu 1 dG by d tau + q by mu 1. q by mu 1 is = 0. Okay. So, this is our second equation. So, let us solve this.

This is simple ordinary differential equation. We can easily solve it. And we get G is = a constant C – 1 by mu 1 square times q tau. So, this is a solution for G in terms of q of course. Okay. So, G is = C, C – 1 upon mu 1 square q tau. Okay.

(Refer Slide Time: 26:03)



Now, we apply the boundary condition. So, boundary condition in solving this at tau is = 0. We have i 1 + is = 1 upon 4 pi G + q by mu 1 is = J 1 upon pi. So, we have written the intensity in terms of G and q. So, I 1 + and tau is = tau L. The intensity I i – is = 1 upon 4 pi G – q upon mu 1 is = j 2 upon pi. Okay. So, we have to solve for these quantities. So, let us substitute. So, this equation, let us call it equation 3 and 4. Okay.

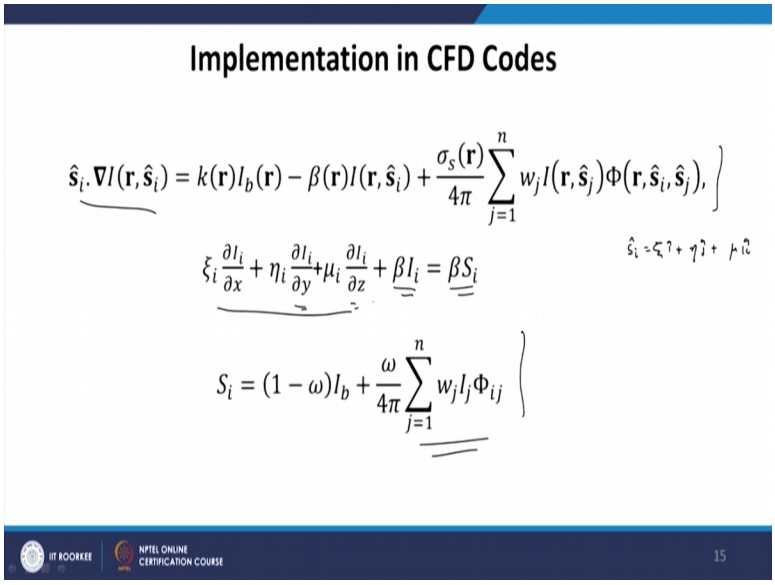
And together with this solution G. So, we have this equation. Let us call this equation 1. We have already; let us call it 2 prime. So, substituting the value of G in the boundary condition, we can solve for q. And the unknown coefficient C. So, what we do is, we write like this: 4 times J 1 is = G + q upon mu 1. Substituting for G; so, at tau is = 0, from this equation, at tau is = 0, G is simply C.

G e is = C. So, this will become C + q upon mu 1. So, 4 J 1 is = C + q by mu 1. So, this is the 1 equation. And similarly, at tau is = tau L, we get 4 times J 2 is = G – q by mu 1 is = C – 1 upon mu 1 square q tau L. Okay. And then, – q by mu 1. Okay. Now, we have obtained the final equations 5 and 6. The unknown here is q. So, we can eliminate C from 5 and 6. So, eliminate C from 5 and 6.

We can write down non-dimensional heat flux as, in terms of radiosities; q by J 1 by J 2 as 2 mu 1 upon 1 +; be this 1 upon mu 1 square. So, this will be 1 upon mu 1 square, mu 1 tau L by 2. Or simply this will be = 2 mu 1. Mu 1 is canceled. So, we have 1 + tau L by 2 mu 1. Okay. So, this is the expression for the non-dimensional radiative heat flux. Psi is = 2 mu 1 upon 1 + tau L by 2 mu 1.

So, this is the expression for the radiative heat flux for the case of 1-dimensional medium using the S 2 approximation. Now, how do we implement it in CFD codes. So, together with spherical harmonics method P 1, the spherical ordinate method is one of the most popular method used in CFD codes.

(Refer Slide Time: 29:46)



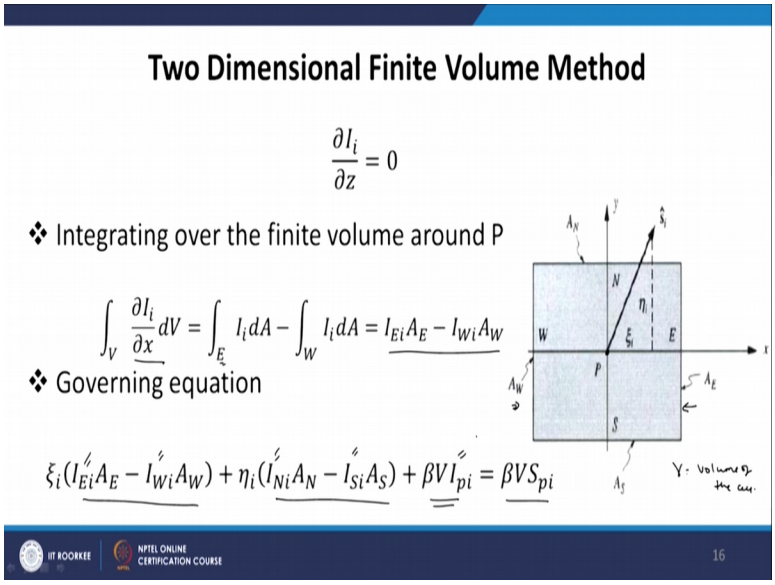
So, many commercial packages like Ansys have implementation of spherical harmonics as well as discrete ordinates method. The accuracy of the 2 method is more or less the same. Both the methods, spherical harmonics and this discrete ordinal method give good results in optically thick case. But the accuracy is relatively low in optically thin case. Now, how do we implement?

