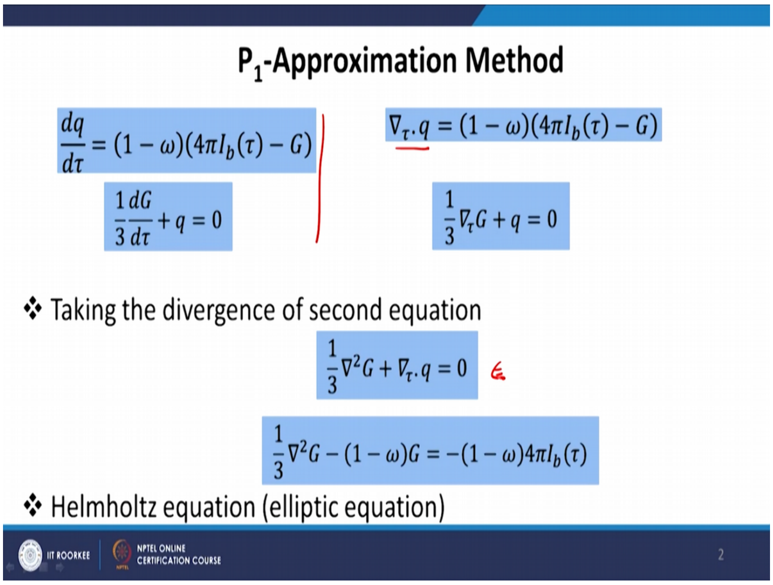
Radiative Heat Transfer
Prof. Ankit Bansal
Department of Mechanical and Industrial Engineering
Indian Institute of Technology - Roorkee

Module - 5
Lecture - 23
The Method of Spherical Harmonics (P N Approximation) - II

Hello friends, in the previous lecture we discussed the method of spherical harmonics, the P N approximation method. We derived equations and boundary conditions for the method, the first order method, that is P 1 method. We discussed the boundary conditions for 1-dimensional medium. We also argued that the method is general and it can be applied to any problem, that is any complex geometry we can apply the method. So, we will take it to general coordinate system in this lecture. The equation that we derived in the method, the spherical harmonics method, where there were 2 equations.

(Refer Slide Time: 01:07)



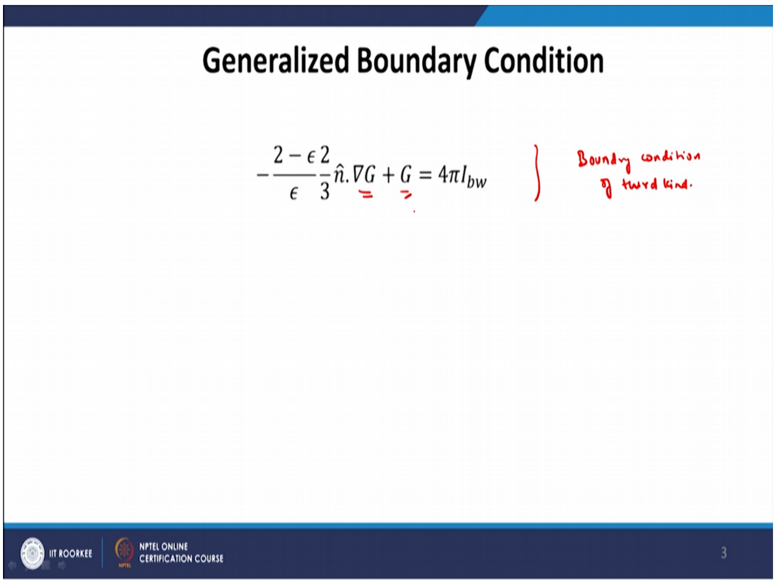
The first equation is dq by d tau is = 1 – omega 4 pi I b – G. This is the first equation. And second equation is, 1 by 3 dG by d tau + q. So, these equation, we derived in the framework of 1-dimensional plane parallel slab. But they are general, because the only assumptions while driving these equations was that the intensity does not vary with azimuthal angle. And we took first 2 terms of the spherical harmonics in deriving those equations.