The implementation of spherical harmonics method was relatively easy as we discussed in the previous lecture. However, the implementation of discrete ordinate method is little difficult. So, what we do is, we start again the discretized radiative transfer equation in a direction I. So, this is the radiative transfer equation in a given direction I. Now the given direction I can be written in terms of the direction cosine. Okay.

So, we write the left-hand side of this radiative transfer equation by expanding the left-hand side. We get zeta i del I by del x + eta i del I by del y + mu i del I by del z, where zeta, eta and mu are direction cosines for a given direction I. Now, we have to, this is, the second term is basically the absorption term and the right-hand side is the source term, where we have source term given by 1 – omega I b + omega upon 4 pi and summed over all the discrete direction.

So, this is the summation over all the discrete directions. And phi i j is the scattering phase function. Now, what we will do is, we have to discretize the partial derivatives appearing in this equation.

(Refer Slide Time: 31:38)



So, we will take a special case of 2-dimensional method. 2-dimensional domain. The method can easily be extended to 3-dimension. Just we are understanding how it is implemented in 2-dimension. We take a special case where we have, del I by del z is = 0. That means, in 2-dimensional problem, the intensity does not vary in the z direction. Because the z direction is infinite. So, the intensity will not change with respect to z direction.

And this term will be 0. Now, as we do in any finite volume scheme, we first integrate the terms over a finite volume. So, we integrate, let us say the first term del I by del x over a finite volume. And then, we approximate using the Gauss theorem. We convert the volume integral into a surface integral. So, del I by del x is converted into intensity I subscript i d A – integration over phase w I i d A.

So, the intensity at any point we are interested in, the point is P. In the finite volume cell, we are interested in finding the intensity at point P. So, this del i by del x dV, that is integral over this volume, has been converted into flux of intensity at phase E. So, E represents the phase with area A E east phase. And w represents the west phase with area A W. And we can write down this as I Ei A E – I Wi A W.

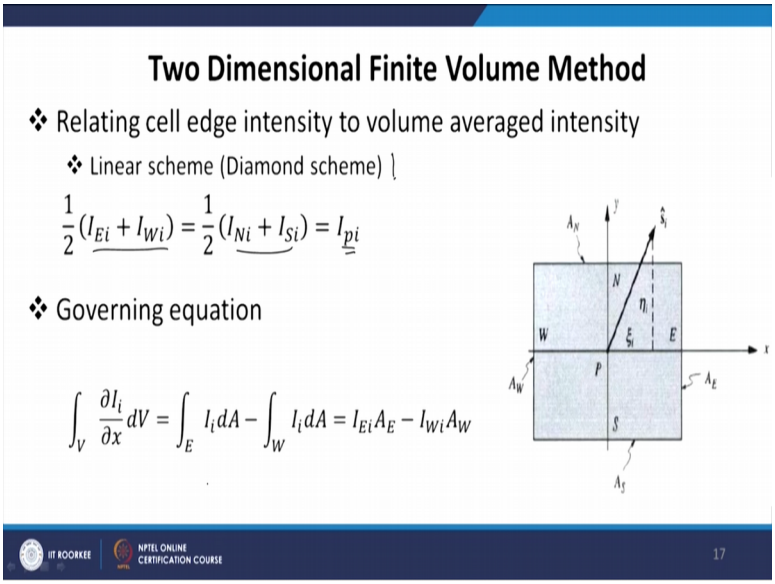
Where A E and A W are the phase areas. And I subscript Ei and I subscript Wi are intensities averaged over these phase areas. So, over the entire phase, this the average intensity I Ei and I Wi. This is similar to what we do in any finite volume discretization scheme. So similarly, we can discretize the second term del I by del y. And then, we put del I by del z as 0. And our

governing equation is now converted as, zeta i the discretize derivative with respect to x; eta i, discretize derivative with respect to y.

And then, integrated over the volume, the intensity at point P, I pi; so, I pi is the average intensity over the volume over the finite volume, is = beta V S pi. Where S pi is again now average source term over the volume. And V is the volume of the cell. In finite volume method, V represent; so, here we have unknowns I Ei, I Wi, I Ni, I Si and I pi. So, we have introduced more unknowns.