So, there is no restriction on the dimensionality of the problem as such on this method. So, we can write down dq by d tau which is the divergence as del dot q, where divergence is calculated in the non-dimensional optical coordinate tau. And the right-hand side is 1 – omega 4 pi I b – G. So, this is the equation in vector form for the del dot q. Similarly, the second term, we write 1 by 3 del G. Now, we have G is a scalar.

So, we have to take gradient of this scalar G. And again, the coordinates are non-dimensional optical coordinates + q is = 0. So, q is a vector. So, when we take del dot q, we are getting a scalar. And G is a scalar. When we take gradient, it becomes a vector. So, the first equation is basically a scalar equation and the second equation is basically the vector equation. Now, what we do is; so, these are 2 coupled differential equation, 2 coupled partial differential equation if we are solving this problem in complex geometry.

So, what we will do is, we will eliminate q from these equations. So, what we take, we take divergence of the second equation and we get 1 by 3 del square G + del dot q after taking the divergence of the second equation. And then substitute for del dot q from the first equation in this. So, we get 1 by 3 del square G – 1 – omega G is = 1 – omega – 1 – omega 4 pi I b. So, this equation looks familiar very similar to the Laplace equation. This equation is called Helmholtz equation. It is an elliptic equation. We can solve this equation numerically very easily.

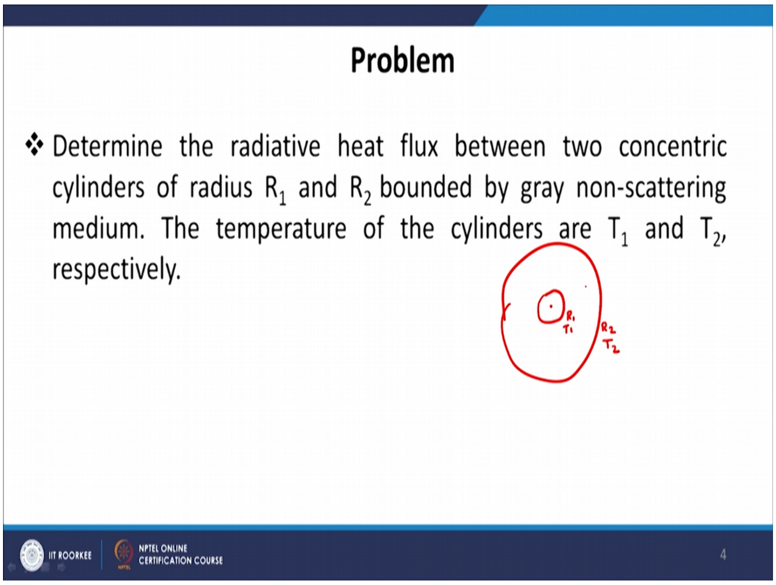
(Refer Slide Time: 03:48)



The generalized boundary condition for a gray surface in 3-dimension can be written as, – 2 – epsilon upon epsilon, where epsilon is the emittance of the surface. 2 by 3 n dot gradient G, where n dot is the vector at the local surface + G is = 4 pi I bw. Okay. Now, this type of boundary condition is called boundary condition of mixed type or third kind. So, this is boundary condition of third kind, where we have gradient and the variable appearing in the equation. So, we have gradient of G as well as the variable appearing in this equation. That is why it is boundary condition of third kind.

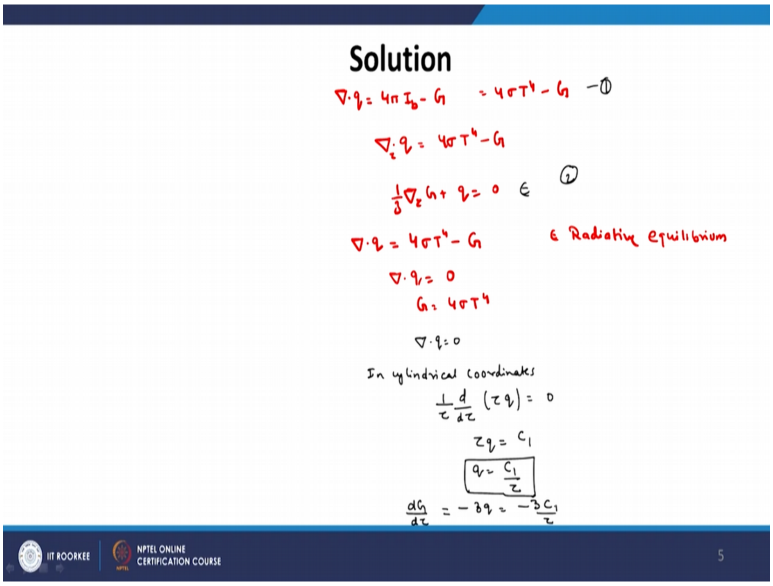
But this can be easily solved using finite volume method. I will give you an outline how this method can be applied to any problem in complex geometry. We will solve, let us solve 1 problem. So, we have a cylindrical medium, 2 concentric cylinders of radius R 1 and R 2 maintained at isothermal temperatures T 1 and T 2.

(Refer Slide Time: 04:59)



The medium inside the cylinder is gray and non-scattering. So, we have to find out the heat flux between the 2 cylinders. So, we have cylinder 1; second cylinder; this is the center. This cylinder is radius R 1 and temperature is T 1. And this is R 2 temperature is T 2. We have to find out how much heat flux is basically happening between these 2 cylinders. So, let us solve this problem.

(Refer Slide Time: 05:40)



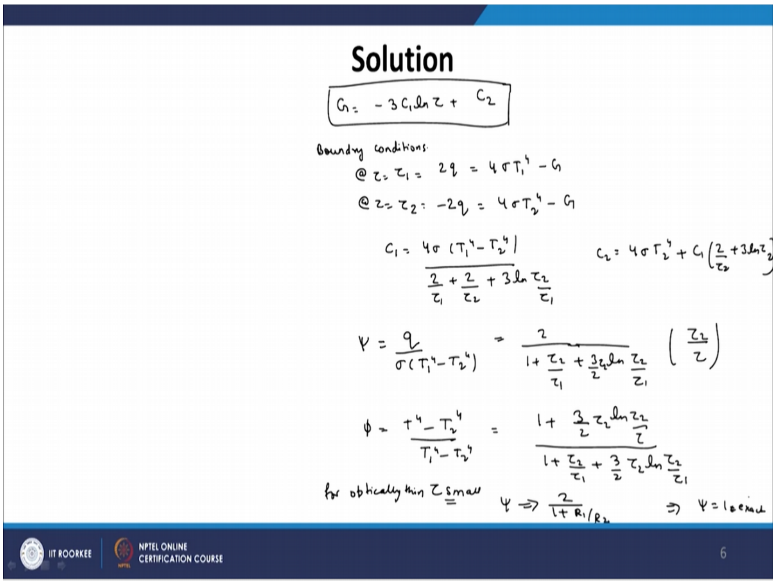
The governing equation we will, we have already derived. Okay. So, in a vector form, we will take this equation. The first equation as we derived, del dot q is = 1 – omega; omega is 0, so non-scattering medium, so 4 pi I b – G, 4 pi I b – G or simply 4 sigma T power 4 – G. So, we have gray absorption coefficient kappa. So, we can just simply integrate and find out with the total values of q and G.

So, the first equation reduces to del dot q is = 4 sigma T power 4 – G. And this is in optical coordinate. The second equation is 1 by 3 gradient of G + q is = 0. So, these are the 2 equation. Now, let us find out the solution of these equations. So, we have del dot q is = 4 sigma T power 4 – G. Now, we are assuming radiative equilibrium. So, under radiative equilibrium, del dot q will be simply = 0 and G will be simply = 4 sigma T power 4.

So, if we know G, we can calculate the temperature inside the medium. So, G will be simply = 4 sigma T power 4. If we know G, then we can solve for the unknown temperature inside the medium. Okay. Now, from this, what we get, del dot q is = 0. So, we get, in cylindrical coordinates we can write down divergence operator as 1 by tau d by d tau, tau times q. And this will be = 0.