And in finite volume method, we connect this unknowns at the phase to values at the center. So, that is typically what we do in finite volume method. We connect the phase values with the phase, with the center.

(Refer Slide Time: 34:49)



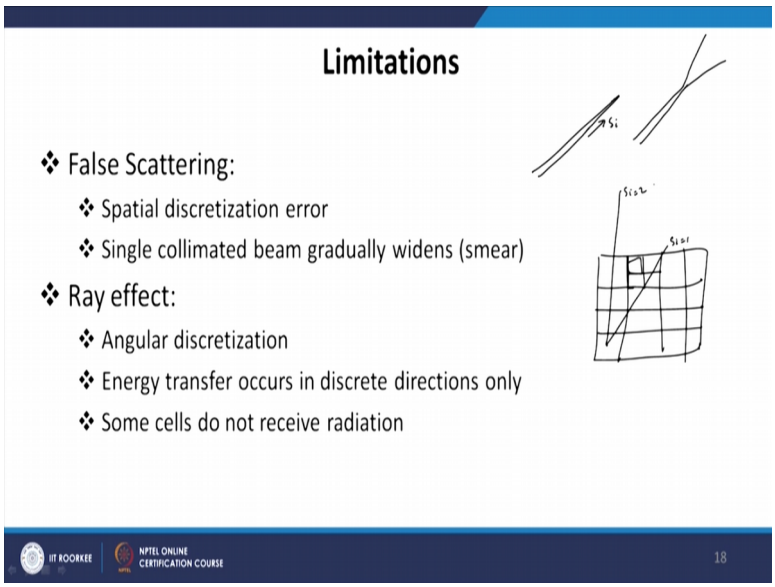
So, we can take a linear scheme. In linear scheme, we connect; this phase is in the x direction I Ei and I Wi. So, average of these 2 intensities is nothing but the intensity at the center. And similarly, North and South intensities are related to the cell center well intensity using an average value. This particular interpolation scheme in finite volumes method; we call it interpolation scheme.

This particular interpolation scheme is linear and it is called diamond scheme in radiative transfer. So, this is called diamond scheme, where we are interpolating the intensities linearly. Okay. So, our governing equation now has become relativity simplified. And we can easily

implement this in any finite volume framework. We have to repeat this for all the directions. So, what we have discussed is, in a given direction.

But, this we have to repeat for all the discrete directions. So, in a sense, we are implementing the finite volume method in space. But, we are implementing finite difference method in angular direction. We can also implement finite volume method in a angular space. Then, the method is called finite volume method, rather than the discrete ordinate method. The finite volume method of radiative transfer discretize the angular directions also in finite volume schemes.

(Refer Slide Time: 36:17)



There are certain limitations of this method such as false scattering and ray effect. And this is again what we observe in a typical finite volume method. False scattering is analogous to numerical diffusion in finite volume schemes. So, it is a result of spatial discretization error. Because we have approximated our phase center intensities with cell center intensities. We have assumed that the intensity is averaged over the phase.

So, we get spatial discretization error on intensity. And this is called false scattering. Now, how it basically looks like. We have let us say a pencil of rays in a discrete direction S i. Now, as this ray move, it will basically smear out. That means, it will scatter the radiation. So, we will have some kind of scattering effect in the radiation. So, the wave or the ray will kind of widens. It will smear in angular direction. It will not be sharp.

It will basically smear in the angular direction. It will not be confined to small solid angle, rather it will be smeared over a range of solid angles. So, this is called false scattering. Similarly, where we have a discretization error in angular direction. In angular direction, we have used finite difference method. And there are only discrete directions. So, let us take a 2-dimensional case.

We are interested in finding how the radiation is transferred between 1 cell to another. Let us say there is 1 ray we are modelling in a direction S i. Okay. So, what we see is, if this cell is let us say divided into 4 cells, this cell is not touched by this intensity. Because it is only in discrete directions. Okay. So, let us say this is S i is = 1 and this is S i is = 2. The angular directions that we are discretizing.

So, there may be some cells in the region which are not basically touched by or affected by the discrete intensity directions. Okay. So, this is called angular discretization error. And it is called ray effect. Okay. Because the energy transfer is taking place in discrete directions only, some cells do not receive radiation. And we get some kind of error. Now, definitely this error can be minimized if we increase the number of directions.

That is, more number of points in finite difference scheme results in smaller error. Same thing we can do here also. Or we can go for finite volume method, where the ray effect is minimized. So, this is all on the discrete ordinate method, is a powerful method and finds lot of applications in a radiative transfer in combustion. Many commercial packages come with this method.

In the next lecture, we will discuss the zone method, which is not a solution method for radiative transfer equation, but rather is based on principle of energy conservation. Thank you for your time. We will continue in the next lecture.