So, when we integrate this, we get tau q is simply = C 1 or q is = C 1 by tau. That is the heat flux varies inversely with respect to tau. Okay. Now, same thing we, what we do, we substitute for q and into this equation. This is equation 1, this is equation 2. So, putting the expression of q in equation 2, we get dG by d tau is = – 3q is = – 3 C 1 by tau. Okay.

(Refer Slide Time: 08:46)



So, we get G is = – 3 times C 1 ln tau + C 2. So, this is the expression for G. Okay. Now, the boundary conditions; so, boundary condition, we will take general boundary conditions as given by this equation. Epsilon is 1, because we have black cylinders. So, epsilon is = 1. So, our boundary condition will be: At tau is = tau 1, the boundary condition is simply = 2q is = 4 sigma T 1 4 – G. And at tau is = tau 2, we have the boundary condition – 2q is = 4 sigma T 2 power 4 – G. Okay.

So, we have got 2 boundary conditions and 2 unknowns here, C1 and C2. So, what we can do, we can solve for this unknown C 1 and C 2. So, C 1 comes out to be = 4 sigma T 1 power 4 – T 2 power 4, subtracting the 2 boundary conditions will eliminate the G and you can solve for this. So, you get C 1 is = 4 sigma T 1 power 4 – T 2 power 4 upon 2 by tau 1 + 2 by tau 2 + 3 ln tau 2 by tau 1. This is the value of C1.

And C 2 is = 4 sigma T 2 power 4 + C 1 2 by tau 2 + 3 ln tau 2. So, this is the second constant. So, C 1 and C 2; sorry, it seems they are mathematically complicated. But we can still, we could solve this equation. So, we can define non-dimensional heat flux as q upon sigma T 1 power 4 – T 2 power 4. So, we have already calculated the expression for flux. So, flux is simply C 1 by tau. So, this will be simply = C 1 by tau.

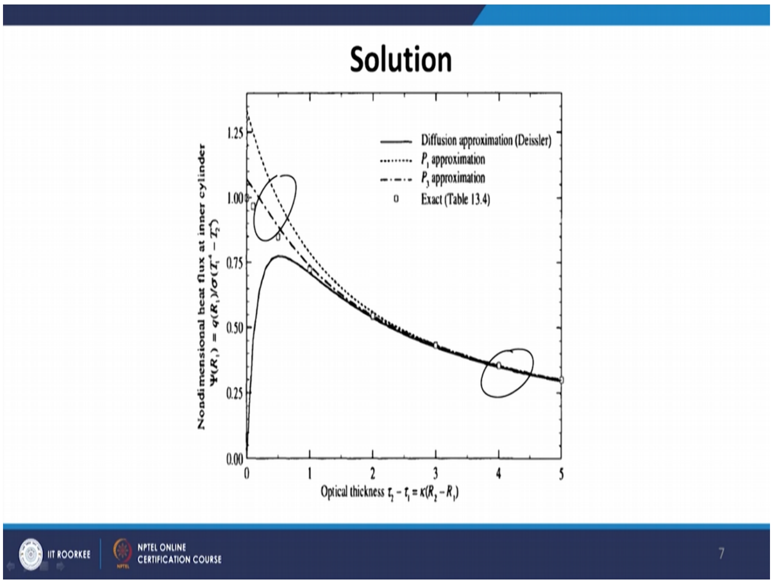
So, we get 2 upon 1 + tau 2 by tau 1 by tau 2, tau 2 by tau 1. We can just take tau 2 out; + 3 by 2 ln. In this it is tau 2 by tau 1. And this will be tau 2 common; so, tau. Okay. So, this is the heat flux, non-dimensional heat flux. And similarly, non-dimensional temperature distribution or the heat source term T 4 – T 2 4 upon T 1 4 – T 2 4. That is non-dimensional emissive power of the medium or temperature power 4 is = 1 + 3 by 2 tau 2 ln tau 2 by tau.

And this will be =, 1 + tau 2 by tau 1 + 3 by 2 tau 2 ln tau 2 by tau 1. Okay. So, this is how we have calculated the radiative heat flux, non-dimensional radiative heat flux and non-dimensional temperature power 4 or the emissive power for this concentric cylinders. Now, we will see that for optically thick case, the heat flux goes to correct value, but for optically thin case, the result is not accurate.

So, for optically thin case; optically thin case means we have tau small; kappa is small, tau is small. So, our heat flux goes to, for optically thin cases as 2 upon 1 + R 1 by R 2. So, 2 upon 1 + R 1 by R 2. While for optically thin case, the correct value is, psi is = 1. This is the exact value. Okay. So, we see that if R 1 is, if R 2 is much much larger than R 1, then this value will go to 2, psi will go to 2. So, this is not exact.

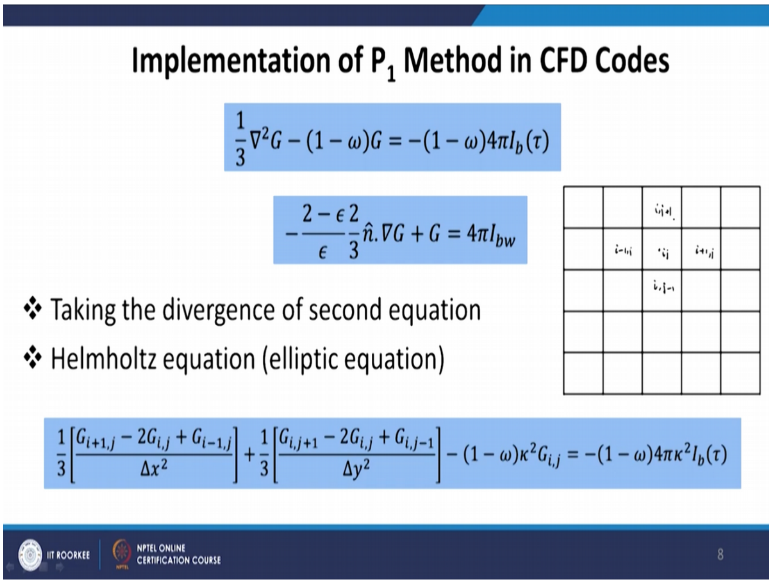
In fact, the error is 100%. While if the cylinders are almost the same dimension, R 1 is = R 2, then it will go to correctly correct optically thin limit. So, if the gap between the these 2 cylinder is small, it goes to optically thin limit correctly, but if the gap is large, then the error will be 100% for this P 1 method. So, P 1method, although very popular, but it has its own limitation in optically thin medium. And this is the result plotted for this case.

(Refer Slide Time: 14:24)



What you see here is a, the non-dimensional heat flux at the inner cylinder versus optically, optical thickness. And you see that under optically thick condition, the P 1 method, P 3 method goes to correct limit, correct values. While for optically thin case, the error is large. The P 3 method is more accurate than the P 1 method as is expected. It detains more terms. But still, the error is large for optically thin cases. So, there is inherent limitation in this spherical harmonics method to calculate heat flux in optically thin cases.

(Refer Slide Time: 15:04)



Now, how to implement this method in safety codes. So, as engineers, we all know how to program or discretize the partial differential equations. This is elliptic type of partial differential equation. We call it Helmholtz equation. We can discretize this on any domain. I have taken this here. A cartesian system on in a 2-dimensional domain, cartesian mesh, you can discretize it using central differencing scheme.

So, the first term del square G in 2-dimensional can be discretized as in terms of the values i j, i + 1 j, i – 1 j, i j – 1, i j + 1. So, this is basically a stencil. In terms of this stencil values, we can discretize the Helmholtz equation which is elliptic. And we can solve it using a techniques already developed for CFD, for this type of equations. So, normally, in a combustion applications, the many researchers have used and are using the spherical harmonics method along with the equations of momentum and energy.

But one should always keep in mind while calculating the heat flux, relative heat flux in such applications, that the method has problems in optically thin cases and the heat flux calculated using this method in optically thin cases may not be accurate. While for optically thick case, the method is in good agreement with the exact results. In the next lecture, we will discuss another approximate method. That is the discrete ordinate method to solve the radiative transfer equation.

So, together with the method of spherical harmonics, the discrete ordinate method is one of the most popular methods available in commercial safety codes. So, we will study this method in the next lecture. Thank you.